C&O 781 / QIC 890 Topics in Quantum Information
Recent advances in Quantum Information
University of Waterloo, Fall 2013
Instructor: Ashwin Nayak
Assignment 2, Oct. 25, 2013
Due: by noon, Nov. 8, 2013

Question 1. Let n be an odd integer ≥ 3. Let $\mathrm{Maj}_n : \{0,1\}^n \to \{0,1\}$ denote the majority of n bits $\mathrm{Maj}_n(x) = 1$ iff $\sum_{i=1}^n x_i > n/2$. Using the adversary method, find the largest bound you can on the bounded error query complexity of Maj_n.

Question 2. Let $S, T \subseteq \{0,1\}^n$ be non-empty and disjoint. Let $A = (a_{xy})$ be a $|S| \times |T|$ matrix over $\mathbb{R}$. Consider the following optimization problem over vector variables $|u_x\rangle \in \mathbb{R}^d$, for some $d \geq 1$, $x \in S \cup T$:

$$\sup \quad \sum_{x \in S, y \in T} a_{xy} \langle u_x | u_y \rangle$$

subject to:

$$\|u_x\| = 1 \ , \quad \text{for all } x \in S \cup T$$

Formulate this as an SDP. Relax any equality constraint in your SDP to obtain an equivalent SDP with only inequalities (i.e., an SDP whose optimum remains the same). Find the dual of the relaxation, and prove that strong duality holds for this SDP.

Question 3. Let $f, g : \{0,1\}^n \to \{0,1\}$ be two boolean functions. Let $h : \{0,1\}^{2n} \to \{0,1\}$ be a composition of the two defined as $h(xy) = f(x) \vee g(y)$, where $x, y \in \{0,1\}^n$. Suppose we are given span programs for f, g with complexity C_f, C_g, respectively. Construct a span program for h with complexity $\|C\| = \left(C_f^2 + C_g^2\right)^{1/2}$.

Note that the naïve composition of span programs would give us complexity $\sqrt{2} \cdot \max\{C_f, C_g\}$. The above construction generalizes to the OR of m different functions, and gives us an alternative to an algorithm for "variable time amplitude amplification" due to Ambainis. Another implication is that any "read-once" formula over n variables and gate set $\{\vee, \wedge, \neg\}$ (with possibly unbounded fan-in) has quantum query complexity of order $\sqrt{n}$.

Question 4. Let $\mathcal{K}_1, \mathcal{K}_2$ be arbitrary subspaces of a finite dimensional Hilbert space $\mathcal{H}$. Let Π_i be the orthogonal projection operator onto $\mathcal{K}_i$. Consider $U = (2\Pi_2 - \mathbb{I})(2\Pi_1 - \mathbb{I})$, the product of the reflections through $\mathcal{K}_1$ and $\mathcal{K}_2$. Let $D = \Pi_2 \Pi_1$ be the corresponding *discriminant* matrix, and let $D = \sum_{i=1}^n \sigma_i |u_i\rangle\langle v_i|$ be its singular value decomposition. Prove that

1. The subspace $\mathcal{H}_i = \mathrm{Span}\{|u_i\rangle, |v_i\rangle\}$ is invariant under U, and
2. If $\dim(\mathcal{H}_i) = 2$, then the restriction of U to $\mathcal{H}_i$ is a rotation through angle $2\theta_i$, where $\cos\theta_i = \sigma_i$. Further, the eigenvalues of U in the eigenspaces in $\mathcal{H}_i$ are $\exp(\pm 2\imath\theta_i)$, where $\imath = \sqrt{-1}$.