C&O 781 Topics in Quantum Information

Quantum Information Theory, Error-correction, and Cryptography

University of Waterloo

Fall 2006

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Assignment 1

Due: Oct 20, 2006

Total: 40 marks. In questions 1–4, $\mathcal{H}$ and $\mathcal{K}$ are finite dimensional Hilbert spaces.

Question 1. [5 marks] Let $M \in \mathrm{L}(\mathcal{H} \otimes \mathcal{K})$ be any linear operator on $\mathcal{H} \otimes \mathcal{K}$. Let $\{|e\rangle\}$ be any orthonormal basis for $\mathcal{H}$. Recall that the partial trace operation is a linear transformation from $\mathrm{L}(\mathcal{H} \otimes \mathcal{K})$ onto $\mathrm{L}(\mathcal{K})$, and is defined as:

$$\mathrm{Tr}_{\mathcal{H}}(M) \quad = \quad \sum_e (\langle e| \otimes I) M (|e\rangle \otimes I).$$

Prove that the partial trace operation is well-defined. I.e., it is independent of the choice of basis $\{|e\rangle\}$.

Question 2. [5 marks] Let $|\psi\rangle \in \mathcal{H} \otimes \mathcal{K}$ be a bipartite pure state such that $\mathrm{Tr}_{\mathcal{H}}(\psi) = \rho$. If $\mathcal{E} = \{p_i, \psi_i\}$ is any mixed state over $\mathcal{K}$ with the same density matrix ρ, show that there is a measurement on $\mathcal{H}$ which when performed on $|\psi\rangle$, results in the mixed state $\mathcal{E}$ in $\mathcal{K}$.

Question 3. [5 marks] Let $|\psi\rangle \in \mathcal{H} \otimes \mathcal{K}$ be a bipartite pure state such that $\mathrm{Tr}_{\mathcal{H}}(\psi) = \rho$, and $\rho = \sum_i \lambda_i |v_i\rangle\langle v_i|$ in diagonal form. Show that there is an orthonormal set $\{|u_i\rangle\}$ in $\mathcal{H}$ such that $|\psi\rangle = \sum_i \sqrt{\lambda_i}|u_i\rangle|v_i\rangle$.

Question 4. [5 marks] Find the relation between two sets of operation elements $\{A_k\}$ and $\{B_l\}$ that correspond to the same TCP map $\mathcal{E} : \mathrm{L}(\mathcal{H}) \to \mathrm{L}(\mathcal{K})$. I.e.,

$$\mathcal{E}(\rho) = \sum_k A_k \rho A_k^\dagger = \sum_l B_l \rho B_l^\dagger \quad \text{for} \quad k \leq l$$

Question 5. [5 marks] Consider the "amplitude damping channel" $\mathcal{E}$ acting on a single qubit S. The isometric extension to the qubit S and an environment qubit E (the isometry in the unitary representation) acts on the input state $|\psi\rangle = a|0\rangle + b|1\rangle$ as

$$U\ |\psi\rangle_S |0\rangle_E = a|00\rangle + b\left[\ \sqrt{1-\gamma}\ |10\rangle + \sqrt{\gamma}\ |01\rangle\ \right]$$

Find the Kraus representation of $\mathcal{E}$, with the minimal number of operation elements. What event does each of these operation elements represent?

Question 6. [15 marks]

(a) Let C be the 5-qubit stabilizer code with generators $IXZZX$, $XZZXI$, $ZZXIX$, $ZXIXZ$. How does it correct for up to 1 Pauli error [3 marks]? How does it protect an encoded qubit against the amplitude damping noise $\mathcal{E}^{\otimes 5}$ (up to order γ) [2 marks]?

(b) Consider instead the stabilizer code with generators $ZZZZ, XXII, IIXX$. Write down a simple basis for the codespace [2 marks]. What can be done to an encoded qubit if the amplitude damping noise $\mathcal{E}^{\otimes 4}$ occurs (assuming small γ)? [6 marks]. Interpret your result [2 mark].

(c) Open problem: Does a 3-qubit amplitude damping code exist?