# One-Shot Quantum State Redistribution and Quantum Markov Chains

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***Abstract*—We revisit the task of quantum state redistribution in the one-shot setting, and design a protocol for this task with communication cost in terms of a measure of distance from quantum Markov chains. More precisely, the distance is defined in terms of quantum max-relative entropy and quantum hypothesis testing entropy.**

**Our result is the first to operationally connect one-shot quantum state redistribution and quantum Markov chains, and can be interpreted as an operational interpretation for a possible one-shot analogue of quantum conditional mutual information. The communication cost of our protocol is lower than all previously known ones and asymptotically achieves the well-known rate of quantum conditional mutual information. Thus, our work takes a step towards the important open question of near-optimal characterization of the one-shot quantum state redistribution.**

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## I. Introduction

The connection between conditional mutual information and Markov chains has led to a rich body of results in classical computer science and information theory. It is well known that for any tripartite distribution $P^{RBC}$,

$$\mathrm{I}(R : C \,|\, B)_P = \min_{Q^{RBC} \,\in\, \mathsf{MC}_{R-B-C}} \mathrm{D}\left(P^{RBC} \middle\| Q^{RBC}\right) \ ,$$

where $\mathsf{MC}_{R-B-C}$ is the set of Markov distributions $Q$, i.e., those that satisfy $\mathrm{I}(R : C \,|\, B)_Q = 0$. In fact, one can choose a distribution $Q$ achieving the minimum above with $Q^{RB} = P^{RB}$, $Q^{BC} = P^{BC}$. In the quantum case, the above identity fails drastically. For an example presented in Ref. [1] (see also Ref. [2, Section VI]), the right-hand side is a constant, whereas the left-hand side approaches zero as the system size increases. Given this, it is natural to ask if there is an extension of the classical identity to the quantum case. This is shown to be true in a sense that for any tripartite quantum state $\psi^{RBC}$, it holds that

$$\mathrm{I}(R : C|B)_\psi = \min_{\sigma^{RBC} \in \mathsf{QMC}_{R-B-C}} \left(\mathrm{D}\big(\psi^{RBC} \| \sigma^{RBC}\big) - \mathrm{D}\big(\psi^{BC} \| \sigma^{BC}\big)\right) \tag{I.1}$$

where $\mathsf{QMC}_{R-B-C}$ is the set of quantum states $\sigma$ satisfying $\mathrm{I}(R : C|B)_\sigma = 0$, $\psi^{RB} = \sigma^{RB}$. The proof of the above equation is implicit in Ref. [3, Lemma 1]. The difference between the quantum and the classical expressions can now be understood as follows. For the classical case, the closest Markov chain $Q$ to a distribution $P$ (in relative entropy) satisfies the aforementioned relations $Q^{RB} = P^{RB}$ and $Q^{BC} = P^{BC}$. Thus, the second relative entropy term vanishes in Eq. (I.1). In quantum case, due to monogamy of entanglement we cannot in general ensure that $\sigma^{BC} = \psi^{BC}$. Thus, the quantum relative entropy distance to quantum Markov chains can be bounded away from the quantum conditional mutual information.

In this work, we prove a one-shot analogue of Eq. (I.1). This is achieved in an operational manner, by showing that a one-shot analogue of the right-hand side in Eq. (I.1) is the achievable communication cost in *quantum state redistribution* of $|\psi\rangle^{RABC}$, a purification of $\psi^{RBC}$. In the task of quantum state redistribution, a pure quantum state $|\psi\rangle^{RABC}$ known to two parties, Alice and Bob, is shared between Alice (who has registers $AC$), Bob (who has $B$), and the referee (who has $R$). Additionally, Alice and Bob may share an arbitrary (pure) entangled state. The goal is to transmit the content of register $C$ to Bob using a communication protocol involving only Alice and Bob, in such a way that all correlations, including those with the referee, are approximately preserved. (See Figure 1 for an illustration of state redistribution.) Given

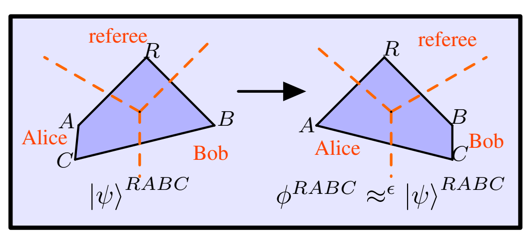


Fig. 1. An illustration of quantum state redistribution.

a quantum state $\phi^{RBC}$, we identify a natural subset of Markov extensions of $\phi^{RB}$, which we denote by $\mathsf{ME}^{\epsilon,\phi}_{R-B-C}$ and define formally in Section II-B. We establish the following result.

**Theorem I.1.** *For any pure quantum state $|\psi\rangle^{RABC}$, the quantum communication cost of redistributing the register $C$ from Alice (who initially holds $AC$) to Bob (who initially holds $B$) with error $10\sqrt{\epsilon}$ is at most*

$$\frac{1}{2} \min_{\psi' \in \mathsf{B}^{\epsilon}(\psi^{RBC})} \min_{\sigma^{RBC} \in \mathsf{ME}^{\epsilon^2/4, \psi'}_{R-B-C}} \left[ \mathrm{D}_{\max}\left(\psi'^{RBC} \| \sigma^{RBC}\right) - \mathrm{D}^{\epsilon}_{\mathrm{H}}\left(\psi'^{BC} \| \sigma^{BC}\right) \right] + \mathrm{O}\left(\log \frac{1}{\epsilon}\right) \quad . \quad (\text{I}.2)$$

The difference between minimizing over the set $\mathsf{ME}^{\epsilon^2/4,\psi'}_{R-B-C}$ versus $\mathsf{QMC}_{R-B-C}$ appears to be minor, and is best understood from the definitions in Section II-A. We believe the above result can be stated in terms of a minimization over all of $\mathsf{QMC}_{R-B-C}$. As far as we know, this result is the first that operationally connects the cost of quantum state redistribution to Markov chains (even in the asymptotic i.i.d. setting). Among the previous works [4]–[6], the best previously known achievable one-shot bound for the communication cost of state redistribution, namely,

$$\frac{1}{2} \inf_{\sigma^C} \inf_{\psi' \in \mathsf{B}^{\epsilon}(\psi^{RBC})} \left( \mathrm{D}_{\max}\left(\psi'^{RBC} \| \psi'^{RB} \otimes \sigma^{C}\right) - \mathrm{D}^{\epsilon^2}_{\mathrm{H}}\left(\psi'^{BC} \| \psi'^{B} \otimes \sigma^{C}\right) \right) + \log \frac{1}{\epsilon^2} \quad , \quad (\text{I}.3)$$

when the state $|\psi\rangle^{RABC}$ is redistributed with error $\mathrm{O}(\epsilon)$ was due to Anshu, Jain, and Warsi [6]. Note that $\sigma^C := \psi'^C$ is a nearly optimal solution for Eq. (I.3) as discussed in Ref. [7], and the product state $\psi'^{RB} \otimes \psi'^C$ is a Markov state in the set $\mathsf{ME}^{\epsilon^2/4,\psi'}_{R-B-C}$. So, the bound in Theorem I.1 is smaller than Eq. (I.3) in the sense that the minimization is over a larger set. In the special case where $\psi^{RBC}$ is a quantum Markov chain, our protocol has near-zero communication. This feature is not present in other protocols and their communication may be as large as $(1/2)\log|C|$.

*Techniques*

The protocol we design is most easily understood by considering a folklore protocol for redistributing quantum Markov states. In the case that $\psi^{RBC}$ is a Markov state, its purification $|\psi\rangle^{RABC}$ can be transformed through local isometry operators $V_A : A \to A^R J' A^C$ and $V_B : B \to B^R J B^C$ into the following:

$$V_A \otimes V_B \, |\psi\rangle^{RABC} = \sum_j \sqrt{p(j)}\, |\psi_j\rangle^{RA^RB^R} \otimes |jj\rangle^{JJ'} \otimes |\psi_j\rangle^{A^CB^CC} \quad . \quad (\text{I}.4)$$

The existence of isometries $V_A$ and $V_B$ is a consequence of the special structure of quantum Markov states proved by Hayden, Josza, Petz, and Winter [8]. Note that after the above transformation, conditioned on registers $J$ and $J'$, systems $RA^RB^R$ are decoupled from systems $A^CCB^C$. So using the embezzling technique due to van Dam and Hayden [9], conditioned on $J$ and $J'$, Alice and Bob can first embezzle-out systems $A^CCB^C$ and then embezzle-in the same systems such that at the end the global state is close to the state in Eq. (I.4) and system $C$ is with Bob. This protocol incurs no communication; see Fig. 2 for an illustration.

The protocol we design is a more sophisticated version of the above. The key technique is a reduction procedure using embezzling quantum states, that allows us to use a protocol due to Anshu, Jain, and Warsi [6] as a subroutine.

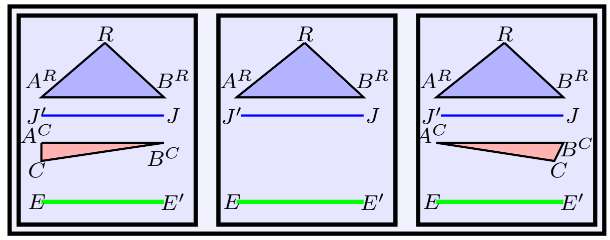


Fig. 2. An illustration of the zero-cost protocol for redistributing Markov states. Left: Registers $RA^RB^RJJ'A^CCB^C$ are in the state given in Eq. (I.4) and registers $E$ and $E'$ contain Alice and Bob's share of an embezzling state, respectively. Middle: Using embezzling registers, Alice and Bob jointly embezzled out registers $A^CCB^C$ via local unitary operations. Right: Using embezzling registers, conditioned on $J$ and $J'$, Alice and Bob embezzled in $|\psi_j\rangle^{A^CCB^C}$ such that registers $C$ and $B^C$ are with Bob and register $A^C$ is with Alice. This step also only contains local unitary operations and no communication.

## II. Preliminaries

### A. Mathematical notation and background

For a thorough introduction to basics of quantum information, we refer the reader to the book by Watrous [10]. In this section, we briefly review the notation and some results that we use in this article.

For the sake of brevity, we denote the set $\{1, 2, \ldots, k\}$ by $[k]$. We denote physical quantum systems ("registers") with capital letters, like $A$, $B$ and $C$. The state space corresponding to a register is a finite-dimensional Hilbert space. We denote (finite dimensional) Hilbert spaces by capital script letters like $\mathcal{H}$ and $\mathcal{K}$, and the Hilbert space corresponding to a register $A$ by $\mathcal{H}^A$. We sometimes refer to the space corresponding to the register $A$ by the name of the register.

We use the Dirac notation, i.e., "ket" and "bra", for unit vectors and their adjoints, respectively. We denote the set of all unitary operators by $\mathsf{U}(\mathcal{H})$, and the set of all quantum states (or "density operators") over $\mathcal{H}$ by $\mathsf{D}(\mathcal{H})$. The identity operator on space $\mathcal{H}$ or register $A$, is denoted by $\mathbb{1}^{\mathcal{H}}$ or $\mathbb{1}^A$, respectively. Similarly, we use superscripts to indicate the registers on which an operator acts.

We denote quantum states by lowercase Greek letters like $\rho, \sigma$. We use the notation $\rho^A$ to indicate that register $A$ is in quantum state $\rho$. We denote the *partial trace* over register $A$ by $\mathrm{Tr}_A$. We say $\rho^{AB}$ is an *extension* of $\sigma^A$ if $\mathrm{Tr}_B(\rho^{AB}) = \sigma^A$. A *purification* of a quantum state $\rho$ is an extension of $\rho$ with rank one. For the Hilbert space $\mathbb{C}^S$ for some set $S$, we refer to the basis $\{|x\rangle : x \in S\}$ as the canonical basis for the space. We say the register $X$ is *classical* in a quantum state $\rho^{XB}$ if $\rho^{XB}$ is block-diagonal in the canonical basis of $X$, i.e., $\rho^{XB} = \sum_x p(x)|x\rangle\langle x|^X \otimes \rho^B_x$ for some probability distribution $p$ on $X$. For a non-trivial register $B$, we say $\rho^{XB}$ is a *classical-quantum* state if $X$ is classical in $\rho^{XB}$. We say a unitary operator $U^{AB} \in \mathsf{U}(\mathcal{H}^A \otimes \mathcal{H}^B)$ is *read-only* on register $A$ if it is block-diagonal in the canonical basis of $A$, i.e., $U^{AB} = \sum_a |a\rangle\langle a|^A \otimes U_a^B$ where each $U_a^B$ is a unitary operator.

The *fidelity* between two quantum states $\rho$ and $\sigma$ is defined as

$$\mathrm{F}(\rho, \sigma) \quad := \quad \mathrm{Tr}\sqrt{\sqrt{\rho}\,\sigma\sqrt{\rho}} \ .$$

Fidelity can be used to define a useful metric called the *purified distance* [11], [12] between quantum states:

$$\mathrm{P}(\rho, \sigma) \quad := \quad \sqrt{1 - \mathrm{F}(\rho, \sigma)^2} \ .$$

This metric is also known as *sine distance* in literature [13]–[15]. For a quantum state $\rho \in \mathsf{D}(\mathcal{H})$ and $\epsilon \in [0,1]$, we define

$$\mathsf{B}^\epsilon(\rho) \quad := \quad \{\widetilde{\rho} \in \mathsf{D}(\mathcal{H}) : \ \mathrm{P}(\rho, \widetilde{\rho}) \leq \epsilon\}$$

as the ball of quantum states that are within purified distance $\epsilon$ of $\rho$. Note that in some works, the states in the set $\mathsf{B}^\epsilon(\rho)$ are allowed to be sub-normalized. But here, we require the states in the ball to have trace equal to one.

**Theorem II.1** (Uhlmann [16])**.** *Consider quantum states $\rho^A, \sigma^A \in \mathsf{D}(\mathcal{H}^A)$. Suppose $|\xi\rangle^{AB}, |\theta\rangle^{AB} \in \mathsf{D}(\mathcal{H}^A \otimes \mathcal{H}^B)$ are arbitrary purifications of $\rho^A$ and $\sigma^A$, respectively. Then, there exists some unitary operator $V^B \in \mathsf{U}(\mathcal{H}^B)$ such that*

$$\mathrm{P}\Big(|\xi\rangle^{AB}, (\mathbb{1} \otimes V^B)\,|\theta\rangle^{AB}\Big) \quad = \quad \mathrm{P}(\rho^A, \sigma^A) \ .$$

Let $\rho \in \mathsf{D}(\mathcal{H})$ be a quantum state over the Hilbert space $\mathcal{H}$. The *von Neumann entropy* of $\rho$ is defined as

$$\mathrm{S}(\rho) \quad := \quad -\mathrm{Tr}\,(\rho \log \rho) \ .$$

This coincides with Shannon entropy for a classical state. The *relative entropy* of two quantum states $\rho, \sigma \in \mathsf{D}(\mathcal{H})$ is defined as

$$\mathrm{D}(\rho \| \sigma) \quad := \quad \mathrm{Tr}\,(\rho\,(\log \rho - \log \sigma)) \ ,$$

when $\mathrm{supp}(\rho) \subseteq \mathrm{supp}(\sigma)$, and is $\infty$ otherwise. The *max-relative entropy* [17] of $\rho$ with respect to $\sigma$ is defined as

$$\mathrm{D}_{\max}(\rho \| \sigma) \quad := \quad \min\{\lambda : \rho \leq 2^\lambda \sigma\} \ ,$$

when $\mathrm{supp}(\rho) \subseteq \mathrm{supp}(\sigma)$, and is $\infty$ otherwise. For $\epsilon \in [0,1]$, the *$\epsilon$-smooth max-relative entropy* [17] of $\rho$ with respect to $\sigma$ is defined as

$$\mathrm{D}^\epsilon_{\max}(\rho \| \sigma) \quad := \quad \min_{\rho' \in \mathsf{B}^\epsilon(\rho)} \mathrm{D}_{\max}(\rho' \| \sigma) \ .$$

For $\epsilon \in [0,1]$, the *$\epsilon$-hypothesis testing relative entropy* [18]–[20] of $\rho$ with respect to $\sigma$ is defined as

$$\mathrm{D}^\epsilon_{\mathrm{H}}\,(\rho \| \sigma) \quad := \quad \sup_{0 \preceq \Pi \preceq \mathbb{1},\, \mathrm{Tr}(\Pi\rho) \geq 1-\epsilon} \log\left(\frac{1}{\mathrm{Tr}(\Pi\sigma)}\right) \ .$$

Suppose that $\rho^{AB} \in \mathsf{D}(\mathcal{H}^A \otimes \mathcal{H}^B)$ is the joint state of registers $A$ and $B$, then the *mutual information* of $A$ and $B$ is denoted by

$$\mathrm{I}(A:B)_\rho \quad := \quad \mathrm{D}\big(\rho^{AB} \|\ \rho^A \otimes \rho^B\big) \ .$$

When the state is clear from the context, the subscript $\rho$ may be omitted. Let $\rho^{RBC} \in \mathsf{D}(\mathcal{H}^{RBC})$ be a tripartite quantum state. The *conditional mutual information* of $R$ and $C$ given $B$ is defined as

$$\mathrm{I}(R:C\,|\,B) \quad := \quad \mathrm{I}(RB:C) - \mathrm{I}(B:C) \ .$$

### B. Quantum Markov states

A tripartite quantum state $\sigma^{RBC} \in \mathsf{D}(\mathcal{H}^{RBC})$ is called a *quantum Markov state* if there exists a quantum operation $\Lambda : \mathsf{L}\big(\mathcal{H}^B\big) \to \mathsf{L}\big(\mathcal{H}^{BC}\big)$ such that $(\mathbb{1} \otimes \Lambda)(\sigma^{RB}) = \sigma^{RBC}$, equivalently, if $\mathrm{I}(R:C\,|\,B) = 0$. This is the quantum analogue of the notion of *Markov chains* for classical registers. Classical registers $YXM$ form a *Markov chain* in this order (denoted as $Y - X - M$) if registers $Y$ and $M$ are independent given $X$. Hayden, Josza, Petz, and Winter [8] showed that an analogous property holds for quantum Markov states. In particular, they showed that a state $\sigma^{RBC} \in \mathsf{D}(\mathcal{H}^R \otimes \mathcal{H}^B \otimes \mathcal{H}^C)$ is a Markov state if and only if there is a decomposition of the space $\mathcal{H}^B$ into a direct sum of tensor products as

$$\mathcal{H}^B \quad = \quad \bigoplus_j \mathcal{H}^{B_j^R} \otimes \mathcal{H}^{B_j^C} \ , \qquad \text{(II.1)}$$

such that

$$\sigma^{RBC} \quad = \quad \bigoplus_j p(j)\, \sigma_j^{RB_j^R} \otimes \sigma_j^{B_j^C C} \ , \qquad \text{(II.2)}$$

where $\sigma_j^{RB_j^R} \in \mathsf{D}\left(\mathcal{H}^R \otimes \mathcal{H}^{B_j^R}\right)$, $\sigma_j^{B_j^C C} \in \mathsf{D}\left(\mathcal{H}^{B_j^C} \otimes \mathcal{H}^C\right)$ and $p$ is a probability distribution.

For a state $\psi^{RBC}$, we say that $\sigma^{RBC}$ is a *Markov extension* of $\psi^{RB}$ if $\sigma^{RB} = \psi^{RB}$ and $\sigma^{RBC}$ is a Markov state. We denote the set of all Markov extensions of $\psi^{RB}$ by $\mathsf{QMC}^\psi_{R-B-C}$. Note that $\mathsf{QMC}^\psi_{R-B-C}$ is non-empty, as we may take $\sigma^{RBC} := \psi^{RB} \otimes \psi^C$.

For a Markov extension $\sigma \in \mathsf{QMC}^\psi_{R-B-C}$, let $\Pi^\sigma_j$ be the orthogonal projection operator onto the $j$-th subspace of the register $B$ given by the decomposition corresponding to the Markov state $\sigma$ as described above. In other words, $\Pi^\sigma_j$ is the projection onto the Hilbert space $\mathcal{H}^{B_j^R} \otimes \mathcal{H}^{B_j^C}$ in Eq. (II.1). For a quantum state $\psi^{RBC}$, we define

$$\mathsf{ME}^{\epsilon,\psi}_{R-B-C} := \Big\{ \sigma \in \mathsf{QMC}^\psi_{R-B-C} \ \Big| \ \text{for all } j, \ \sigma_j^{B_j^C C} \in \mathsf{B}^\epsilon\Big(\mathrm{Tr}_{B_j^R}\left[(\Pi^\sigma_j \otimes \mathbb{1})\psi^{BC}(\Pi^\sigma_j \otimes \mathbb{1})\right]\Big)\Big\} \ . \qquad \text{(II.3)}$$

Informally, this is the subset of Markov extensions $\sigma$ of $\psi$ such that the restrictions of $\sigma$ and $\psi$ to the $j$-th subspace in the decomposition of $\sigma$ agree well on the registers $B_j^C C$. Again, the state $\sigma^{RBC} := \psi^{RB} \otimes \psi^C$ belongs to $\mathsf{ME}^{\epsilon,\psi}_{R-B-C}$ for every $\epsilon \geq 0$, so the set is non-empty.

### C. Quantum state redistribution

Consider a pure state $|\psi\rangle^{RABC}$ shared between the referee ($R$), Alice ($AC$) and Bob ($B$). In an $\epsilon$-error *quantum state redistribution* protocol, Alice and Bob share an entangled state $|\theta\rangle^{E_A E_B}$, register $E_A$ with Alice and register $E_B$ with Bob. Alice applies an encoding operation $\mathcal{E} : \mathsf{L}(\mathcal{H}^{ACE_A}) \to \mathsf{L}(\mathcal{H}^{AQ})$, and sends the register $Q$ to Bob. Then, Bob applies a decoding operation $\mathcal{D} : \mathsf{L}(\mathcal{H}^{QBE_B}) \to \mathsf{L}(\mathcal{H}^{BC})$. The output of the protocol is the state $\phi^{RABC}$ with the property that $\mathrm{P}(\psi^{RABC}, \phi^{RABC}) \leq \epsilon$, and the communication cost of the protocol is $\log |Q|$.

To derive the upper bound in Theorem I.1, we use an existing protocol due to Anshu, Jain and Warsi [6] which we call the AJW protocol in the sequel. The following theorem states the communication cost and the error in the final state of their protocol.

**Theorem II.2** (Ref. [6], Theorem 1)**.** *Let $\epsilon \in (0,1)$, and $|\psi\rangle^{RABC}$ be a pure quantum state shared between the referee ($R$), Alice ($AC$) and Bob ($B$). There exists an entanglement-assisted one-way protocol operated by Alice and Bob which starts in the state $|\psi\rangle^{RABC}$, and outputs a state $\phi^{RABC} \in \mathsf{B}^{9\epsilon}(\psi^{RABC})$, and the number of qubits communicated by Alice and Bob is upper bounded by*

$$\frac{1}{2}\inf_{\sigma^C} \inf_{\psi' \in \mathsf{B}^{\epsilon}(\psi^{RBC})} \Big(\mathrm{D}_{\max}\big(\psi'^{RBC} \| \psi'^{RB} \otimes \sigma^C\big) - \mathrm{D}_{\mathrm{H}}^{\epsilon^2}\big(\psi'^{BC} \| \psi'^{B} \otimes \sigma^C\big)\Big) + \log\frac{1}{\epsilon^2} \quad . \quad \text{(II.4)}$$

## III. Our protocol

In this section, we explain a protocol for redistributing $|\psi\rangle^{RABC}$ with error $9\sqrt{\epsilon}$ and cost at most

$$\min_{\sigma^{RBC} \in \mathsf{ME}_{R-B-C}^{\epsilon^2/4,\psi}} \big[\mathrm{D}_{\max}\big(\psi^{RBC} \| \sigma^{RBC}\big) - \mathrm{D}_{\mathrm{H}}^{\epsilon}\big(\psi^{BC} \| \sigma^{BC}\big)\big] + \mathrm{O}\bigg(\log\frac{1}{\epsilon}\bigg) \quad , \quad \text{(III.1)}$$

which is a non-smoothed version of Eq. (I.2). Then, Theorem I.1 follows since for every $|\psi'\rangle \in \mathsf{B}^{\epsilon}(|\psi\rangle^{RABC})$, Alice and Bob can assume that the global state is $|\psi'\rangle^{RABC}$, and run the protocol for $|\psi'\rangle$. This protocol redistributes the state $|\psi\rangle$ with up to an $\epsilon$ additional error.

Let $\sigma^{RBC}$ be a quantum Markov extension of $\psi^{RB}$. We now describe a reduction procedure which allows us to use the AJW protocol as a subroutine. This procedure is a method which decouples $C$ from $RB$ when applied to $\sigma^{RBC}$, while preserving $\psi^{RB}$ when applied to $\psi^{RBC}$. This procedure is similar to the *conditional erasure* task [21], [22] except that, here, the decoupling and negligible disturbance properties are desired for two possibly different quantum states.

### A. The GHZ state example

To elaborate on the procedure, consider an example where $\psi^{RBC}$ is the GHZ state $\frac{1}{\sqrt{d}}\sum_{j=1}^{d} |j\rangle^R |j\rangle^B |j\rangle^C$, and the Markov extension $\sigma^{RBC}$ of $\psi^{RB}$ is

$$\tfrac{1}{d}\sum_{j=1}^{d} |j\rangle\langle j|^R \otimes |j\rangle\langle j|^B \otimes |j\rangle\langle j|^C \quad . \quad \text{(III.2)}$$

A naive way to decouple register $C$ from registers $RB$ in $\sigma^{RBC}$ is to coherently erase register $C$ conditioned on register $B$. However, the same operation applied to $\psi^{RBC}$ changes $\psi^{RB}$. To overcome this problem, first, we coherently measure register $B$ by adding an extra system $T$ and making another "copy" of $|j\rangle^B$ in system $T$ using Heisenberg-Weyl operators. This operation measures the register $B$ in $\psi^{RBC}$, keeps $\sigma^{RBC}$ unchanged, and leaves $\Psi^T$ in tensor product with registers $RB$ in both $\psi$ and $\sigma$. Then, conditioned on register $B$, we can coherently erase register $C$ in $\sigma^{RBC}$; this operation applied to $\psi$ does not change the state $\psi^{RB}$. In particular, the reduction is a two-step process as follows:

*1) Coherent measurement of register $B$:* Let $T$ be a register with $|T| = d$, and $\{|t\rangle\}_{t=0}^{d-1}$ be a basis for $\mathcal{H}^T$. For $a, b \in \{0, \ldots, d-1\}$, let $P_{a,b} \in \mathsf{U}(\mathcal{H}^T)$ be the *Heisenberg-Weyl* operator defined as $P_{a,b} \coloneqq \sum_t \exp\big(\frac{2\pi i t b}{d}\big)|t+a\rangle\langle t|^T$. Define the unitary operator $U_1 \in \mathsf{U}(\mathcal{H}^{BT})$ as $U_1 \coloneqq \sum_j |j\rangle\langle j|^B \otimes P_{j,1}^T$. Let $|\Psi\rangle^{TT'}$ be the maximally entangled state over registers $T$ and $T'$, and $|\kappa_1\rangle^{RABCTT'}$ and $\tau_1^{RBCT}$ be the states after applying $U_1$ to $|\psi\rangle^{RABC} \otimes |\Psi\rangle^{TT'}$ and $\sigma^{RBC} \otimes \Psi^T$, respectively. Since the set of Heisenberg-Weyl operators is closed under multiplication (up to global phases) and each $P_{a,b}$ is traceless unless $a = b = 0$, the unitary operator $U_1$ acts trivially on $\sigma$ while it coherently measures register $B$ in $\psi^{RBC}$. In particular,

$$\tau_1^{RBCT} \quad = \quad \sigma^{RBC} \otimes \frac{\mathbb{1}^T}{d} \quad , \quad \text{(III.3)}$$

and

$$\kappa_1^{RBC} \quad = \quad \frac{1}{d} \quad \sum_j |j\rangle\langle j|^R \otimes |j\rangle\langle j|^B \otimes |j\rangle\langle j|^C \quad . \quad \text{(III.4)}$$

*2) Decoupling $C$ from $RB$ in $\sigma$:* Let $U_2 \in \mathsf{U}(\mathcal{H}^{BC})$ be a unitary operator that is read-only on $B$ and maps $|j\rangle^C$ to $|0\rangle^C$ if system $B$ is in the state $|j\rangle$. Let $|\kappa_2\rangle^{RABCTT'}$ and $\tau_2^{RBCT}$ be the states after applying $U_2$ to $|\kappa_1\rangle^{RABCTT'}$ and $\tau_1^{RBCT}$, respectively. Eq. (III.3) implies that

$$\tau_2^{RBCT} \quad = \quad \psi^{RB} \otimes |0\rangle\langle 0|^C \otimes \frac{\mathbb{1}^T}{d}$$

and $\kappa_2^{RB} = \psi^{RB}$ since register $B$ is classical in $\kappa_1^{RBC}$ and $U_2$ is read-only on $B$.

Therefore, the reduction procedure is essentially adding the maximally mixed state $\Psi^T$ and applying $U_2 U_1$. Note that running this procedure on both $\psi$ and $\sigma$ does not change the max-relative entropy and the hypothesis testing entropy and we have

$$\begin{aligned} &\mathrm{D}_{\max}\big(\psi^{RBC} \| \sigma^{RBC}\big) - \mathrm{D}_{\mathrm{H}}^{\epsilon}\big(\psi^{BC} \| \sigma^{BC}\big) \\ &= \quad \mathrm{D}_{\max}\big(\kappa_2^{RBCT} \| \tau_2^{RBCT}\big) - \mathrm{D}_{\mathrm{H}}^{\epsilon}\big(\kappa_2^{BCT} \| \tau_2^{BCT}\big) \quad \text{(III.5)} \end{aligned}$$

where $\tau_2^{RBCT} = \kappa_2^{RB} \otimes |0\rangle\langle 0|^C \otimes \frac{\mathbb{1}^T}{d}$. Hence, if Alice and Bob locally map $|\psi\rangle$ to $|\kappa_2\rangle$, then they can run the AJW protocol to transfer registers $CT$ to Bob and finally get back to $|\psi\rangle$ by applying $U_1^{-1} U_2^{-1}$. Note that the reduction procedure cannot be used directly for the local transformation of $|\psi\rangle$ to $|\kappa_2\rangle$ since initially, register $C$ is with Alice and register $B$ is with Bob. However, since $\psi^{RB} = \kappa_2^{RB}$, there exists an isometry operator $V : \mathcal{H}^{AC} \to \mathcal{H}^{ACTT'}$ which maps $|\psi\rangle^{RABC}$ to $|\kappa\rangle^{RABCTT'}$, guaranteed by the Uhlmann theorem. Therefore, the protocol works as follows:

1) Alice applies the isometry $V$ on her registers, and transforms the global state to the state $|\kappa_2\rangle^{RABCTT'}$ such that registers $(ACTT')$, $(B)$ and $(R)$ are with Alice, Bob and the referee, respectively.
2) Choosing $\sigma^{CT} = |0\rangle\langle 0|^C \otimes \frac{\mathbb{1}^T}{d}$, Alice and Bob run the AJW protocol to transfer registers $CT$ to Bob with error at most $9\sqrt{\epsilon}$. Let $\hat{\kappa}_2^{RABCTT'}$ be the state over registers $RABCTT'$ at the end of this step.
3) Bob applies $U_1^{-1}U_2^{-1}$ on his registers.
4) The final state is obtained in registers $RABC$.

By Theorem II.2 and Eq. (III.5), the cost of the above protocol is at most

$$\mathrm{D}_{\max}\big(\psi^{RBC}\|\sigma^{RBC}\big) - \mathrm{D}_{\mathrm{H}}^{\epsilon}\big(\psi^{BC}\|\sigma^{BC}\big) + \log\frac{1}{\epsilon} \ ,$$

and $\mathrm{P}\big(\kappa_2^{RABCTT'}, \hat{\kappa}_2^{RABCTT'}\big) \leq 9\sqrt{\epsilon}$. Let $\phi^{RABC}$ be the final state in registers $RABC$. We have

$$\begin{aligned}\mathrm{P}\big(\psi^{RABC}, \phi^{RABC}\big) &\leq \mathrm{P}\Big(\psi^{RABC}\otimes\Psi^{TT'}, \phi^{RABCTT'}\Big)\\ &= \mathrm{P}\Big(\kappa_2^{RABCTT'}, \hat{\kappa}_2^{RABCTT'}\Big)\\ &\leq 9\sqrt{\epsilon}\ .\end{aligned}$$

### B. General states

Now consider a general state $|\psi\rangle^{RABC}$ and a quantum Markov extension $\sigma^{RBC}$ of $\psi^{RB}$. Recall that $\sigma^{RBC}$ can be decomposed as in Eq. (II.2). Thus, for sufficiently large systems $B^R$ and $B^C$, there exists an isometry $U_{\mathrm{i}} : \mathcal{H}^B \to \mathcal{H}^{B^R J B^C}$ mapping $\sigma^{RBC}$ to the following classical-quantum state

$$\sigma^{RB^RJB^CC} := \sum_j p(j)\ \sigma_j^{RB^R} \otimes |j\rangle\langle j|^J \otimes \sigma_j^{B^CC}\ .$$

Let $|\psi\rangle^{RAB^RJB^CC}$ be the state after applying the same operation on $|\psi\rangle^{RABC}$. It is sufficient to design a protocol for redistributing register $C$ in $|\psi\rangle^{RAB^RJB^CC}$ when initially $AC$ are with Alice, $B^RJB^C$ are with Bob.

Notice that $\psi^{RB^RJB^C} = \sigma^{RB^RJB^C}$ since $\psi^{RB} = \sigma^{RB}$, and so $\psi^{RB^RJB^C}$ is a quantum Markov state with $RB^R - J - B^C$. Therefore, Bob can send $B^C$ to Alice with zero communication using the protocol depicted in Fig. 2. Now registers $B^CC$ are with Alice, and she wants to send them to Bob. To achieve this goal, we design a procedure that decouples $B^CC$ from $RB^RJ$ in $\sigma$ while keeping $\psi^{RB^RJ}$ intact when applied to $\psi$. Then, Alice and Bob can combine this procedure with the AJW protocol to send $B^CC$ to Bob with the desired cost (as described in the previous section). So, it remains to explain the decoupling procedure.

First, Bob coherently measures register $J$ using Heisenberg-Weyl operators to make sure that $\psi^{RB^RJ}$ does not change during the decoupling procedure. Notice that the coherent measurement does not change $\sigma$ since register $J$ is classical in $\sigma$. Then, one can decouple registers $B^CC$ from $RB^RJ$ in $\sigma$ using a unitary operator that conditioned on $j$, maps all $\sigma_j^{B^CC}$ to a fixed state. However, in general, such a unitary operation may not exist since the spectrum of $\sigma_j^{B^CC}$ are not necessarily the same for all $j \in [d]$. To overcome this problem, we first approximately *flatten* each $\sigma_j^{B^CC}$ through a unitary procedure. This task can be achieved via the technique of coherent flattening via embezzlement due to Anshu and Jain [23] such that the outcome is close to a flat state in the max-relative entropy. After flattening, the dimension of the support of systems $B^CC$ no more depends on $j$ and so the states in registers $B^CC$ can be all rotated to a flat state over a fixed subspace. Hence, $B^CC$ gets decoupled from $RB^RJ$ in the state $\sigma$.

**Remark:** As opposed to the GHZ example, the above mentioned decoupling procedure is not exact. Thus, in order to bound the cost of our protocol by Eq. (III.1), we need to choose $\sigma^{RBC}$ from the subset $\mathsf{ME}_{R-B-C}^{\epsilon^2/4,\psi}$ of quantum Markov extensions of $\psi^{RB}$ and use the triangle inequality for max-relative entropy in addition to the unitary invariance property of max-relative entropy and hypothesis testing entropy. We refer the reader to the full-version of this article for a detailed proof.

## IV. Conclusion and Outlook

In this article, we revisited the task of one-shot quantum state redistribution, and introduced a new protocol achieving this task with communication cost

$$\frac{1}{2}\inf_{\psi'\in\mathsf{B}^{\epsilon}(\psi^{RBC})}\ \inf_{\sigma^{RBC}\in\mathsf{ME}_{R-B-C}^{\epsilon,\psi'}}\Big[\mathrm{D}_{\max}\big(\psi'^{RBC}\|\sigma^{RBC}\big) - \mathrm{D}_{\mathrm{H}}^{\epsilon}\big(\psi'^{BC}\|\sigma^{BC}\big)\Big] + O(\log\frac{1}{\epsilon})\ ,$$

with error parameter $\epsilon$. This is the first result connecting the communication cost of state redistribution with Markov chains and it provides an operational interpretation for a one-shot representation of quantum conditional mutual information as explained in Sec I. In the special case where $\psi^{RBC}$ is a quantum Markov chain, our protocol leads to near-zero communication which was not known for the previous protocols. Moreover, the communication cost of our protocol is lower than all previously known one-shot protocols and it achieves the optimal cost of $\mathrm{I}(R : C \,|\, B)$ in the asymptotic i.i.d. setting. Our protocol also achieves the near-optimal result of Ref. [24] in the case when $\psi^{RBC}$ is classical.

A question of interest is whether the communication cost of our one-shot protocol can be bounded with $\mathrm{I}(R : C \,|\, B)$. In the quantum communication complexity setting, such a bound would imply the possibility of compressing the communication of bounded-round quantum protocols to their information content which would lead to a direct-sum theorem for bounded-round quantum communication complexity [25].

Another question that we have not addressed in this article is whether our bound is near-optimal. There are several known lower bounds in the literature for the communication cost of entanglement-assisted quantum state redistribution, such as in Ref. [4] and Ref. [26]. However, it is not clear if our bound achieves any of them. Near-optimal bound for one-shot quantum state redistribution is a major open question.

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