

Accessible versus Holevo Information for a Binary Random Variable

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Abstract

The *accessible information* $I_{\text{acc}}(\mathcal{E})$ of an ensemble \mathcal{E} is the maximum mutual information between a random variable encoded into quantum states, and the probabilistic outcome of a quantum measurement of the encoding. Accessible information is extremely difficult to characterize analytically; even bounds on it are hard to place. The celebrated *Holevo bound* states that accessible information cannot exceed $\chi(\mathcal{E})$, the quantum mutual information between the random variable and its encoding. However, for general ensembles, the gap between the $I_{\text{acc}}(\mathcal{E})$ and $\chi(\mathcal{E})$ may be arbitrarily large.

We consider the special case of a binary random variable, which often serves as a stepping stone towards other results in information theory and communication complexity. We give explicit lower bounds on the the accessible information $I_{\text{acc}}(\mathcal{E})$ of an ensemble $\mathcal{E} \triangleq \{(p, \rho_0), (1-p, \rho_1)\}$, with $0 \leq p \leq 1$, as functions of p and $\chi(\mathcal{E})$. The bounds are incomparable in the sense that they surpass each other in different parameter regimes.

Our bounds arise by measuring the ensemble according to a complete orthogonal measurement that preserves the fidelity of the states ρ_0, ρ_1 . As an intermediate step, therefore, we give new relations between the two quantities $I_{\text{acc}}(\mathcal{E}), \chi(\mathcal{E})$ and the fidelity $B(\rho_0, \rho_1)$.

1 Introduction

Let X be a classical random variable taking values in a finite set $\{0, \dots, n-1\}$ such that $\Pr(X = i) \triangleq p_i$. Let M be an encoding of X into (possibly mixed) quantum states in a finite dimensional, say d -dimensional, Hilbert space \mathbb{C}^d , such that $M = \rho_i$ when $X = i$. This gives rise to an ensemble of quantum states $\mathcal{E} \triangleq \{(p_i, \rho_i)\}$.

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The mapping $i \mapsto \rho_i$ may be viewed as a quantum communication channel, and it is natural to ask how much information about X can be obtained from the transmitted signal M . The answer to this question depends heavily on the way we quantify the notion of “information”. For example, one may seek to maximize the probability of guessing, via a measurement, the value i given an unknown state ρ_i from the ensemble \mathcal{E} [9]. This quantity frequently arises in quantum communication, but has no simple description in terms of the ensemble. For a boolean random variable, the answer is related to the trace distance of the two density operators [9, pp. 106–108]. While no analytical expression for this probability is known in the general case, we can still place meaningful bounds on it (see, e.g., Ref. [18]).

A different way of quantifying the information content of an ensemble arises in Quantum Information Theory. Consider a classical random variable $Y^{\mathcal{M}}$ that represents the result of a measurement of the encoding M according to the POVM \mathcal{M} . The *accessible information* $I_{\text{acc}}(\mathcal{E})$ of the ensemble \mathcal{E} is defined as the maximum mutual information $I(X : Y^{\mathcal{M}})$ obtainable via a quantum measurement \mathcal{M} :

$$I_{\text{acc}}(\mathcal{E}) \triangleq \max_{\text{POVM } \mathcal{M}} I(X : Y^{\mathcal{M}}). \quad (1)$$

Accessible information is extremely difficult to characterize analytically, even for a binary random variable (see, e.g., Ref. [6, page 1222], where it is referred to as *Shannon Distinguishability*). In a celebrated result, Holevo [10] bounded the accessible information for an ensemble \mathcal{E} by the quantum mutual information between the random variable X and its encoding M :

$$I_{\text{acc}}(\mathcal{E}) \leq \chi(\mathcal{E}) \triangleq S(\mathbb{E}_i[\rho_i]) - \mathbb{E}_i[S(\rho_i)] = I(X : M), \quad (2)$$

where $S(\rho)$ denotes the von Neumann entropy of a density matrix ρ , and $I(A : B) = S(A) + S(B) - S(AB)$ denotes the mutual information of a bipartite quantum system AB , and \mathbb{E}_i denotes the expectation (i.e., average), taken according to the distribution $\{p_i\}$ in the ensemble \mathcal{E} . The quantity $\chi(\mathcal{E})$ has come to be called the *Holevo information* of the ensemble.

The Holevo bound is attained by ensembles of commuting states (equivalently, for classical mixtures). In fact, Ruskai [22, Section VII.B] (see also Ref. [21, Section 4.3]) proves that these are the only ensembles for which $I(X : Y^{\mathcal{M}}) = I(X : M)$ is possible for some fixed measurement. A theorem due to Davies [3, Theorem 3] establishes that there is a measurement with at most d^2 outcomes that achieves accessible information for an ensemble of d -dimensional states. (See also Ref. [23, Lemma 5], and the conjecture in Refs. [17, 20] regarding this measurement.) Consequently, the Holevo bound is attained only by ensembles of commuting states. There are ensembles for which $I_{\text{acc}}(\mathcal{E})$ may be arbitrarily smaller than $\chi(\mathcal{E})$: a uniformly random ensemble of n states in a d -dimensional space (where d is suitably smaller than n) has this property with high probability.

Lower bounds on I_{acc} are hard to derive even for specific ensembles (cf. Refs. [4, 14]). In the simplest case of a binary random variable, Fuchs and van de Graaf [6] relate accessible information to the trace distance and fidelity of the density matrices. Fuchs and Caves [5] also consider the case of a binary random variable, stopping short of an explicit lower bound. It is also conjectured that the two-outcome measurement achieving the trace-distance (or fidelity) between two pure states occurring with equal probability also achieves accessible information [17]. This was numerically verified, but not formally proven, by Osaki *et al.* [20]. We revisit this special case, and give a lower bound for accessible information in terms of Holevo information. We show the following:

Theorem 1.1 *Let $\mathcal{E} \triangleq \{(p, \rho_0), (1 - p, \rho_1)\}$ be an binary ensemble over quantum states of any (finite)*

dimension. Then

$$I_{\text{acc}}(\mathcal{E}) \geq H(p) - \sqrt{4p(1-p) - \chi(\mathcal{E})^2}, \text{ and} \quad (3)$$

$$I_{\text{acc}}(\mathcal{E}) \geq -\log_2 \left[p^2 + (1-p)^2 + 2p(1-p) \left\{ 1 - \frac{\chi(\mathcal{E})^2}{4p(1-p)} \right\}^{1/2} \right]. \quad (4)$$

These lower bounds arise from a specific kind of measurement of the quantum states, namely, a measurement that preserves fidelity [5]. As an intermediate step, we therefore relate both accessible information and Holevo information to fidelity in different ways so as to give the stated bounds.

Note that the first bound above, Eq. (3), may be negative for certain parameter regimes. For example, the bound is negative for an ensemble of two pure states at an angle of $\pi/4$, as the probability p approaches 0. This occurs because our measurement completely disregards the prior probability p , and because of the approximations we make in order to get an explicit relationship with Holevo information. The second bound is meaningful for all parameter regimes. On the other hand, for any two orthogonal states, the first bound is greater than the second for $p \notin \{0, 1, 1/2\}$.

The motivation for this work is to investigate the extent to which the Holevo bound $I_{\text{acc}}(\mathcal{E}) \leq \chi(\mathcal{E})$ is tight for binary ensembles. Note also that $\chi(\mathcal{E}) = I(X : M) \leq H(p)$ by the Lanford-Robinson inequality [26, Eq. (2.3), page 238] (also [19, Theorem 11.10, page 518]). Our main theorem may thus be seen as a sharpening of these bounds. None of the previously known bounds provide a lower bound on accessible information as an explicit function of Holevo information.

Relations between different measures of information in the binary case often form a stepping stone in results in information theory. For example, as explained in Ref. [4], the Holevo bound may be derived by studying the case of the binary ensemble. In complexity theory, these relations form the basis of strong, sometimes optimal, lower bounds for the quantum communication complexity of computing functions, as in Refs. [12, 16]. They have also found application in the study of quantum interactive proof systems [15] and quantum coin flipping [1, 24]. We expect that our inequalities provide a more operationally useful view of accessible information, and find similar application.

2 Comparison with previous bounds

A number of lower bounds on accessible information $I_{\text{acc}}(\mathcal{E})$ are already known. Jozsa, Robb, and Wootters [14] bounded it for arbitrary (possibly non-binary) ensembles by the expected mutual information resulting from a uniformly random complete orthogonal measurement. The latter quantity depends solely on the average density matrix $\rho = \mathbb{E}_i \rho_i$ corresponding to the ensemble, and was defined as the *subentropy* $Q(\rho)$ of the density matrix ρ . They also showed that for every density matrix ρ (of arbitrarily large finite dimension), subentropy expressed in bits is bounded as $Q(\rho) \leq (1 - \gamma) \log_2 e \leq 0.60995$, where γ is the Euler constant. Thus, the best possible lower bound resulting from the subentropy bound is 0.60995. Our bound, Theorem 1.1 (Eq. (3)), is more sensitive to the ensemble. For example, it gives the sharper lower bound of $1 - \sqrt{2\epsilon}$ for an equally weighted binary ensemble \mathcal{E} for which $\chi(\mathcal{E}) = 1 - \epsilon$, for any $\epsilon \in [0, 1]$. Holevo χ information arbitrarily close to 1 is achieved, among others, by a uniform distribution over two pure states whose inner product is arbitrarily close to 0. Thus our bound better reflects the ensemble-dependence of accessible information.

Fuchs and Caves [4] considered a measurement of a binary ensemble \mathcal{E} that arises in one proof of the Holevo bound. This is a measurement \mathcal{M} that minimizes the second derivative of the mutual informa-

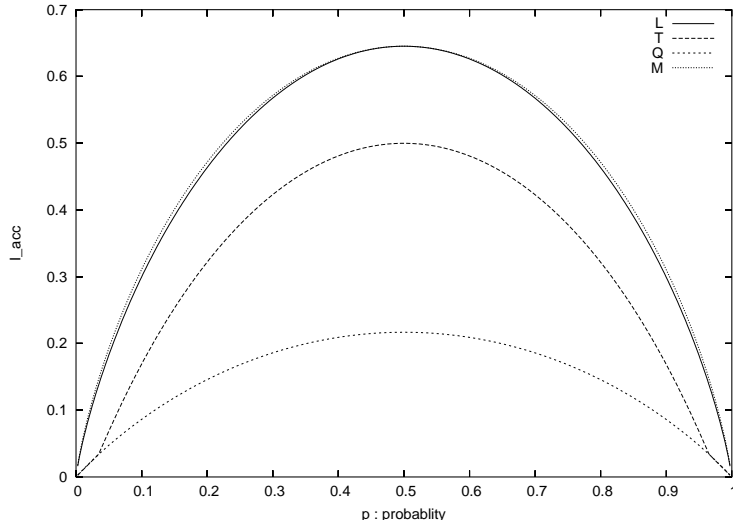


Figure 1: Bounds for an ensemble $\mathcal{E} = \{(p, \rho_0), (1 - p, \rho_1)\}$ of two pure states with inner-product $1/2$. The bounds on $I_{\text{acc}}(\mathcal{E})$ are plotted as p varies in $[0, 1]$.

tion $I(X : Y^{\mathcal{M}})$ (as defined in Section 1) with respect to the variable p . The measurement may be computed numerically given any ensemble, and leads to a lower bound on accessible information for binary ensembles. In a few special cases with single qubit states, such as when the states are pure, or commute, the Fuchs-Caves measurement has a familiar description. In all these special cases, the measurement is in a basis that diagonalizes $p\rho_0 - (1-p)\rho_1$; precisely the measurement that maximizes the probability of guessing i correctly given a state ρ_i picked from the ensemble \mathcal{E} . This measurement also preserves fidelity between the states ρ_0, ρ_1 when $p = 1/2$, and the lower bound with which we start matches this (see the proof of Lemma 3.6). Our final lower bounds are however weaker, since we make further approximations in order to relate accessible information to the Holevo χ quantity. For $p \neq 1/2$, even our initial lower bound may be worse, since fidelity preserving measurements are independent of the value of p .

Hall [7, Eq. 24, page 104] considered the lower bound on accessible information for arbitrary (possibly non-binary) ensembles that arises from the “pretty good measurement”. A special case of this measurement, when the ensemble consists of pure states, was defined by Hughston, Jozsa, and Wootters [11], later named so and studied by Hausladen and Wootters [8]. The measurement may be viewed as a dual to ensembles, and is a natural candidate for state discrimination. The bound that arises from the pretty good measurement may also be computed numerically for any given ensemble, as with the previous two bounds. In all the cases of an ensemble with two-states that we studied numerically, the Hall bound follows the Fuchs and Caves bound very closely. We are not aware of any formal connection between the two, though. Our numerical findings differ from the ones originally presented by Hall, and have been verified by him in a private communication. (Figure 1 in [7, p. 104] is therefore in error.)

Typically for applications in complexity theory and cryptography such as those mentioned in Section 1, we seek simple functional relationships between different measures of information. All the three bounds discussed here can be evaluated numerically on a computer but seem not to lead to an explicit functional connection with Holevo- χ information. Our work attempts to fill this gap.

In Figures 1–3, we present a numerical comparison of the abovementioned bounds with the maximum of

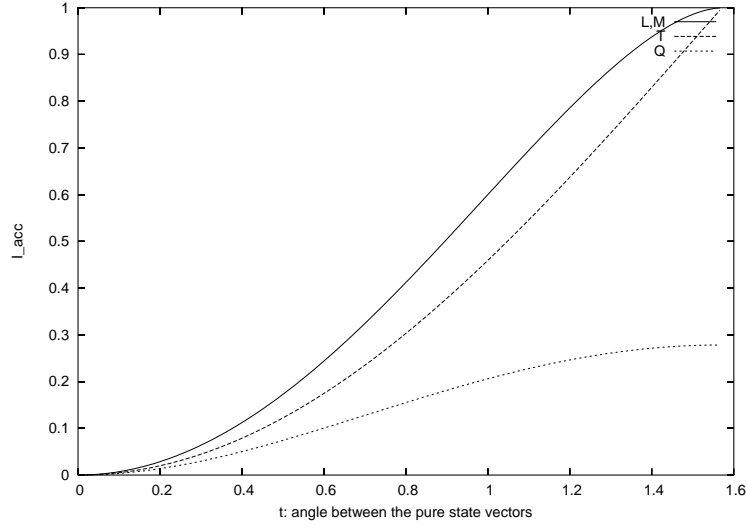


Figure 2: Bounds for a uniform ensemble $\mathcal{E} = \{(1/2, \rho_0), (1/2, \rho_1)\}$ of two pure states at an angle t with each other, as t varies in $[0, \pi/2]$.

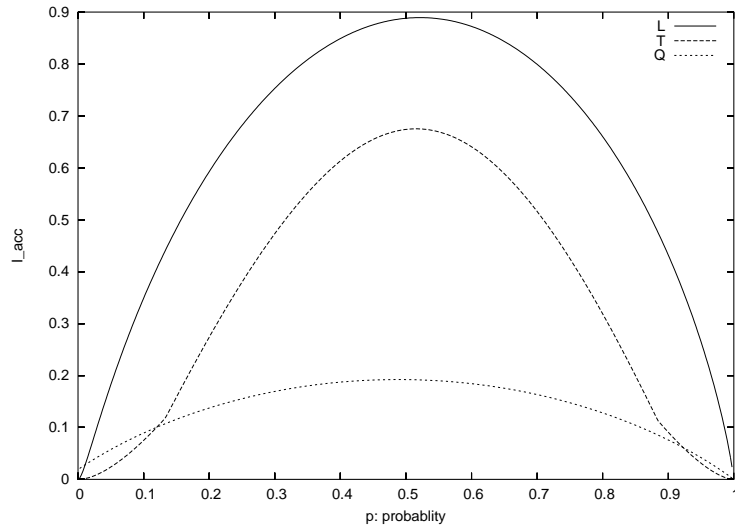


Figure 3: Bounds for an ensemble $\mathcal{E} = \{(p, \rho_0), (1-p, \rho_1)\}$ of a mixed qutrit state $\rho_0 = 0.01|0\rangle\langle 0| + 0.01|1\rangle\langle 1| + 0.98|2\rangle\langle 2|$ and a pure qutrit state $\rho_1 = |v\rangle\langle v|$, where $v = \sqrt{0.02}|0\rangle + \sqrt{0.96}|1\rangle + \sqrt{0.02}|2\rangle$. The bound M is not plotted here. The bounds on $I_{\text{acc}}(\mathcal{E})$ are plotted as p varies in $[0, 1]$.

our lower bounds, which is denoted by T. The Jozsa *et al.* bound is marked as Q, the Fuch-Caves bound is marked as M, and the Hall bound is marked as L. In all the cases we plotted, our bound T is below M and L. Except when the probability p is close to 0 or 1 in Figure 3, T is well above Q. For pure state ensembles with $p = 1/2$, the lower bound with which we start in Lemma 3.6 equals L and M. However, our final bound T is an approximation to this bound, and is lower (see Figure 2).

3 Lower bounds on accessible information

3.1 Preliminaries

We quickly summarize some information theoretic concepts and notation we use in this article. For a more comprehensive treatment, we refer the reader to a text such as [19].

All logarithms in this article are taken to the base 2. For $0 \leq p \leq 1$, $H(p) \triangleq -p \log p - (1-p) \log(1-p)$ denotes the binary entropy function. The following fact (see, e.g., [6, page 1224, Fig. 1]) bounds the binary entropy function close to 1/2:

Fact 3.1 For $\delta \in [-\frac{1}{2}, \frac{1}{2}]$, $H(\frac{1}{2} + \delta) \leq \sqrt{1 - (2\delta)^2}$.

Let \mathcal{H}, \mathcal{K} be Hilbert spaces. For a quantum state $\rho \in L(\mathcal{H})$, we call a pure state $|\phi\rangle \in \mathcal{H} \otimes \mathcal{K}$ a *purification* of ρ if $\text{Tr}_{\mathcal{K}} |\phi\rangle\langle\phi| = \rho$. The *fidelity* between (mixed) quantum states ρ, σ is defined as $B(\rho, \sigma) \triangleq \sup |\langle\phi|\psi\rangle|$, where the optimization is over states $|\phi\rangle, |\psi\rangle$ which are purifications of ρ and $|\sigma\rangle$, respectively. The *trace norm* of an operator $A \in L(\mathcal{H})$ is defined as $\|A\|_t \triangleq \text{Tr} \sqrt{A^\dagger A}$.

The fidelity of two mixed states may be expressed in terms of the trace norm.

Theorem 3.2 (Uhlmann [13, 25]) For any quantum states ρ and σ , $B(\rho, \sigma) = \|\sqrt{\rho}\sqrt{\sigma}\|_t$.

The *von-Neumann* entropy of a quantum state ρ is defined as $S(\rho) \triangleq -\text{Tr} \rho \log \rho$. The *Kullback-Leibler divergence* or *relative entropy* between two quantum states ρ, σ is defined as $S(\rho||\sigma) \triangleq \text{Tr} \rho (\log \rho - \log \sigma)$. The following fact relates fidelity and relative entropy:

Lemma 3.3 (Dacunha-Castelle [2]) For any quantum states ρ, σ , we have $S(\rho||\sigma) \geq -2 \log B(\rho, \sigma)$.

The following relation follows from the definitions:

Lemma 3.4 Let X be a random variable over a finite sample space and let $p_x \triangleq \text{Pr}(X = x)$. Let M be a quantum encoding of X such that $M = \rho_x$ when $X = x$. Let $\rho \triangleq \mathbb{E}_x[\rho_x] = \sum_x p_x \rho_x$. Then $I(X : M) = \mathbb{E}_x[S(\rho_x||\rho)]$.

3.2 The lower bounds

We begin with a property of fidelity that we later use.

Lemma 3.5 Let $\mathcal{E} \triangleq \{(p, \rho_0), (1-p, \rho_1)\}$ be an ensemble of commuting quantum states of arbitrary (finite) dimension, and let $\rho = p\rho_0 + (1-p)\rho_1$. Then

$$pB(\rho_0, \rho) + (1-p)B(\rho_1, \rho) \leq [p^2 + (1-p)^2 + 2p(1-p)B(\rho_0, \rho_1)]^{1/2}.$$

Proof: For any two commuting positive semi-definite matrices A, B , we have

$$\|AB\|_t = \text{Tr } AB.$$

Since ρ_0, ρ_1, ρ all commute, we have

$$\begin{aligned} & pB(\rho_0, \rho) + (1-p)B(\rho_1, \rho) \\ &= p\|\sqrt{\rho_0}\sqrt{\rho}\|_t + (1-p)\|\sqrt{\rho_1}\sqrt{\rho}\|_t && \text{By Theorem 3.2} \\ &= p\text{Tr}[\sqrt{\rho_0}\sqrt{\rho}] + (1-p)\text{Tr}[\sqrt{\rho_1}\sqrt{\rho}] \\ &= \text{Tr}[(p\sqrt{\rho_0} + (1-p)\sqrt{\rho_1}) \cdot \sqrt{\rho}] \\ &\leq \left[\text{Tr}(p\sqrt{\rho_0} + (1-p)\sqrt{\rho_1})^2\right]^{1/2} \cdot [\text{Tr } \rho]^{1/2}, && \text{By Cauchy-Schwartz} \\ &= [p^2 + (1-p)^2 + 2p(1-p)\text{Tr}(\sqrt{\rho_0}\sqrt{\rho_1})]^{1/2}, \end{aligned}$$

which is the inequality we seek. ■

We now bound the accessible information from below by the mutual information achieved by a measurement that preserves fidelity.

Lemma 3.6 Let $\mathcal{E} \triangleq \{(p, \rho_0), (1-p, \rho_1)\}$ be an ensemble of quantum states of arbitrary (finite) dimension. Then

$$I_{\text{acc}}(\mathcal{E}) \geq H(p) - 2\sqrt{p(1-p)}B(\rho_0, \rho_1), \text{ and} \tag{5}$$

$$I_{\text{acc}}(\mathcal{E}) \geq -\log_2 [p^2 + (1-p)^2 + 2p(1-p)B(\rho_0, \rho_1)]. \tag{6}$$

Proof: Let ρ'_0, ρ'_1 be the classical distributions resulting from a measurement that achieves the fidelity between ρ_0 and ρ_1 (see Ref. [5]), so that $B(\rho'_0, \rho'_1) = B(\rho_0, \rho_1)$. Let $\rho' = p\rho'_0 + (1-p)\rho'_1$. Let M' be the encoding of X such that $M' = \rho'_0$ when $X = 0$ and $M' = \rho'_1$ when $X = 1$. Let

$$q_0(m) \triangleq \Pr(M' = m | X = 0)$$

$$q(m) \triangleq \Pr(M' = m), \text{ and}$$

$$r_0(m) \triangleq \Pr(X = 0 | M' = m).$$

We similarly define $q_1(m)$. Note that

$$q(m)r_0(m) = pq_0(m) \quad \text{and} \quad q(m)(1-r_0(m)) = (1-p)q_1(m). \tag{7}$$

We now have

$$\begin{aligned}
I_{\text{acc}}(\mathcal{E}) &\geq I(X : M') \\
&= H(p) - \sum_m q(m) H(r_0(m)) \\
&\geq H(p) - \sum_m q(m) \cdot 2\sqrt{r_0(m)(1-r_0(m))} && \text{From Fact 3.1} \\
&= H(p) - \sum_m 2\sqrt{p(1-p)q_0(m)q_1(m)} && \text{From Eq. (7)} \\
&= H(p) - 2\sqrt{p(1-p)} B(\rho'_0, \rho'_1).
\end{aligned}$$

This proves the bound in Eq. (5).

For the second part, Eq. (6), we have

$$\begin{aligned}
I_{\text{acc}}(\mathcal{E}) &\geq I(X : M') \\
&= pS(\rho'_0 \parallel \rho') + (1-p)S(\rho'_1 \parallel \rho') && \text{From Lemma 3.4} \\
&\geq -2p \log B(\rho'_0, \rho') - 2(1-p) \log B(\rho'_1, \rho') && \text{From Lemma 3.3} \\
&\geq -2 \log (p B(\rho'_0, \rho') + (1-p) B(\rho'_1, \rho')) && \text{By convexity} \\
&\geq -\log (p^2 + (1-p)^2 + 2p(1-p) B(\rho'_0, \rho'_1)) && \text{By Lemma 3.5}
\end{aligned}$$

Since $B(\rho'_0, \rho'_1) = B(\rho_0, \rho_1)$, the bound follows. ■

The bounds in the above lemma are incomparable, in that they surpass each other in different parameter regimes as described in Section 1.

Next we observe that for pure quantum states, there is a direct relationship between fidelity and the Holevo- χ quantity.

Lemma 3.7 *Let $\mathcal{E} \triangleq \{(p, \rho_0), (1-p, \rho_1)\}$ be an ensemble such that ρ_0, ρ_1 are pure states. Then,*

$$\chi(\mathcal{E}) \leq 2\sqrt{p(1-p)(1-B(\rho_0, \rho_1)^2)}.$$

Proof: Let $\theta \in [0, \pi/2]$ be the angle between the pure states ρ_0 and ρ_1 so that $B(\rho_0, \rho_1) = \cos \theta$. Let $\rho = p\rho_0 + (1-p)\rho_1$. By a direct calculation we see that the eigenvalues of ρ are

$$\frac{1 \pm \sqrt{1 - 4p(1-p)\sin^2 \theta}}{2}.$$

Therefore from Fact 3.1 we have,

$$\begin{aligned}
\chi(\mathcal{E}) &= S(\rho) = H\left(\frac{1 + \sqrt{1 - 4p(1-p)\sin^2 \theta}}{2}\right) \\
&\leq (2 \sin \theta) \sqrt{p(1-p)} = 2\sqrt{p(1-p)(1-B(\rho_0, \rho_1)^2)}.
\end{aligned}$$
■

As a corollary of the above lemma, we get:

Corollary 3.8 Let $\mathcal{E} \triangleq \{(p, \rho_0), (1-p, \rho_1)\}$ be an ensemble where ρ_0, ρ_1 may be mixed states. Then,

$$\chi(\mathcal{E}) \leq 2\sqrt{p(1-p)(1 - \mathbb{B}(\rho_0, \rho_1)^2)}.$$

Proof: As before, let $|\phi_0\rangle, |\phi_1\rangle$ be purifications of ρ_0, ρ_1 which achieve fidelity between the two states. Let us consider the encoding M' of X such that $M' = |\phi_0\rangle$ when $X = 0$ and $M' = |\phi_1\rangle$ when $X = 1$. From the strong sub-additivity property of von Neumann entropy it follows that $\mathbb{I}(X : M) \leq \mathbb{I}(X : M')$. Using Lemma 3.7 we have

$$\begin{aligned} \chi(\mathcal{E}) &= \mathbb{I}(X : M) \leq \mathbb{I}(X : M') \\ &\leq 2\sqrt{p(1-p)(1 - \mathbb{B}(|\phi_0\rangle\langle\phi_0|, |\phi_1\rangle\langle\phi_1|)^2)} \\ &= 2\sqrt{p(1-p)(1 - \mathbb{B}(\rho_0, \rho_1)^2)}, \end{aligned}$$

as required. ■

Finally we get our main result, Theorem 1.1.

Proof of Theorem 1.1: From Corollary 3.8 we have

$$\mathbb{B}(\rho_0, \rho_1) \leq \left[1 - \frac{\chi(\mathcal{E})^2}{4p(1-p)} \right]^{1/2}.$$

The inequalities follow by combining the above with those in Lemma 3.6. ■

4 Concluding remarks

In Theorem 1.1 we bounded the accessible information of an arbitrary ensemble corresponding to a binary random variable from below, by relating it to the Holevo χ quantity. By a theorem of Ruskai [22, Section VII.B], whenever the states in the ensemble are not orthogonal (or equal), no measurement achieves Holevo information. This also rules out the possibility of the two quantities being equal in the limit of more and more refined measurements, since the number of outcomes in the optimal measurement on a finite dimensional space may be bounded by the Davies Theorem [3, Theorem 3]. This implies that the Holevo bound is strict for ensembles of non-orthogonal states. The significance of our lower bounds is that they quantify the extent to which accessible information may be smaller than Holevo information.

In Section 2, we pointed out some advantages of our bounds vis-a-vis previously known lower bounds. Along with the intermediate bounds we obtain in Lemma 3.6 and Corollary 3.8, Theorem 1.1 also adds to the suite of relations between different distinguishability measures such as those identified by Fuchs and van de Graaf [6]. These relations have found applications in a variety of areas—information theory, cryptography, and communication complexity. We anticipate that our bounds find similar application. Finally, we believe that our bounds may be further tightened. Due to the basic nature of the question, the existence of a tighter bound would be of interest regardless of potential applications.

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