ON MINOR-CLOSED CLASSES OF MATROIDS WITH EXPONENTIAL GROWTH RATE

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Abstract. Let $\mathcal{M}$ be a minor-closed class of matroids that does not contain arbitrarily long lines. The growth rate function, $h : \mathbb{N} \to \mathbb{N}$ of $\mathcal{M}$ is given by

$$h(n) = \max\{|M| : M \in \mathcal{M} \text{ is simple, and } r(M) \le n\}.$$

The Growth Rate Theorem shows that there is an integer c such that either: $h(n) \le c\,n$, or $\binom{n+1}{2} \le h(n) \le c\,n^2$, or there is a prime-power q such that $\frac{q^n-1}{q-1} \le h(n) \le c\,q^n$; this separates classes into those of linear density, quadratic density, and base-q exponential density. For classes of base-q exponential density that contain no (q^2+1)-point line, we prove that $h(n) = \frac{q^n-1}{q-1}$ for all sufficiently large n. We also prove that, for classes of base-q exponential density that contain no (q^2+q+1)-point line, there exists $k \in \mathbb{N}$ such that $h(n) = \frac{q^{n+k}-1}{q-1} - q\frac{q^{2k}-1}{q^2-1}$ for all sufficiently large n.

1. Introduction

We prove a refinement of the Growth Rate Theorem for certain exponentially dense classes. We call a class of matroids *minor-closed* if it is closed under both minors and isomorphism. The *growth rate function*, $h_{\mathcal{M}} : \mathbb{N} \to \mathbb{N} \cup \{\infty\}$ for a class $\mathcal{M}$ of matroids is defined by

$$h_{\mathcal{M}}(n) = \max\{|M| : M \in \mathcal{M} \text{ is simple, and } r(M) \le n\}.$$

The following striking theorem summarizes the results of several papers, [1,2,4].

Theorem 1.1 (Growth Rate Theorem)**.** *Let $\mathcal{M}$ be a minor-closed class of matroids, not containing all simple rank-2 matroids. Then there is an integer c such that either:*

(1) $h_{\mathcal{M}}(n) \le cn$ for all $n \ge 0$, or

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(2) $\binom{n+1}{2} \le h_{\mathcal{M}}(n) \le cn^2$ *for all* $n \ge 0$*, and* $\mathcal{M}$ *contains all graphic matroids, or*

(3) *there is a prime power* q *such that* $\frac{q^n-1}{q-1} \le h_{\mathcal{M}}(n) \le cq^n$ *for all* $n \ge 0$*, and* $\mathcal{M}$ *contains all* $\mathrm{GF}(q)$*-representable matroids.*

In particular, the theorem implies that $h_{\mathcal{M}}(n)$ is finite for all n if and only if $\mathcal{M}$ does not contain all simple rank-2 matroids. If $\mathcal{M}$ is a minor-closed class satisfying (3), then we say that $\mathcal{M}$ is *base-q exponentially dense*. Our main theorems precisely determine, for many such classes, the eventual value of the growth rate function:

Theorem 1.2. *Let* q *be a prime power. If* $\mathcal{M}$ *is a base-q exponentially dense minor-closed class of matroids such that* $U_{2,q^2+1} \notin \mathcal{M}$*, then*

$$h_{\mathcal{M}}(n) = \frac{q^n - 1}{q - 1}$$

for all sufficiently large n*.*

Consider, for example, the class $\mathcal{M}$ of matroids with no $U_{2,\ell+2}$-minor, where $\ell \ge 2$ is an integer. By the Growth Rate Theorem, this class is base-q exponentially dense, where q is the largest prime-power not exceeding ℓ. Clearly $q^2 > \ell$, so, by Theorem 1.2, $h_{\mathcal{M}}(n) = \frac{q^n-1}{q-1}$ for all large n. This special case is the main result of [3], which essentially also contains a proof of Theorem 1.2.

Theorem 1.3. *Let* q *be a prime power. If* $\mathcal{M}$ *is a base-q exponentially dense minor-closed class of matroids such that* $U_{2,q^2+q+1} \notin \mathcal{M}$*, then there is an integer* $k \ge 0$ *such that*

$$h_{\mathcal{M}}(n) = \frac{q^{n+k} - 1}{q - 1} - q\frac{q^{2k} - 1}{q^2 - 1}$$

for all sufficiently large n*.*

Consider, for example, any proper minor-closed subclass $\mathcal{M}$ of the $\mathrm{GF}(q^2)$-representable matroids that contains all $\mathrm{GF}(q)$-representable matroids. Such classes are all base-q exponentially dense and do not contain U_{2,q^2+2}, so Theorem 1.3 applies; this special case is the main result of [8].

If the hypothesis of Theorem 1.3 is weakened to allow $U_{2,q^2+q+1} \in \mathcal{M}$, then the conclusion no longer holds. Consider the class $\mathcal{M}_1$ defined to be the set of truncations of all $\mathrm{GF}(q)$-representable matroids; note that $U_{2,q^2+q+2} \notin \mathcal{M}_1$ and $h_{\mathcal{M}_1}(n) = \frac{q^{n+1}-1}{q-1}$ for all $n \ge 2$.

More generally, for each $k \ge 0$, if $\mathcal{M}_k$ is the set of matroids obtained from $\mathrm{GF}(q)$-representable matroids by applying k truncations, then

$h_{\mathcal{M}_k}(n) = \frac{q^{n+k}-1}{q-1}$ for all $n \geq 2$. This expression differs from that in Theorem 1.3 by only the constant $q\frac{q^{2k}-1}{q^2-1}$. It is conjectured [8,9] that, for each k, these are the extremes in a small spectrum of possible growth rate functions:

Conjecture 1.4. *Let q be a prime power, and $\mathcal{M}$ be a base-q exponentially dense minor-closed class of matroids. There exist integers k and d with $k \geq 0$ and $0 \leq d \leq \frac{q^{2k}-1}{q^2-1}$, such that $h_{\mathcal{M}}(n) = \frac{q^{n+k}-1}{q-1} - qd$ for all sufficiently large n.*

We conjecture further that, for every allowable q, k and d, there exists a minor-closed class with the above as its eventual growth rate function.

There is a stronger conjecture [9] regarding the exact structure of the extremal matroids. For a non-negative integer k, a *k-element projection* of a matroid M is a matroid of the form N/C, where $N \backslash C = M$, and C is a k-element set of N.

Conjecture 1.5. *Let q be a prime power, and $\mathcal{M}$ be a base-q exponentially dense minor-closed class of matroids. There exists an integer $k \geq 0$ such that, if $M \in \mathcal{M}$ is a simple matroid of sufficiently large rank with $|M| = h_{\mathcal{M}}(r(M))$, then M is the simplification of a k-element projection of a projective geometry over* GF(q).

We will show, as was observed in [9], that this conjecture implies the previous one; see Lemma 3.1.

2. Preliminaries

A matroid M is called (q,k)*-full* if

$$\varepsilon(M) \geq \frac{q^{r(M)+k}-1}{q-1} - q\frac{q^{2k}-1}{q^2-1};$$

moreover, if strict inequality holds, M is (q,k)*-overfull*.

Our proof of Theorem 1.3 follows a strategy similar to that in [8]; we show that, for any integer $n > 0$, every (q,k)-overfull matroid in $\text{EX}(U_{2,q^2+q+1})$, with sufficiently large rank, contains a $(q,k+1)$-full rank-n minor. The Growth Rate Theorem tells us that a given base-q exponentially dense minor-closed class cannot contain (q,k)-full matroids for arbitrarily large k, so this gives the result. Theorem 1.2 is easier and will follow along the way.

We follow the notation of Oxley [10]; flats of rank 1, 2 and 3 are respectively *points*, *lines* and *planes* of a matroid. If M is a matroid, and $X, Y \subseteq E(M)$, then $\sqcap_M(X,Y) = r_M(X) + r_M(Y) - r_M(X \cup Y)$

is the *local connectivity* between X and Y. If $\sqcap_M(X,Y) = 0$, then X and Y are *skew* in M, and if $\mathcal{X}$ is a collection of sets in M such that each $X \in \mathcal{X}$ is skew to the union of the sets in $\mathcal{X} - \{X\}$, then $\mathcal{X}$ is a *mutually skew* collection of sets. A pair (F_1, F_2) of flats in M is *modular* if $\sqcap_M(F_1, F_2) = r_M(F_1 \cap F_2)$, and a flat F of M is *modular* if, for each flat F' of M, the pair (F, F') is modular. In a projective geometry each pair of flats is modular and, hence, each flat is modular.

For a matroid M, we write $|M|$ for $|E(M)|$, and $\varepsilon(M)$ for $|\operatorname{si}(M)|$, the number of points in M. Thus, $h_{\mathcal{M}}(n) = \max(\varepsilon(M) : M \in \mathcal{M}, r(M) \leq n)$. Two matroids are equal *up to simplification* if their simplifications are isomorphic. We let $\mathrm{EX}(M)$ denote the set of matroids with no M-minor; Theorems 1.2 and 1.3 apply to subclasses of $\mathrm{EX}(U_{2,q^2+1})$ and $\mathrm{EX}(U_{2,q^2+q+1})$ respectively. The following theorem of Kung [5] bounds the density of a matroid in $\mathrm{EX}(U_{2,\ell+2})$:

Theorem 2.1. *Let $\ell \geq 2$ be an integer. If $M \in \mathrm{EX}(U_{2,\ell+2})$, then $\varepsilon(M) \leq \frac{\ell^{r(M)}-1}{\ell-1}$.*

The next result is an easy application of the Growth Rate Theorem.

Lemma 2.2. *There is a real-valued function $\alpha_{2.2}(n, \beta, \ell)$ so that, for any integers $n \geq 1$ and $\ell \geq 2$, and real number $\beta > 1$, if $M \in \mathrm{EX}(U_{2,\ell+2})$ is a matroid such that $\varepsilon(M) \geq \alpha_{2.2}(n, \beta, \ell)\beta^{r(M)}$, then M has a $\mathrm{PG}(n-1, q)$-minor for some $q > \beta$.*

The following lemma was proved in [8]:

Lemma 2.3. *Let λ, μ be real numbers with $\lambda > 0$ and $\mu > 1$, let $t \geq 0$ and $\ell \geq 2$ be integers, and let A and B be disjoint sets of elements in a matroid $M \in \mathrm{EX}(U_{2,\ell+2})$ with $r_M(B) \leq t < r(M)$ and $\varepsilon(M|A) > \lambda\mu^{r_M(A)}$. Then there is a set $A' \subseteq A$ that is skew to B and satisfies $\varepsilon(M|A') > \lambda\left(\frac{\mu-1}{\ell}\right)^t \mu^{r_M(A')}$.*

3. Projections

Recall that a k-element projection of a matroid M is a matroid of the form N/C, where C is a k-element set of a matroid N satisfying $N \backslash C = M$.

In this section we are concerned with projections of projective geometries. Consider a k-element set C in a matroid N such that $N \backslash C = \mathrm{PG}(n+k-1, q)$ and let $M = N/C$. Thus M is a k-element projection of $\mathrm{PG}(n+k-1, q)$. Below are easy observations that we use freely.

- If C is not independent, then M is a $(k-1)$-element projection of $\mathrm{PG}(n+k-1, q)$.

- If C is not coindependent, then M is a $(k-1)$-element projection of $\mathrm{PG}(n+k-1,q)$.
- If C is not closed in N, then M is, up to simplification, a $(k-1)$-element projection of $\mathrm{PG}(n+k-2,q)$.
- M has a $\mathrm{PG}(r(M)-1,q)$-restriction.

Our next result gives the density of projections of projective geometries; given such a projection M, this density is determined to within a small range by the minimum k for which M is a k-element projection. As mentioned earlier, this lemma also tells us that Conjecture 1.5 implies Conjecture 1.4.

Lemma 3.1. *Let q be a prime power, and $k \geq 0$ be an integer. If N is a matroid, and C is a rank-k flat of N such that $N\backslash C \cong \mathrm{PG}(r(N)-1,q)$, then $\varepsilon(N/C) = \varepsilon(N\backslash C) - qd$ for some $d \in \{0,1,\dots,\frac{q^{2k}-1}{q^2-1}\}$.*

Proof. Each point P of N/C is a flat of the projective geometry $N\backslash C$, so $|P| = \frac{q^{r_N(P)}-1}{q-1} = 1 + q\frac{q^{r_N(P)-1}-1}{q-1}$. Therefore $\varepsilon(N\backslash C) - \varepsilon(N/C)$ is a multiple of q.

Let $\mathcal{P}$ denote the set of all points in N/C that contain more than one element, and let F be the flat of $N\backslash C$ spanned by the union of these points. Choose a minimal set $\mathcal{P}_0 \subseteq \mathcal{P}$ of points spanning F in N/C (so $|\mathcal{P}_0| = r_{N/C}(F)$); if possible choose $\mathcal{P}_0$ so that it contains a set in $P \in \mathcal{P}$ with $r_N(P) > 2$. Note that: (1) the points in $\mathcal{P}_0$ are mutually skew in N/C, (2) each pair of flats of $N\backslash C$ is modular, and (3) C is a flat of N. It follows that $\mathcal{P}_0$ is a mutually skew collection of flats in $N\backslash C$. Now, for each $P \in \mathcal{P}_0$, $r_N(P) > r_{N/C}(P)$. Therefore, since $r(N) - r(N/C) = k$, we have $r_{N/C}(F) = |\mathcal{P}_0| \leq k$. Moreover, if $r_{N/C}(F) = k$, then each set in $\mathcal{P}_0$ is a line of $N\backslash C$, and, hence, by our choice of $\mathcal{P}_0$, each set in $\mathcal{P}$ is a line in $N\backslash C$.

If $r_{N/C}(F) = k$, then we have $|F| = \frac{q^{2k}-1}{q-1}$ and $|\mathcal{P}| \leq \frac{|F|}{q+1}$. This gives $\varepsilon(N\backslash C) - \varepsilon(N/C) \leq q\frac{|F|}{q+1} = q\frac{q^{2k}-1}{q^2-1}$, as required.

If $r_{N/C}(F) < k$, then $\varepsilon(N\backslash C) - \varepsilon(N/C) \leq |F| \leq \frac{q^{2k-1}-1}{q-1}$. It is routine to verify that $\frac{q^{2k-1}-1}{q-1} < q\frac{q^{2k}-1}{q^2-1}$, which proves the result. □

The next two lemmas consider single-element projections, highlighting the importance of U_{2,q^2+1} and U_{2,q^2+q+1} in Theorems 1.2 and 1.3.

Lemma 3.2. *Let q be a prime power and let e be an element of a matroid M such that $M\backslash e \cong \mathrm{PG}(r(M)-1,q)$. Then there is a unique minimal flat F of $M\backslash e$ that spans e. Moreover, if $r(M) \geq 3$ and $r_M(F) \geq 2$, then M/e contains a U_{2,q^2+1}-minor, and if $r_M(F) \geq 3$, then M/e contains a U_{2,q^2+q+1}-minor.*

Proof. If F_1 and F_2 are two flats of $M \backslash e$ that span e, then, since $r_M(F_1 \cap F_2) + r_M(F_1 \cup F_2) = r_M(F_1) + r_M(F_2)$, it follows that $F_1 \cap F_2$ also spans e. Therefore there is a unique minimal flat F of $M \backslash e$ that spans e. The uniqueness of F implies that e is freely placed in F.

Suppose that $r_M(F) \geq 3$. Thus $(M/e)|F$ is the truncation of a projective geometry of rank ≥ 3. So M/e contains a truncation of $\mathrm{PG}(2, q)$ as a minor; therefore M/e has a U_{2,q^2+q+1}-minor.

Now suppose that $r(M) \geq 3$ and that $r_M(F) = 2$. If F' is a rank-3 flat of $M \backslash e$ containing F, then $\varepsilon((M/e)|F') = q^2 + 1$, so M/e has a U_{2,q^2+1}-minor. □

An important consequence is that, if M is a simple matroid with a $\mathrm{PG}(r(M) - 1, q)$-restriction R and no U_{2,q^2+q+1}-minor, then every $e \in E(M) - E(R)$ is spanned by a unique line of R. The next result describes the structure of the projections in $\mathrm{EX}(U_{2,q^2+q+1})$.

Lemma 3.3. *Let q be a prime power, and $M \in \mathrm{EX}(U_{2,q^2+q+1})$ be a simple matroid, and $e \in E(M)$ be such that $M \backslash e \cong \mathrm{PG}(r(M) - 1, q)$. If L is the unique line of $M \backslash e$ that spans e, then L is a point of M/e, and each line of M/e containing L has $q^2 + 1$ points and is modular.*

Proof. Let L' be a line of M/e containing L. Then L' is a plane of $M \backslash e$, so, by Lemma 3.2, L' has $q^2 + 1$ points in M/e.

Note that e is freely placed on the line $L \cup \{e\}$ in M. It follows that M is $\mathrm{GF}(q^2)$-representable. Now L' is a $(q^2 + 1)$-point line in the $\mathrm{GF}(q^2)$-representable matroid M/e; hence, L' is modular in M/e. □

4. Dealing with long lines

This section contains two lemmas that construct a U_{2,q^2+q+1}-minor of a matroid M with a $\mathrm{PG}(r(M)-1, q)$-restriction R and some additional structure.

Lemma 4.1. *Let q be a prime power, and M be a simple matroid of rank at least 7 such that*

- *M has a $\mathrm{PG}(r(M) - 1, q)$-restriction R, and*
- *M has a line L containing at least $q^2 + 2$ points, and*
- *$E(M) \neq E(R) \cup L$,*

then M has a U_{2,q^2+q+1}-minor.

Proof. We may assume that $E(M) = E(R) \cup L \cup \{z\}$, where $z \notin L \cup E(R)$. Let F be a minimal flat of R that spans $L \cup \{z\}$. It follows easily from Lemma 3.2, that either M has a U_{2,q^2+q+1}-minor or $r_M(F) \leq 6$. To simplify the proof we will prove the lemma with the weaker hypothesis that $r(M) \geq 1 + r_M(F)$, in place of the hypothesis that $r(M) \geq 7$, and we will suppose that (M, R, L) forms a minimum rank counterexample under these weakened hypotheses.

Let L_z denote the line of R that spans z in M. Since $z \notin L$, we have $r_M(L \cup L_z) \geq 3$. We may assume that $r_M(L \cup L_z) = 3$, since otherwise we could contract a point in $F - (L \cup L_z)$ to obtain a smaller counterexample. Similarly, we may assume that $r_M(F) = 3$ and $r(M) = 4$, as otherwise we could contract an element of $F - \mathrm{cl}_M(L \cup L_z)$ or $E(M) - \mathrm{cl}_M(F)$.

By Lemma 3.3, L_z is a point of $(M/z)|R$ and each line of $(M/z)|R$ is modular and has q^2+1 points. One of these lines is F, and, since F spans L, F spans a line with q^2+2 points in M/z. Let $e \in cl_{M/z}(F)$ be an element that is not in parallel with any element of F. Since F is a modular line in $(M/z)|R$, the point e is freely placed on the line $F \cup \{e\}$ in $(M/z)|(R \cup \{e\})$. Therefore $\varepsilon(M/\{e, z\}) \geq \varepsilon((M/\{z\})|R) - q^2 = 1 + q^2(q+1) - q^2 = q^3 + 1$, contradicting the fact that $M \in \mathrm{EX}(U_{2,q^2+q+1})$. □

Lemma 4.2. *Let q be a prime power, and $k \geq 3$ be an integer. If M is a matroid of rank at least $k+7$, with a $PG(r(M)-1, q)$-restriction, and a set $X \subseteq E(M)$ with $r_M(X) \leq k$ and $\varepsilon(M|X) > \frac{q^{2k}-1}{q^2-1}$, then M has a U_{2,q^2+q+1}-minor.*

Proof. Let M_0 be a matroid satisfying the hypotheses, with a $\mathrm{PG}(r(M_0) - 1, q)$-restriction R_0. We may assume that $M_0 \in \mathrm{EX}(U_{2,q^2+q+1})$, and by choosing a rank-k set containing X, we may also assume that $r_{M_0}(X) = k$. By Lemma 3.2, R_0 has a flat F_0 of rank at most $2k$ such that $X \subseteq \mathrm{cl}_{M_0}(F_0)$. By contracting at most k points in $F_0 - \mathrm{cl}_{M_0}(X)$, we obtain a minor M of M_0, of rank at least 7, such that $r_M(X) = k$, and M has a $\mathrm{PG}(r(M)-1, q)$-restriction R, and there is a rank-k flat F of R such that $X \subseteq \mathrm{cl}_M(F)$.

We may assume that M is simple and that X is a flat of M, so $F \subseteq X$. Let $n = |F| = \frac{q^k-1}{q-1}$. By Lemma 3.2, each point of X is spanned in M by a line of $R|F$. There are $\binom{n}{2}/\binom{q+1}{2}$ such lines, each containing $q+1$ points of F. If each of these lines spans at most (q^2-q) points of $X - F$, then

$$|X| = |F| + |X - F| \leq \frac{q^k - 1}{q - 1} + \frac{(q^2 - q)\binom{n}{2}}{\binom{q+1}{2}} = \frac{q^{2k} - 1}{q^2 - 1},$$

contradicting the definition of X. Therefore, some line L of $M|X$ contains at least q^2+2 points. We also have $|L| \leq q^2+q$, so a calculation gives $|X - L| > \frac{q^{2k}-1}{q^2-1} - (q^2+q) \geq \frac{q^k-1}{q-1} = |F|$, so $X \neq F \cup L$. Applying Lemma 4.1 to $M|(E(R) \cup X)$ gives the result. □

5. Matchings and unstable sets

For an integer $k \geq 0$, a *k-matching* of a matroid M is a mutually skew k-set of lines of M. Our first theorem was proved in [8], and also follows routinely from the much more general linear matroid matching theorem of Lovász [7]:

Theorem 5.1. *There is an integer-valued function $f_{5.1}(q,k)$ so that, for any prime power q and integers $n \geq 1$ and $k \geq 0$, if $\mathcal{L}$ is a set of lines in a matroid $M \cong \mathrm{PG}(n-1,q)$, then either*

(i) $\mathcal{L}$ contains a $(k+1)$-matching of M, or
(ii) there is a flat F of M with $r_M(F) \leq k$, and a set $\mathcal{L}_0 \subseteq \mathcal{L}$ with $|\mathcal{L}_0| \leq f_{5.1}(q,k)$, such that every line $L \in \mathcal{L}$ either intersects F, or is in $\mathcal{L}_0$. Moreover, if $r_M(F) = k$, then $\mathcal{L}_0 = \varnothing$.

We now define a property in terms of a matching in a spanning projective geometry. Let q be a prime power, $M \in \mathrm{EX}(U_{2,q^2+q+1})$ be a simple matroid with a $\mathrm{PG}(r(M)-1,q)$-restriction R, and $X \subseteq E(M \backslash R)$ be a set such that $M|(E(R) \cup X)$ is simple. Recall that, by Lemma 3.2, each $x \in X$ lies in the closure of exactly one line L_x of R. We say that X is *R-unstable* in M if the lines $\{L_x : x \in X\}$ are a matching of size $|X|$ in R.

Lemma 5.2. *There is an integer-valued function $f_{5.2}(q,k)$ so that, for any prime power q and integer $k \geq 0$, if $M \in \mathrm{EX}(U_{2,q^2+q+1})$ is a matroid of rank at least 3 with a $\mathrm{PG}(r(M)-1,q)$-restriction R, then either*

(i) there is an R-unstable set of size $k+1$ in M, or
(ii) R has a flat F with rank at most k such that $\varepsilon(M/F) \leq \varepsilon(R/F) + f_{5.2}(q,k)$.

Proof. Let q be a prime power, and $k \geq 0$ be an integer. Set $f_{5.2}(q,k) = (q^2+q)f_{5.1}(q,k)$. Let M be a matroid with a $\mathrm{PG}(r(M)-1,q)$-restriction R. We may assume that M is simple, and that the first outcome does not hold. Let $\mathcal{L}$ be the set of lines L of R such that $|\,\mathrm{cl}_M(L)| > |\,\mathrm{cl}_R(L)|$. If $\mathcal{L}$ contains a $(k+1)$-matching of R, then choosing a point from $\mathrm{cl}_M(L) - \mathrm{cl}_R(L)$ for each line L in the matching gives an R-unstable set of size $k+1$. We may therefore assume that $\mathcal{L}$ contains no such matching. Thus, let F and $\mathcal{L}_0$ be the sets defined in the second outcome of Theorem 5.1. Let $D = \cup_{L \in \mathcal{L}_0} \mathrm{cl}_M(L)$. We have $|D| \leq (q^2+q)|\mathcal{L}_0| \leq f_{5.2}(q,k)$. By Lemma 3.2, each element of $M \backslash D$ either lies the closure of a line in $\mathcal{L}$ or in a point of R, so is parallel in M/F to an element of R. Therefore, $\varepsilon(M/F) \leq \varepsilon(R/F) + |D|$; the result now follows. $\square$

We use an unstable set to construct a dense minor. Recall that (q,k)-full and (q,k)-overfull were defined at the start of Section 2.

Lemma 5.3. *Let q be a prime power, and $k \geq 1$ and $n > k$ be integers. If $M \in \mathrm{EX}(U_{2,q^2+q+1})$ is a matroid of rank at least $n+k$ with a $\mathrm{PG}(r(M)-1,q)$-restriction R, and X is an R-unstable set of size k in M, then M has a rank-n (q,k)-full minor N with a U_{2,q^2+1}-restriction.*

Proof. We may assume by taking a restriction if necessary that $r(M) = n+k$, and $E(M) = E(R) \cup X$; we show that $N = M/X$ has the required properties. For each $x \in X$, let L_x denote the line of R that spans X; thus $\{L_x : x \in X\}$ is a matching. By the definition of instability, it is clear that X is independent, so $r(N) = n$. Let $x \in X$, and P be a plane of R that contains L_x and is skew to $X - \{x\}$. By Lemma 3.3, $(M/x)|P$ has a U_{2,q^2+1}-restriction. Since $X - \{x\}$ is skew to P, M/X also has a U_{2,q^2+1}-restriction.

To complete the proof it is enough, by Lemma 3.1, to show that $\mathrm{cl}_M(X)$ is disjoint from R. This is trivial if X is empty, so consider $x \in X$ and let $R' = \mathrm{si}(R/L_x)$. Note that $R' \cong \mathrm{PG}(n+k-3,q)$ is a spanning restriction of M/L_x and $X - \{x\}$ is R'-unstable. Inductively, we may assume that $\mathrm{cl}_{M/L_x}(X - \{x\})$ is disjoint from R/L_x, but this implies that $\mathrm{cl}_M(X)$ is disjoint from R, as required. □

6. The spanning case

In this section we consider matroids that are spanned by a projective geometry.

Lemma 6.1. *There is an integer-valued function $f_{6.1}(n,q,k)$ such that, for any prime power q and integers $k \geq 0$ and $n > k+1$, if $M \in \mathrm{EX}(U_{2,q^2+q+1})$ is a matroid of rank at least $f_{6.1}(n,q,k)$ such that*

- *M has a $\mathrm{PG}(r(M)-1,q)$-restriction R, and*
- *M is (q,k)-overfull,*

then M has a rank-n $(q,k+1)$-full minor N with a U_{2,q^2+1}-restriction.

Proof. Let $k \geq 0$ and $n > k+1$ be integers, and q be a prime power. Let $m > \max(k+7, n+k+1)$ be an integer such that

$$\frac{q^{r+k}-1}{q-1} - q\frac{q^{2k}-1}{q^2-1} > \frac{q^{r+j}-1}{q-1} + \max(q^2+q, (q^2-q)f_{5.1}(q,k))$$

for all $r \geq m$ and $0 \leq j < k$. We set $f_{6.1}(n,q,k) = m$.

Let $M \in \mathrm{EX}(U_{2,q^2+q+1})$ be a (q,k)-overfull matroid of rank at least m, and let R be a $\mathrm{PG}(r(M)-1,q)$-restriction of M. We will show that M has the required minor N; we may assume that M is simple.

6.1.1. *If $k \geq 1$, then no line of M contains more than q^2+1 points.*

Proof of claim: Let L be a line of M containing at least q^2+2 points. We have $|L| \leq q^2+q$, so $|E(R) \cup L| \leq \frac{q^{r(M)}-1}{q-1} + q^2 + q < |M|$ by the definition of m. Therefore, there is a point of M in neither R nor L. By Lemma 4.1, M has a U_{2,q^2+q+1}-minor, a contradiction. $\square$

Let $\mathcal{L}$ be the set of lines of R, and $\mathcal{L}^+$ be the set of lines of R that are not lines of M; note that each $L \in \mathcal{L}^+$ contains exactly $q+1$ points of R, and spans an extra point in M. By Lemma 3.2, every point of $M \backslash E(R)$ is spanned by a line in $\mathcal{L}^+$.

6.1.2. *$\mathcal{L}^+$ contains a $(k+1)$-matching of R.*

Proof of claim: If $k=0$, then since $|M| > |R|$, we must have $\mathcal{L}^+ \neq \varnothing$, so the claim is trivial. Thus, assume that $k \geq 1$ and that there is no such matching. Let $F \subseteq E(R)$ and $\mathcal{L}_0 \subseteq \mathcal{L}$ be the sets defined in Theorem 5.1. Let $j = r_M(F)$; we know that $0 \leq j \leq k$, and that $\mathcal{L}_0$ is empty if $j = k$. Let $\mathcal{L}_F = \{L \in \mathcal{L} : |L \cap F| = 1\}$. By definition, every point of $M \backslash R$ is in the closure of F, or the closure of a line in $\mathcal{L}_F \cup \mathcal{L}_0$.

Every point of $R \backslash F$ lies on exactly $|F|$ lines in $\mathcal{L}_F$, and each such line contains exactly q points of $R \backslash F$, so

$$|\mathcal{L}_F| = \frac{|F||R \backslash F|}{q} = \frac{(q^j-1)(q^{r(M)}-q^j)}{q(q-1)^2}.$$

Furthermore, each line in $\mathcal{L}$ contains $q+1$ points of R, and its closure in M contains at most q^2-q points of $M \backslash R$ by the first claim. We argue that $|\operatorname{cl}_M(F)| \leq \frac{q^{2j}-1}{q^2-1}$; if $j \leq 2$, then this follows from the first claim, and otherwise, we have $r(M) \geq m \geq k+7$, so the bound follows by applying Lemma 4.2 to M and $\operatorname{cl}_M(F)$. We now estimate $|M|$.

$$\begin{aligned} |M| &= |R| + |M \backslash E(R)| \\ &\leq |R| + \sum_{L \in \mathcal{L}_F \cup \mathcal{L}_0} |\operatorname{cl}_M(L) - E(R)| + |\operatorname{cl}_M(F) - F| \\ &\leq \frac{q^{r(M)}-1}{q-1} + (q^2-q)(|\mathcal{L}_F| + |\mathcal{L}_0|) + \left(\frac{q^{2j}-1}{q^2-1} - \frac{q^j-1}{q-1}\right). \end{aligned}$$

Now, a calculation and our value for $\mathcal{L}_F$ obtained earlier together give $|M| \leq \frac{q^{r(M)+j}-1}{q-1} - q\frac{q^{2j}-1}{q^2-1} + (q^2-q)|\mathcal{L}_0|$. If $j < k$, then, since $r(M) \geq m$ and $|\mathcal{L}_0| \leq f_{5.1}(q,k)$, we have $|M| \leq \frac{q^{r(M)+k}-1}{q-1} - q\frac{q^{2k}-1}{q^2-1}$ by definition of m. If $j = k$, then $|\mathcal{L}_0| = 0$, so the same inequality holds. In either case, we contradict the fact that M is (q,k)-overfull. $\square$

Now, $\mathcal{L}^+$ has a matching of size $k+1$, so by construction of $\mathcal{L}^+$, there is an R-unstable set X of size $k+1$ in M. Since $r(M) \geq m > n+k+1$, the required minor N is given by Lemma 5.3. □

7. Connectivity

A matroid M is *weakly round* if there is no pair of sets A, B with union $E(M)$, such that $r_M(A) \leq r(M)-2$ and $r_M(B) \leq r(M)-1$. Any matroid of rank at most 2 is clearly weakly round. This is a variation on *roundness*, a notion equivalent to infinite vertical connectivity introduced by Kung [6] under the name of 'non-splitting'. Weak roundness is preserved by contraction; the following lemma is easily proved, and we use it freely.

Lemma 7.1. *If M is a weakly round matroid, and $e \in E(M)$, then M/e is weakly round.*

The first step in our proof of the main theorems will be to reduce to the weakly round case; the next two lemmas give this reduction.

Lemma 7.2. *If M is a matroid, then M has a weakly round restriction N such that $\varepsilon(N) \geq \varphi^{r(N)-r(M)}\varepsilon(M)$, where $\varphi = \frac{1}{2}(1+\sqrt{5})$.*

Proof. We may assume that M is not weakly round, so $r(M) > 2$, and there are sets A, B of M such that $r_M(A) = r(M)-2$, $r_M(B) = r(M)-1$, and $E(M) = A \cup B$. Now, since $\varphi^{-1} + \varphi^{-2} = 1$, either $\varepsilon(M|A) \geq \varphi^{-2}\varepsilon(M)$ or $\varepsilon(M|B) \geq \varphi^{-1}\varepsilon(M)$; in the first case, by induction $M|A$ has a weakly round restriction N with $\varepsilon(N) \geq \varphi^{r(N)-r(M|A)}\varepsilon(M|A) \geq \varphi^{r(N)-r(M)+2}\varphi^{-2}\varepsilon(M) = \varphi^{r(N)-r(M)}\varepsilon(M)$, giving the result. The second case is similar. □

Lemma 7.3. *Let q be a prime-power, and $k \geq 0$ be an integer. If $\mathcal{M}$ is a base-q exponentially dense minor-closed class of matroids that contains (q,k)-overfull matroids of arbitrarily large rank, then $\mathcal{M}$ contains weakly round, (q,k)-overfull matroids of arbitrarily large rank.*

Proof. Note that $\varphi < 2 \leq q$; by the Growth Rate Theorem, there is an integer $t > 0$ such that

$$\varepsilon(M) \leq \left(\frac{q}{\varphi}\right)^t \frac{q^{r(M)+k}-1}{q-1} - q\frac{q^{2k}-1}{q^2-1},$$

for all $M \in \mathcal{M}$.

For any integer $n > 0$, consider a (q,k)-overfull matroid $M \in \mathcal{M}$ with rank at least $n+t$. By Lemma 7.2, M has a weakly round restriction

N such that $\varepsilon(N) \geq \varphi^{-s}\varepsilon(M)$, where $s = r(M) - r(N)$. We have

$$\begin{aligned} \varepsilon(N) &\geq \varphi^{-s}\varepsilon(M) \\ &> \varphi^{-s}\left(\frac{q^{r(M)+k}-1}{q-1} - q\frac{q^{2k}-1}{q-1}\right) \\ &> \left(\frac{q}{\varphi}\right)^{s}\frac{q^{r(N)+k}-1}{q-1} - q\frac{q^{2k}-1}{q^2-1}. \end{aligned}$$

Thus N is (q,k)-overfull. Moreover, by the definition of t, we have $s < t$ and, hence, $r(N) > n$. □

8. Exploiting connectivity

We now exploit weak roundness by showing that any interesting low-rank restriction can be contracted into the span of a projective geometry.

Lemma 8.1. *There is an integer-valued function $f_{8.1}(n,q,t,\ell)$ so that, for any prime power q, and integers $n \geq 1, \ell \geq 2$ and $t \geq 0$, if $M \in \mathrm{EX}(U_{2,\ell+2})$ is a weakly round matroid with a $\mathrm{PG}(f_{8.1}(n,q,t,\ell)-1,q)$-minor, and T is a restriction of M of rank at most t, then there is a minor N of M of rank at least n, such that T is a restriction of N, and N has a $\mathrm{PG}(r(N)-1,q)$-restriction.*

Proof. Let $n \geq 1, \ell \geq 2$ and $t \geq 0$ be integers. Let $n' = \max(n, t+1)$, and set $f_{8.1}(n,q,t,\ell)$ to be an integer m such that $m \geq 2t$, and

$$\frac{q^m-1}{q-1} \geq \alpha_{2.2}(n', q-\tfrac{1}{2}, \ell)\left(\frac{\ell(q-\frac{1}{2})}{q-\frac{3}{2}}\right)^{t}(q-\tfrac{1}{2})^m.$$

Let $M \in \mathrm{EX}(U_{2,\ell+2})$ be a weakly round matroid with a $\mathrm{PG}(m-1,q)$-minor $S = M/C\backslash D$, where $r(S) = r(M) - r_M(C)$. Let T be a restriction of M of rank at most t; we show that the required minor exists.

8.1.1. *There is a weakly round minor M_1 of M, such that T is a restriction of M_1, and M_1 has a $\mathrm{PG}(n'-1,q)$-restriction R_1.*

Proof of claim: Let $C' \subseteq C$ be maximal such that T is a restriction of M/C', and let $M' = M/C'$. Maximality implies that $C - C' \subseteq \mathrm{cl}_{M'}(E(T))$, so $r_{M'}(C - C') \leq t$. Now, $r_{M'}(E(S)) = r(S) + r_{M'}(C - C') \leq m + t$. Therefore,

$$\begin{aligned} \varepsilon_{M'}(E(S)) &= \frac{q^m-1}{q-1} \\ &\geq \alpha_{2.2}(n', q-\tfrac{1}{2}, \ell)\ell^t(q-\tfrac{3}{2})^{-t}(q-\tfrac{1}{2})^{m+t} \\ &\geq \alpha_{2.2}(n', q-\tfrac{1}{2}, \ell)(\ell(q-\tfrac{3}{2})^{-1})^t(q-\tfrac{1}{2})^{r_{M'}(E(S))}. \end{aligned}$$

By Lemma 2.3 applied to $E(S)$ and $E(T)$, with $\mu = q - \frac{1}{2}$, there is a set $A \subseteq E(S)$, skew to $E(T)$ in M', such that

$$\varepsilon(M'|A) \geq \alpha_{2.2}(n', q - \tfrac{1}{2}, \ell)(q - \tfrac{1}{2})^{r(M'|A)}.$$

Therefore, Lemma 2.2 implies that $M'|A$ has a $\mathrm{PG}(n'-1, q')$-minor $R_1 = (M'|A)/C_1 \setminus D_1$, for some $q' > q - \frac{1}{2}$. Let $M_1 = M'/C_1$. The set A is skew to $E(T)$ in M', and therefore also skew to $C - C'$, so $M'|A = (M'/(C - C'))|A = S|A$, so $M'|A$ is $\mathrm{GF}(q)$-representable, and so is its minor R_1. Thus, $q' = q$, and R_1 is a $\mathrm{PG}(n'-1, q)$-restriction of M_1. Moreover, $C_1 \subseteq A$, so C_1 is skew to $E(T)$ in M', and therefore M_1 has T as a restriction. The matroid M_1 is a contraction-minor of M, so is weakly round, and thus satisfies the claim. □

Let M_2 be a minor-minimal matroid such that:

- M_2 is a weakly round minor of M_1, and
- T and R_1 are both restrictions of M_2.

If $r(R_1) = r(M_2)$, then $N = M_2$ is the required minor of M. We may therefore assume that $r(M_2) > r(R_1) = n'$. We have $r(T) \leq t \leq n' - 1 \leq r(M_2) - 2$, so by weak roundness of M_2, there is some $e \in E(M_2)$ spanned by neither $E(T)$ nor $E(R_1)$, contradicting minimality of M_2. □

9. Critical elements

An element e in a (q,k)-overfull matroid M is called (q,k)-*critical* if M/e is not (q,k)-overfull.

Lemma 9.1. *Let q be a prime power and $k \geq 0$ be an integer. If e is a (q,k)-critical element in a (q,k)-overfull matroid M, then either*

(i) e is contained in a line with at least $q^2 + 2$ points, or
(ii) e is contained in $\frac{q^{2k}-1}{q^2-1} + 1$ lines, each with at least $q + 2$ points.

Proof. Suppose otherwise. Let $\mathcal{L}$ be the set of all lines of M containing e, and let $\mathcal{L}_1$ be the set of the $\min(|\mathcal{L}|, \frac{q^{2k}-1}{q^2-1})$ longest lines in $\mathcal{L}$. Every line in $\mathcal{L} - \mathcal{L}_1$ has at most $q+1$ points and every line in $\mathcal{L}_1$ has at most

q^2+1 points, so

$$\begin{aligned}\varepsilon(M) &\le 1 + q|\mathcal{L}| + (q^2-q)|\mathcal{L}_1| \\ &\le 1 + q\varepsilon(M/e) + (q^2-q)\frac{q^{2k}-1}{q^2-1} \\ &\le 1 + q\left(\frac{q^{r(M)+k-1}-1}{q-1} - q\frac{q^{2k}-1}{q^2-1}\right) + (q^2-q)\frac{q^{2k}-1}{q^2-1} \\ &= \frac{q^{r(M)+k}-1}{q-1} + q\frac{q^{2k}-1}{q^2-1},\end{aligned}$$

contradicting the fact that M is (q,k)-overfull. □

The following result shows that a large number of (q,k)-critical elements gives a denser minor.

Lemma 9.2. *There is an integer-valued function $f_{9.2}(n,q,k)$ so that, for any prime power q, and integers $k \ge 0$, $n > k+1$, if $m \ge f_{9.2}(n,q,k)$ is an integer, and $M \in \mathrm{EX}(U_{2,q^2+q+1})$ is a (q,k)-overfull, weakly round matroid such that*

- *M has a $\mathrm{PG}(m-1,q)$-minor, and*
- *M has a rank-m set of (q,k)-critical elements,*

then M has a rank-n, $(q,k+1)$-full minor with a U_{2,q^2+1}-restriction.

Proof. Let q be a prime power, and $k \ge 0$ and $n \ge 2$ be integers. Let $n' = \max(k+8, n+k+1)$, let $d = f_{5.2}(q,k)$, let $t = d(d+1)+k+6$, let $s = \frac{q^{2k}-1}{q^2-1}+1$, and set $f_{9.2}(n,q,k) = f_{8.1}(n', q, t(s+1), q^2+q-1)$.

Let $m \ge f_{9.2}(n,q,k)$ be an integer, and let $M \in \mathrm{EX}(U_{2,q^2+q+1})$ be a (q,k)-overfull, weakly round matroid with a $\mathrm{PG}(m-1,q)$-minor and a t-element independent set I of (q,k)-critical elements (note that $t \le m$). We will show that M has the required minor.

By Lemma 9.1, for each element $e \in I$, there is a set $\mathcal{L}_e$ of lines containing e such that either $|\mathcal{L}_e| = 1$ and the single line in $\mathcal{L}_e$ has q^2+2 points, or $|\mathcal{L}_e| = \frac{q^{2k}-1}{q^2-1}+1$ and each line in $\mathcal{L}_e$ has at least $q+2$ points. There is a restriction K of M with rank at most $t(s+1)$ that contains all the lines $(\mathcal{L}_e : e \in I)$. By Lemma 8.1, M has a minor M_1 of rank at least n' that has a $\mathrm{PG}(r(M_1)-1,q)$-restriction R_1, and has K as a restriction. By Lemma 4.1, M_1 has at most one line containing q^2+2 points.

9.2.1. *There is a $(t-5)$-element subset I_1 of I such that, for each $e \in I_1$, we have $r_K(\cup\mathcal{L}_e) \ge k+2$.*

Proof of claim: Note that $|I| = t \ge 5$. If $k=0$, then every $e \in I$ satisfies the required condition, so an arbitrary $(t-5)$-subset of I will do; we may thus assume that $k \geq 1$. Since K contains at most one line with at least $q^2 + 2$ points, there are at most two elements $e \in I$ with $|\mathcal{L}_e| = 1$. If the claim fails, there is therefore an 4-element subset I_2 of I such that $|\mathcal{L}_e| = \frac{q^{2k}-1}{q^2-1} + 1$ and $r_K(\cup\mathcal{L}_e) \leq k+1$ for all $e \in I_2$.

For each $e \in I_2$, let $F_e = \mathrm{cl}_K(\cup\mathcal{L}_e)$. Then $(K|F_e)/e$ has rank at most k and has more than $\frac{q^{2k}-1}{q^2-1}$ points. Since $k \geq 1$, this matroid has rank at least 2. Moreover, M_1/e has rank at least $n' - 1 \geq k + 7$ and has a $\mathrm{PG}(r(M_1/e) - 1, q)$-restriction, so, by Lemma 4.2, $r((K|F_e)/e) = 2$. Hence, $k \geq 2$, F_e is a rank-3 set containing at least q^2+2 lines through e, each with at least $q+2$ points, and $(K|F_e)/e$ is a rank-2 set containing at least $q^2 + 2$ points.

Let $a \in I_2$; since $r_{M_1}(I_2) = 4 > r_{M_1}(F_a)$, there is some $b \in I_2 - F_a$. Now, M_1/b has a line $L = \mathrm{cl}_{M_1/b}(F_b - \{b\})$ containing at least $q^2 + 2$ points, and $(M_1/b)|F_a$ is a rank-3 matroid with at least $1+(q+1)(q^2+2)$ points, and therefore at least $1 + (q+1)(q^2+2) - (q^2+q) > q^2+q+1$ points outside L. However, M_1/b has rank at least $k + 7$, and has a $\mathrm{PG}(r(M_1/b) - 1, q)$-restriction containing at most $q^2 + q + 1$ points in $F_a - L$, so we obtain a contradiction to Lemma 4.1. □

9.2.2. *M_1 has an R_1-unstable set of size $k + 1$.*

Proof of claim. Suppose otherwise. By Lemma 5.2, there is a flat F of R_1 with rank at most k such that $\varepsilon(M_1/F) \leq \varepsilon(R_1/F) + f_{5.2}(q, k) = \varepsilon(R_1/F)+d$. Let $M_2 = M_1/F$; the matroid M_2 has a $\mathrm{PG}(r(M_2)-1, q)$-restriction R_2, and satisfies $E(M_2) = E(R_2) \cup D$, where $|D| \leq d$.

Let $I_2 \subseteq I_1$ be a set of size of size $|I_1| - k$ that is independent in M_2; note that $|I_2| \geq d(d + 1) + 1$. For each $e \in I_2$, we have $r_{M_2}(\cup\mathcal{L}_e) \geq (k+2) - k = 2$, so e is contained in a line L_e with at least $q+2$ points in M_2.

Let $\mathcal{L} = \{L_e : e \in I_2\}$. Each L_e contains e, and at most one other point in I_2, so $|\mathcal{L}| \geq \frac{1}{2}|I_2| > \binom{d+1}{2}$. Each line in $\mathcal{L}$ contains $q+2$ points, so must contain a point of $M_2 \backslash E(R_2)$. However, $|M_2 \backslash E(R_2)| \leq d$, so there are at most $\binom{d}{2}$ lines of M_2 containing two points of $M_2 \backslash E(R_2)$, and by Lemma 3.2, we may assume that there are at most d lines of M_2 containing $q+2$ points, but just one point of $M_2 \backslash E(R_2)$. This gives $|\mathcal{L}| \leq d + \binom{d}{2} = \binom{d+1}{2}$, a contradiction. □

Since $r(M_1) \geq n' \geq n+k+1$, we get the required minor N from the above claim and Lemma 5.3. □

10. The main theorems

The following result implies Theorems 1.2 and 1.3:

Theorem 10.1. *Let q be a prime power, and let $\mathcal{M} \subseteq \mathrm{EX}(U_{2,q^2+q+1})$ be a base-q exponentially dense minor-closed class of matroids. There is an integer $k \geq 0$ such that*

$$h_{\mathcal{M}}(n) = \frac{q^{n+k}-1}{q-1} - q\frac{q^{2k}-1}{q^2-1}$$

for all sufficiently large n. Moreover, if $\mathcal{M} \subseteq \mathrm{EX}(U_{2,q^2+1})$, then $k = 0$.

Proof. By the Growth Rate Theorem, $\mathcal{M}$ contains all projective geometries over $\mathrm{GF}(q)$ and, hence, $\mathcal{M}$ contains $(q,0)$-full matroids of every rank. We may assume that there are $(q,0)$-overfull matroids of arbitrarily large rank, since otherwise the theorem holds. By the Growth Rate Theorem, there is a maximum integer $k \geq 0$ such that $\mathcal{M}$ contains (q,k)-overfull matroids of arbitrarily large rank, and there is an integer $s \geq 0$ such that $\mathrm{PG}(s-1,q') \notin \mathcal{M}$ for all $q' > q$.

To prove the result, it suffices to show that, for all $n > k+1$, there is a rank-n matroid $M \in \mathcal{M}$ that is $(q,k+1)$-full and has a U_{2,q^2+1}-restriction. Suppose for a contradiction that $n > k+1$ is an integer for which this M does not exist.

Let $m = f_{9.2}(n,q,k)$, and $m_4 = \max(m+1, s, f_{6.1}(n,q,k))$. Let m_3 be an integer such that

$$\frac{q^{m_3}-1}{q-1} > \alpha_{2.2}(m_4, q-\tfrac{1}{2}, q^2+q-1)\left(\frac{q^2+q-1}{q-\frac{3}{2}}\right)^m (q-\tfrac{1}{2})^{m_3+m-1}.$$

Let $m_2 = \max(s, m_3 m)$, and choose an integer $m_1 > s$ such that

$$\alpha_{2.2}(m_2, q-\tfrac{1}{2}, q^2+q-1)(q-\tfrac{1}{2})^r \leq \frac{q^{r+k}-1}{q-1} - q\frac{q^{2k}-1}{q^2-1}$$

for all $r \geq m_1$. By Lemma 7.3, $\mathcal{M}$ contains weakly round, (q,k)-overfull matroids of arbitrarily large rank; let $M_1 \in \mathcal{M}$ be a weakly round, (q,k)-overfull matroid with rank at least m_1. By Lemma 2.2, M_1 has a $\mathrm{PG}(m_2-1,q')$ minor N_1 for some $q' > q-\frac{1}{2}$; since $m_2 \geq s$, we have $q' = q$. Let I_1 be an independent set of M_1 such that N_1 is a spanning restriction of M_1/I_1, and choose $J_1 \subseteq I_1$ maximal such that M_1/J_1 is (q,k)-overfull.

Let $M_2 = M_1/J_1$ and let $I_2 = I_1 - J_1$. By our choice of J_1, each element in I_2 is (q,k)-critical in M_2. Since $m_2 \geq m$, Lemma 9.2 gives $|I_2| < m$. Choose a collection $(F_1, \ldots, F_m)$ of mutually skew rank-m_3 flats in the projective geometry N_1; each F_i satisfies $r(M_2|F_i) \leq m_3 + m - 1$ and $\varepsilon(M_2|F_i) = \frac{q^{m_3}-1}{q-1}$. By our choice of m_3, and by Lemma 2.3 with $\mu = q-\frac{1}{2}$ for each $i \in \{1,\ldots,m\}$, there is a flat $F_i' \subseteq F_i$ of M_2 that is skew to I_2 in M_2, and satisfies $\varepsilon(M_2|F_i') \geq \alpha_{2.2}(m_4, q-\frac{1}{2}, q^2+q-1)(q-\frac{1}{2})^{r_{M_2}(F_i')}$. Note that, since the sets $(F_1', \ldots, F_m')$ are mutually skew in M_2/I_2 and each of these sets is skew to I_2 in M_2, the flats $(F'_1, \ldots, F'_m)$ are mutually skew in M_2.

By Lemma 2.2, $M_2|F'_i$ has a $\mathrm{PG}(m_4 - 1, q')$ minor P_i for some $q' > q - \frac{1}{2}$; since $m_4 \geq s$, we have $q' = q$. Let X_i be an independent set of $M_2|F'_2$ such that P_i is a spanning restriction of M_2/X_i. Now choose $Z \subseteq X_1 \cup \cdots \cup X_m$ maximal such that M_2/Z is (q, k)-overfull. Let $M_3 = M_2/Z$. Each element of $X_1 \cup \cdots \cup X_s - Z$ is (q, k)-critical in M_3, and P_i is a minor of M_3 for each i. The X_i are mutually skew in M_3 and hence pairwise disjoint; thus, by Lemma 9.2, there exists $i_0 \in \{1, \ldots, m\}$ such that $X_{i_0} - Z = \varnothing$ and, hence, P_{i_0} is a restriction of M_3; let $R = P_{i_0}$.

Choose a minor M_4 of M_3 that is minimal such that:

- M_4 is weakly round, and (q, k)-overfull,
- M_4 has R as a restriction.

By Lemma 6.1, $r(M_4) > r(R)$. Every element of $E(M_4) - \mathrm{cl}_{M_4}(E(R))$ is (q, k)-critical and, since M_4 is weakly round, $r(M_4 \backslash \mathrm{cl}_{M_4}(E(R))) \geq r(M_4) - 2 \geq m_4 - 1 \geq m$. We now get a contradiction from Lemma 9.2. □

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