QIC 890/891 Selected advanced topics in Quantum Information
University of Waterloo, Spring 2011

Semi-definite programming in quantum information
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Homework 1

Question 1. Let $\mathcal{H}, \mathcal{K}, \mathcal{K}_1, \mathcal{K}_2$ be finite dimensional complex Hilbert spaces, $C \in \mathrm{L}(\mathcal{H}), B \in \mathrm{L}(\mathcal{K}), B_i \in \mathrm{L}(\mathcal{K}_i)$, $i = 1, 2$ be Hermitian, and let $\Phi : \mathrm{L}(\mathcal{H}) \to \mathrm{L}(\mathcal{K})$, $\Phi_i : \mathrm{L}(\mathcal{H}) \to \mathrm{L}(\mathcal{K}_i)$, $i = 1, 2$, be linear operators.

Prove that the following three forms of SDP are equivalent, by recasting each in the other forms. Note that the matrices in the first two may be complex, whereas those in the third are all real.

$$\begin{aligned} \inf \quad & \langle C, X \rangle \\ \text{subject to:} \quad & \\ & \Phi(X) \succeq B \\ & X \succeq 0, \quad X \in \mathrm{L}(\mathcal{H}) \end{aligned} \tag{P1}$$

The second form has both equality and inequality constraints:

$$\begin{aligned} \sup \quad & \langle C, X \rangle \\ \text{subject to:} \quad & \\ & \Phi_1(X) = B_1 \\ & \Phi_2(X) \preceq B_2 \\ & X \succeq 0, \quad X \in \mathrm{L}(\mathcal{H}) \end{aligned} \tag{P2}$$

The following is the standard form of SDP studied in the literature. Let $C' \in \mathrm{L}(\mathbb{R}^n), b \in \mathbb{R}^m$ and let $\Phi' : \mathrm{L}(\mathbb{R}^n) \to \mathbb{R}^m$ be linear.

$$\begin{aligned} \sup \quad & \left\langle C', X \right\rangle \\ \text{subject to:} \quad & \\ & \Phi'(X) = b \\ & X \succeq 0, \quad X \in \mathrm{L}(\mathbb{R}^n) \end{aligned} \tag{P3}$$

Using the dual defined for the program P1, derive the duals to the other two in the simplest form you can (with the fewest variables).

Question 2. Recall that the SDP formulation of the maximum acceptance probability of a verifier in a quantum interactive proof has the following form:

$$\begin{aligned} \sup \quad & \langle \mathbb{I}_{\mathcal{P}} \otimes \Pi_1, \rho_m \rangle \\ \text{subject to:} \quad & \\ & \mathrm{Tr}_{\mathcal{M}}(V_x^\dagger \rho_1 V_x) = |\bar{0}\rangle\langle\bar{0}| \\ & \mathrm{Tr}_{\mathcal{M}}(V_x^\dagger \rho_i V_x) = \mathrm{Tr}_{\mathcal{M}}(\rho_{i-1}), \quad \text{for } i = 2, \ldots, m \\ & \rho_i \succeq 0, \quad \rho_i \in \mathrm{L}(\mathcal{M} \otimes \mathcal{V}), \quad \text{for } i = 1, \ldots, m \end{aligned}$$

Derive its dual. Explain how you may modify the primal without changing its objective function value, so that the feasible region is well-bounded.

Question 3. Consider an XOR non-local game G specified by a distribution π on $S \times T$, and function f. Let the strategy of the two players be given by the joint state $|\psi\rangle$, and observables $(A_s), (B_t)$, respectively. Prove that the bias of the strategy is

$$\sum_{s,t} (-1)^{f(s,t)} \pi(s,t) \langle \psi | (A_s \otimes B_t) | \psi \rangle \ .$$

Let Q denote the matrix $|0\rangle\langle 1| \otimes (\frac{1}{2}P) + |1\rangle\langle 0| \otimes (\frac{1}{2}P^{\mathrm{T}})$, where P is the cost matrix of the game. Recall that the dual SDP for the best strategy is given by

$$\begin{array}{rl} \inf & \langle (u,v), \bar{1} \rangle \\ \text{subject to:} & \\ & \Delta(u,v) \succeq Q \\ & u \in \mathbb{R}^S, \quad v \in \mathbb{R}^T \end{array}$$

Find a Slater point for this SDP.

Question 4. Let $M = |0\rangle\langle 1| \otimes N + |1\rangle\langle 0| \otimes N^{\mathrm{T}}$ be a real symmetric matrix. Prove that $\|M\| \leq 1$ iff $M \preceq \mathbb{I}$.