A quantum information trade-off for Augmented Index

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Privacy in communication

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x Is $x > y$? y

Two millionaires problem [Yao '82]

Determine if $x > y$ without revealing any other information about their wealth

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Impossible without restriction on their computational power

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- Better protocols are possible

  Equality: O(log *n*) one-way protocol, 1/poly(n) error, reveals only O(1) bits about one input [GV'10, FHS'10]

Augmented Index

$x = x_1\, x_2 \ldots x_n$

Is $x_k = b$?

$k,\ x[1, k\text{-}1],\ b$

Variant of Index function

Bob has the prefix $x[1, k\text{-}1]$, and a guess b for the value of x_k .

Index function

Fundamental problem with a rich history

- communication complexity [KN'97]
- data structures [MNSW'98]
- private information retrieval [CKGS'98]
- learnability of states [KNR'95, A'07]
- finite automata [ANTV'99]
- formula size [K'07]
- locally decodable codes [KdW'03]
- sketching e.g., [BJKK'04]
- information causality [PPKSWZ'09]
- non-locality and uncertainty principle [OW'10]
- quantum ignorance [VW'11]

Results

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Theorem [JN'11]

If a quantum protocol computes AI_n with probability $1 - \epsilon$ on the uniform distribution, either

Alice reveals $\Omega(n/t)$ information about x, or

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even when restricted to 0-inputs, where t is the number of messages.

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Stronger theorem for classical protocols [JN'10]

Alice reveals $\Omega(n)$, or Bob reveals $\Omega(1)$ information.

Related work

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Privacy in communication (quantum)

- Klauck'04: w.r.t. hard distribution
- Index function: various flavours [JRS'02, '09; KdW'04; LeG'11]
- Jain, Radhakrishnan, Sen'03: AND(a, b), w.r.t. superposition over 0-inputs

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Augmented Index (classical)

- Magniez, Mathieu, N.'10: In Alice-Bob-Alice classical protocols, Alice reveals $\Omega(n)$, or Bob reveals $\Omega(\log n)$ bits of information, even when restricted to 0-inputs.
- Chakrabarti, Cormode, Kondapalli, McGregor'10: independent and concurrent work, similar classical results as ours.
- Neither technique applies to quantum communication.

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Why privacy w.r.t. 0-inputs ?

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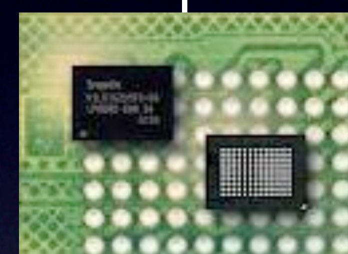

device with small memory

Streaming model

- massive input, cannot be stored entirely in memory
- input arrives sequentially, read one symbol at a time
- device processes each symbol quickly, while maintaining small workspace

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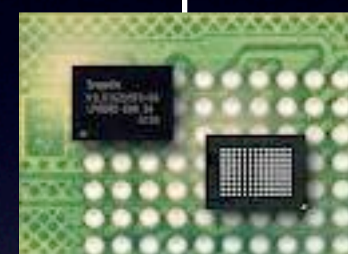

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Attractive model for quantum computation

Streaming quantum algorithms

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Advantage over classical

- Quantum finite automata: streaming algorithms with constant memory and time per symbol. E.g., may be exponentially smaller than classical FA.
- Use exponentially smaller amount of memory for certain problems [LeG'06, GKKRdW'06]

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Advantage for natural problems ?

- For context-free languages: e.g., checking whether a sentence is grammatical.
- For Dyck(2), checking if an expression in two types of parentheses is well-formed ?   Canonical CFL, used in practice.

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Magniez, Mathieu, N.'10:

- A single pass randomized algorithm that uses $O((n \log n)^{1/2})$ space, O(polylog n) time/ symbol
- 2-pass algorithm, uses $O(\log^2 n)$ space, O(polylog n) time/ symbol, second pass in reverse
- Space usage of 1 pass algorithm is optimal, via study of information revealed in classical protocols for Augmented Index.

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Better quantum algorithms ?

- Classical version shows limitations of multiple (unidirectional) passes over input.
- The information cost trade-off would give a similar negative answer, provided a conjectured information inequality holds.

The information cost trade-off

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Intuition behind proof

(2 messages, no private workspace)

$x = x_1\, x_2 \ldots x_n$

M_A

M_B

$k,\; x[1, k\text{-}1],\; b$

↓

output

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Consider uniformly random X, K, let $B = X_K$.

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Consider uniformly random X, K, let $B = X_K$.

- Consider K in [n/2]. If M_A has $o(n)$ information about X, then it is nearly independent of X_L , $L > n/2$. Flipping Alice's L-th bit does not perturb M_A much.

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- If M_B has o(1) information about K, then M_B is nearly the same for most pairs $J \leq n/2$, $L > n/2$. Switching Bob's index from J to L does not perturb M_B much.

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Consequences of Average Encoding Theorem [KNTZ'07, JRS'03]

Intuition continued...

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Alice's input	Bob's input	Protocol state	
0	$X[1, K]$	$\vert\psi\rangle$	0-input
flip L-th bit	same index		
1	$X[1, K]$	$\vert\psi'\rangle \approx \vert\psi\rangle$	

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0		$X[1, K]$		$\vert \psi \rangle$	
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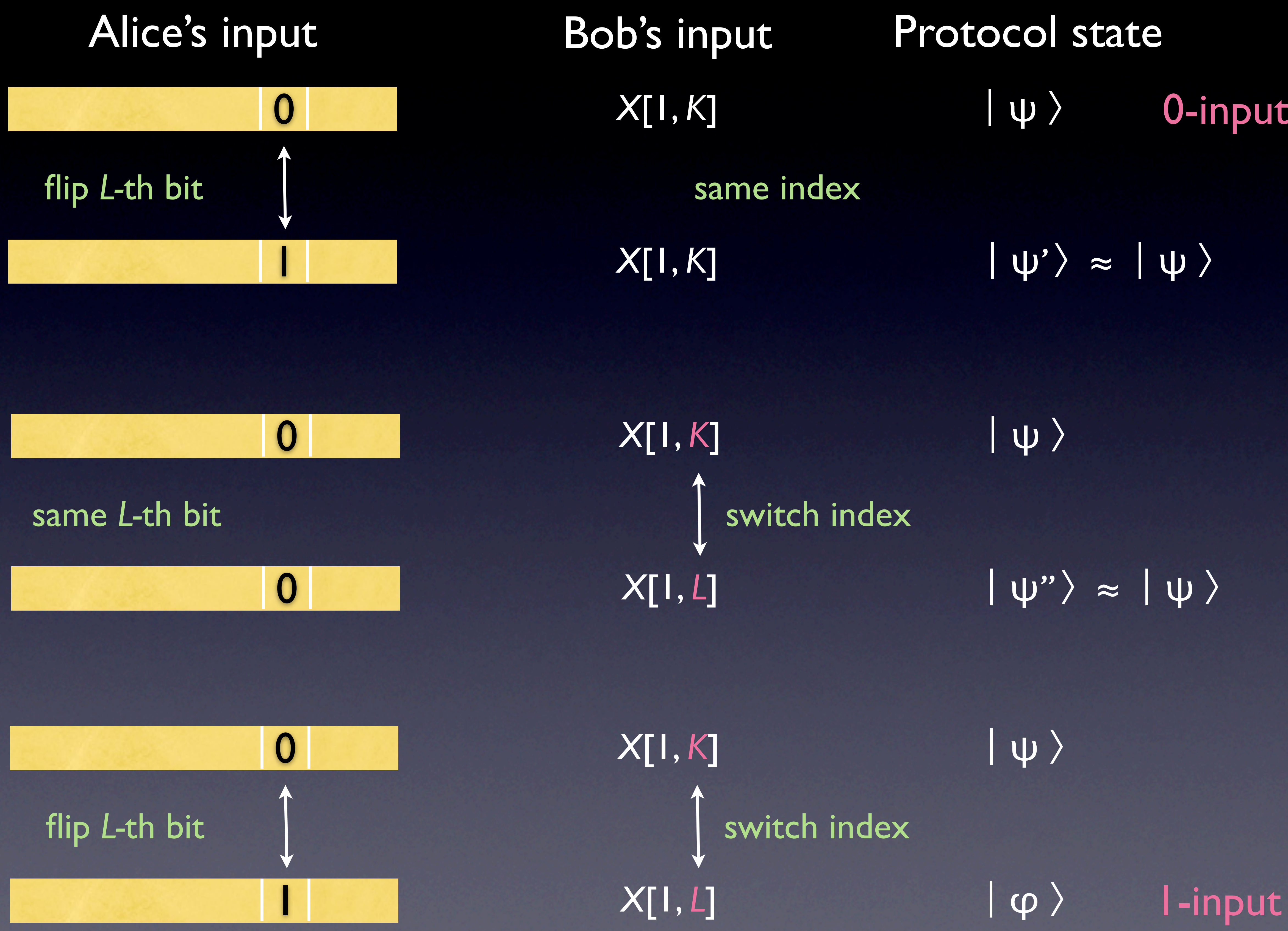

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$$\mid \psi \rangle = V_K U_X \mid 0 \rangle , \quad \mid \psi' \rangle = V_K U_{X'} \mid 0 \rangle , \quad \mid \psi'' \rangle = V_L U_X \mid 0 \rangle$$

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$$\leq \delta + | V_L U_{X'} | 0 \rangle - V_L U_X | 0 \rangle |$$

$$= \delta + | V_K U_{X'} | 0 \rangle - V_K U_X | 0 \rangle |$$

$$= \delta + | \psi - \psi' | \leq 2 \delta$$

Complications swept under the rug

- How we quantify information that is revealed
- Alice and Bob may maintain private workspace
- Information about inputs may increase with each message, penalty for switch increases
- Most of these issues handled à la [JRS'03]
- Leads to a dependence of trade-off on the number of messages
- Connection with streaming algorithms à la [MMN'10] breaks down

Final remarks

- Established a trade-off in quantum information revealed by parties computing Augmented Index
- Stronger results in classical case, with implications for streaming algorithms
- Similar implications likely in the quantum case as well
- Dependence of trade-off on the number of messages unavoidable, without a different notion of information revealed
- Techniques developed for quantum gives conceptually simpler and tighter analysis of classical protocols
- Study of small space (streaming) algorithms is subtle, calls for further exploration