CO 781 / QIC 823 / CS 867 Quantum Algorithms
University of Waterloo, Winter 2017
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Assignment 1, Jan. 23, 2017
Due: by noon, Feb. 6, 2017

Question 1. Recall that in the Element Distinctness problem, we are given as input a positive integer n, and an oracle (or "black-box") for n numbers $x_1, x_2, \ldots, x_n$. The goal is to determine if there is a collision, i.e., if there are distinct indices $i, j \in [n]$ such that $x_i = x_j$.

(a) Let $A \subseteq [n]$ be any set of m indices. Design an $\mathrm{O}(m + \sqrt{n}\,)$-query quantum algorithm that determines whether some element $i \in A$ is part of a collision.

(b) To solve Element Distinctness, it suffices to search for a block of m indices which contains a part of a collision. Using this idea, design an algorithm for Element Distinctness with the smallest quantum query complexity you can.

Question 2. Consider the following Markov Chain on the set of vertices $\{0, 1, \ldots, n-1\}$. In each time step, from vertex i, with probability $1/2$, we stay at the vertex i, and with probability $1/2$, we move to a uniformly random j such that $(i - j) \equiv \pm 1 \pmod{n}$.

(a) Compute the eigenvalues of the Markov Chain, and hence its spectral gap.

(b) Suppose n is even. What is the expected hitting time of vertex 0, if we start at vertex $n/2$?

Question 3. Let $G = (V, E)$ be an undirected graph on n vertices. A 4-cycle in G is a sequence of four distinct vertices v_0, v_1, v_2, v_3, such that $\{v_i, v_{i+1}\}$, for $i \in [2]$ are edges, as is $\{v_3, v_0\}$.

Suppose the adjacency matrix of the graph G is specified through an oracle (or "black box"). Design a quantum algorithm that, given n and an oracle for the edges of the graph, finds a 4-cycle in G with query complexity $\mathrm{o}(n^2)$.

Question 4. Let P be a reversible, ergodic Markov chain over set X. Let $M \subset X$ be a subset of states, and $P_{\bar{M}}$ the restriction of P to $X - M$. Using the notation defined in class, define $\mathcal{A}_{\bar{M}}$ and $\mathcal{B}_{\bar{M}}$ to be the span of the vectors $\{|\phi_x\rangle\}$ and $\{|\psi_y\rangle\}$, for $x, y \in X - M$, respectively. Define $W_{\bar{M}}$ as the product of reflections through $\mathcal{A}_{\bar{M}}$ and $\mathcal{B}_{\bar{M}}$.

(a) Describe the eigenvalues and eigenvectors of $W_{\bar{M}}$ in terms of those of $Q_{\bar{M}}$, the symmetrization of $P_{\bar{M}}$.

(b) Design an algorithm to detect whether the set M is empty or not, using the walk $W_{\bar{M}}$. What is the complexity of your algorithm in terms of the set-up, update, and checking costs, and the spectral properties of $P_{\bar{M}}$?