BANACH SPACE TENSOR PRODUCTS AND NUCLEAR OPERATORS

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Abstract. We begin by defining tensor products as certain linear functionals on spaces of bilinear forms and by proving several elementary properties about them. We recall some basic results from Banach space theory, then proceed to define the projective tensor norm on the tensor product of two Banach spaces and to calculate an example of it. After that, we prove a useful duality theorem that gives another characterization of the projective norm, which is often easier to work with. Finally, we define nuclear operators and discuss the approximation property.

1. Notation

Our notation is almost identical to that of [1], but in the interest of keeping this article relatively self-contained, I go over the notation again. We use X, Y, Z to denote Banach spaces over the fields $\mathbb{R}$ or $\mathbb{C}$. We denote the closed unit ball of a Banach space X by B_X. We use the word *operator* to mean a bounded linear map, and we denote the space of operators from X to Y by $\mathcal{L}(X, Y)$.

We denote the *continuous dual space* (or *dual space* for short) of X by X^*. This is the space of continuous linear functionals on X, and is a subspace of the *algebraic dual space*, which is the space of all linear functionals on X and which we denote $X^\#$.

Given two Banach spaces X, Y, we denote the space of bounded bilinear forms on the product $X \times Y$ by $\mathcal{B}(X, Y)$ and abbreviate this to $B(X)$ when $X = Y$. $\mathcal{B}(X, Y)$ comes equipped with the usual norm $\|B\| = \sup\{|B(x, y)| : x \in B_X, y \in B_Y\}$.

Familiarity with standard examples of Banach spaces, particularly $c_0(X)$ and $\ell^p(X)$ for $1 \leq p \leq \infty$, is assumed.

2. Preliminaries

In what follows, we define the tensor product of vector spaces V and W as a certain subspace of $B(X, Y)^\#$. This approach is convenient because it works in both the finite- and infinite-dimensional settings.

Definition 2.1. Given vector spaces V, W and elements $v \in V, w \in W$, $v \otimes w \in B(X, Y)^\#$ is the unique linear functional given by evaluation at (v, w), i.e., $(v \otimes w)(A) = A(v, w)$ for every $A \in B(X, Y)$. $V \otimes W$ is the subspace of $B(X, Y)^\#$ spanned by elements of the form $v \otimes w$.

Remark 2.2. Even for finite-dimensional vector spaces, a given representation $u = \sum_{i=1}^n v_1 \otimes w_1$ of $u \in V \otimes W$ is not unique. For example, taking $\mathbb{C}^3$ with standard basis $\{e_1, e_2, e_3\}$, we have

$(e_1+e_2)\otimes(e_1-e_2)+(e_2+e_3)\otimes(e_2-e_3)+(e_3+e_1)\otimes(e_3-e_1) = e_1\otimes(e_3-e_2)+e_2\otimes(e_1-e_3)+e_3\otimes(e_2-e_1)$.

This shows that there is not even a unique representation consisting of exactly three terms in this case.

The following two propositions will be useful to us in the following section, but are also important in their own right.

Proposition 2.3. *Let X, Y be vector spaces. We have the following.*
(a) If E and F are linearly independent subsets of X and Y respectively, then $\{x \otimes y | x \in E, y \in F\}$ is a linearly independent subset of $X \otimes Y$.
(b) If $\{e_i | i \in I\}$ and $\{f_j | j \in J\}$ are bases for X and Y respectively, then $\{e_i \otimes f_j | i \in I$ and $j \in J\}$ is a basis for $X \otimes Y$.

Proof. We prove (a); (b) clearly follows. Let $u = \sum_{i=1}^n \lambda_i x_i \otimes y_i = 0$, where $x_i \in E$, $y_i \in F$ for all i. Suppose φ, ψ are linear functionals on X and Y respectively, and let A be the bilinear form given by $A(x, y) := \varphi(x)\psi(y)$. Then, since $u(A) = 0$,

$$\sum_{i=1}^n \lambda_i \varphi(x_i)\psi(y_i) = \psi(\sum_{i=1}^n \lambda_i \varphi(x_i) y_i) = 0.$$

As ψ was arbitrary, we conclude that $\sum_{i=1}^n \lambda_i \varphi(x_i) y_i = 0$, so because F is a linearly independent set, we have $\lambda_i \varphi(x_i) = 0$ for every $\varphi \in X^\#$. However, since E is a linearly independent set as well, each x_i is nonzero so $\lambda_i = 0$ for all i. Therefore, $\{x \otimes y | x \in E, y \in F\}$ is also linearly independent. □ □

Proposition 2.4. *(We retain the notation from the previous proposition.) If*

$$u = \sum_{i=1}^n x_i \otimes y_i \in X \otimes Y,$$

then the following are equivalent:
(i) $u = 0$;
(ii) $\sum_{i=1}^n \varphi(x_i)\psi(y_i) = 0$ for every $\varphi \in X^\#, \psi \in Y^\#$;
(iii) $\sum_{i=1}^n \varphi(x_i) y_i = 0$ for every $\varphi \in X^\#$;
(iv) $\sum_{i=1}^n \psi(y_i) x_i = 0$ for every $\psi \in Y^\#$.

Proof. (i) $\implies$ (ii): Simply define the linear functional $A(x, y) := \varphi(x)\psi(y)$, and this follows as in the proof of the previous proposition since $u(A) = 0$.

(ii) $\implies$ (iii): Write

$$\sum_{i=1}^n \varphi(x_i)\psi(y_i) = \psi(\sum_{i=1}^n \varphi(x_i) y_i)$$

and the result follows.

(iii) $\implies$ (iv): Apply ψ to both sides of the equality in (iii) to get

$$\sum_{i=1}^n \varphi(x_i)\psi(y_i) = 0 = \varphi(\sum_{i=1}^n \psi(y_i) x_i),$$

and the result follows.

(iv) $\implies$ (i): Let $A \in B(X \times Y)$ be arbitrary. Let E, F be the subspaces of X, Y spanned by $\{x_1, ..., x_n\}$ and $\{y_1, ..., y_n\}$ respectively. Let $B := A|_{E\times F}$. As E, F are finite-dimensional, we may choose bases for them and expand B relative to these bases to obtain a representation

$$B(x,y) = \sum_{j=1}^{m} \theta_j(x)\omega_j(y),$$

where $\theta_j \in E^\#, \omega_j \in F^\#$. We may choose algebraic complements G, H for E, F respectively and set $\theta_j|_G := 0, \omega_j|_H := 0$; in this way we extend θ_j and ω_j to X and Y respectively. Now B can be thought of as a bilinear form on $X \times Y$. As A and B agree on $E \times F$, we have

$$u(A) = \sum_{i=1}^{n} A(x_i, y_i) = \sum_{i=1}^{n} B(x_i, y_i)$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{M} \theta_j(x_i)\omega_j(y_i) = \sum_{j=1}^{m} \theta_j(\sum_{i=1}^{n} \omega_j(y_i)x_i) = 0,$$

by (iv). Therefore, $u(A) = 0$ for every $A \in B(X \times Y)$. □

Definition 2.5. A subset S of the algebraic dual $X^\#$ of a Banach space X is called *separating* if for any $x, y \in X$ such that $x \neq y$, there exists $\varphi \in S$ such that $\varphi(x) \neq \varphi(y)$.

Remark 2.6. It is clear that instead of considering $\varphi \in X^\#, \psi \in Y^\#$ in Proposition 2.4, we may take $\varphi \in X_1^\#, \psi \in Y_1^\#$ where $X_1^\# \subseteq X^\#$, $Y_1^\# \subseteq Y^\#$ are separating subsets. The proof of Proposition 2.4 remains essentially unchanged because the following statement still holds, carried over from $X^\#$ to $X_1^\#$ and from $Y^\#$ to $Y_1^\#$: if $f(u) = 0$ for all $f \in X_1^\#, u \in X$, then $u = 0$. This fact is used several times and is the only part of the proof that directly relies on the structure of our spaces of functionals.

The next two propositions are well known. The proof of the first will be omitted as I believe we have used it earlier in the course, but I will prove the second.

Proposition 2.7. *Given Banach spaces X, Y, Z and a bilinear map $B : X \times Y \to Z$, there exists a unique, well-defined linear map $\tilde{A} : X \otimes Y \to Z$ such that $A(x,y) = \tilde{A}(x \otimes Y)$ for all $x \in X, y \in Y$. This linear map is called the linearization of B.*

Proposition 2.8. *Given a bounded linear transformation T from a normed vector space V to a a Banach space X, T can be uniquely extended to a bounded linear transformation $\tilde{T}$ from the completion $\tilde{V}$ of V to X.*

Proof. Suppose (v_n) is a Cauchy sequence in V. Then $\|Tv_n - Tv_m\| \leq \|T\|\|v_n - v_m\|$ and $\|T\| < \infty$ as T is bounded, so (Tv_n) is Cauchy, and by completeness of X it converges. Define $\tilde{T}v := \lim_n Tv_n$ where (v_n) is an arbitrary Cauchy sequence converging to $v \in \tilde{V}$.

$\tilde{T}$ is well-defined because if $v_n^{'} \to v$ as well, then $\|Tv_n - Tv_n^{'}\| \leq \|T\|\|v_n - v_n^{'}\| \to 0$ as $n \to \infty$, so $\lim_n Tv_n = \lim_n Tv_n^{'} = \tilde{T}v$. Taking constant sequences shows that $\tilde{T}|_V = T$. Also,

$$\tilde{T}v = \lim_n Tv_n \leq \|T\| \lim_n \|v_n\| = \|T\| \lim_n \|v_n\|,$$

by continuity of the norm,

$$= \|T\|\|v\|,$$

so $\tilde{T}$ is bounded. Finally, if S is another bounded linear extension of T to $\tilde{V}$, then $\tilde{T}v_n = Sv_n$ for all n, so as $\tilde{T} - S$ is bounded and linear, and therefore continuous, $(\tilde{T} - S)v = 0$ for any $v \in \tilde{V}$, which establishes uniqueness. $\square$

Suppose X, Y are Banach spaces. If we want to make $X \otimes Y$ into a Banach space, we will need to impose a norm on it. But given that the representation of a given element in $X \otimes Y$ is not unique, as described in Remark 2.2, we must define the norm on $X \otimes Y$ in a way that is independent of the representation chosen for any element of $X \otimes Y$. We now present one way of doing this.

3. The Projective Tensor Norm and Duality

3.1. **The Projective Tensor Norm.** We impose an additional algebraic condition to simplify the problem of determining a suitable norm on $X \otimes Y$. If $x \in X, y \in Y$, we impose the condition

$$\|x \otimes y\| \leq \|x\|\|y\|.$$

Let $u \in X \otimes Y$ so that $u = \sum_{i=1}^{n} x_i \otimes y_i$ for some $x_i \in X, y_i \in Y$. (We can represent u in this way because if an element of the form $\lambda x_i \otimes y_i$ appears in the sum, we may rewrite x_i as λx_i, which is still in X.) Then, as $\|\cdot\|$ is assumed to be a norm, we have by the triangle inequality that

$$\|u\| \leq \inf\{\sum_{i=1}^{n} \|x_i\|\|y_i\|\}.$$

This motivates our next definition.

Definition 3.1. Given Banach spaces X and Y, the *projective norm* on $X \otimes Y$ is given by

$$\pi(u) = \inf\{\sum_{i=1}^{n} \|x_i\|\|y_i\| : u = \sum_{i=1}^{n} x_i \otimes y_i\},$$

where the infimum is taken over all possible representations of u.

Proposition 3.2. *π is a norm on $X \otimes Y$ and $\pi(x \otimes y) = \|x\|\|y\|$ for every $x \in X, y \in Y$.*

Proof. Let λ be a scalar. We will first show that $\pi(\lambda u) = |\lambda|\pi(u)$. This clearly holds when $\lambda = 0$, so suppose $\lambda \neq 0$. Suppose also that $u = \sum_{i=1}^{n} x_i \otimes y_i$ is a representation of u. Then

$$\pi(\lambda u) \leq \sum_{i=1}^{n} \|\lambda x_i\|\|y_i\|$$

$$= |\lambda| \sum_{i=1}^{n} \|x_i\|\|y_i\|,$$

and since this holds for every representation of u, $\pi(\lambda u) \leq |\lambda|\pi(u)$. For the same reason, $\pi(u) = \pi(\lambda^{-1}\lambda u) \leq |\lambda|^{-1}\pi(\lambda u)$, so $|\lambda|\pi(u) \leq \pi(\lambda u)$. Therefore, $\pi(\lambda u) = |\lambda|\pi(u)$.

Next, let $u, v \in X \otimes Y$ and $\epsilon > 0$. We will show that π satisfies the triangle inequality. From the definition of π, we may choose representations $u = \sum_{i=1}^{n} x_i \otimes y_i$ and $v = \sum_{j=1}^{m} w_j \otimes z_j$

such that
$$\sum_{i=1}^{n}\|x_i\|\|y_i\| \le \pi(u) + \epsilon/2 \text{ and } \sum_{j=1}^{m}\|w_j\|\|z_j\| \le \pi(v) + \epsilon/2.$$
As $\sum_{i=1}^{m}(x_i \otimes y_i) + \sum_{j=1}^{m}(w_j \otimes z_j)$ represents $u + v$, we have
$$\pi(u+v) \le \sum_{i=1}^{n}\|x_i\|\|y_i\| + \sum_{j=1}^{m}\|w_j\|\|z_j\| \le \pi(u) + \pi(v) + \epsilon.$$
As $\epsilon > 0$ was arbitrary, $\pi(u+v) \le \pi(u) + \pi(v)$ as desired.

Finally, we show that $\pi(u) = 0$ implies $u = 0$. Suppose $\pi(u) = 0$. Then for every $\epsilon > 0$ there is a representation $u = \sum_{i=1}^{n} x_i \otimes y_i$ such that
$$\sum_{i=1}^{n}\|x_i\|\|y_i\| \le \epsilon.$$
It follows that for every $\varphi \in X^*, \psi \in Y^*$,
$$|\sum_{i=1}^{n} \varphi(x_i)\psi(y_i)| \le \epsilon\|\varphi\|\|\psi\|$$
by definition of $\|\varphi\|, \|\psi\|$.

The value of $\sum_{i=1}^{n} \varphi(x_i)\psi(y_i)$ is independent of the representation of u, so we have
$$\sum_{i=1}^{n} \varphi(x_i)\psi(y_i) = 0.$$
By Remark 2.6 and the fact that $X^* \subset X^{\#}, Y^* \subset Y^{\#}$ are separating subsets, we may apply Proposition 1.2 to obtain $u = 0$.

Finally, we will show that $\pi(x \otimes y) = \|x\|\|y\|$. We have $\pi(x \otimes y) \le \|x\|\|y\|$ by definition of π. To prove the other direction, let $\varphi \in B_{X^*}, \psi \in B_{Y^*}$ satisfy $\varphi(x) := \|x\|$ and $\psi(y) := \|y\|$. Let $B(w,z) := \varphi(w)\psi(z)$, which is a bounded bilinear form on $X \times Y$. Let $\tilde{B}$ denote the linearization of B, so that
$$|\tilde{B}(\sum_{i=1}^{n} x_i \otimes y_i)| \le \sum_{i=1}^{n}|\tilde{B}(x_i \otimes y_i)| = \sum_{i=1}^{n}|\varphi(x_i)\psi(y_i)| \le \sum_{i=1}^{n}\|x_i\|\|y_i\|.$$
Then $|\tilde{B}(u)| \le \pi(u)$ for every $u \in X \otimes Y$, so $\|x\|\|y\| = \tilde{B}(x \otimes y) \le \pi(x \otimes y)$, and we are done. □

Definition 3.3. Let $X \otimes_\pi Y := (X \otimes Y, \pi)$ be the space $X \otimes Y$ equipped with the norm π. This space is not complete, so let $X \hat{\otimes}_\pi Y$ denote its completion. We call this completion the *projective tensor product* of X and Y.

Remark 3.4. Recall that if X, W, Y, Z are Banach spaces and $S \in \mathcal{L}(X,W)$, $T \in \mathcal{L}(Y,Z)$ are operators, then there is a unique linear map $S \otimes T : X \otimes Y \to W \otimes Z$ satisfying $(S \otimes T)(x \otimes y) = (Sx) \otimes (Ty)$ for every $x \in X, y \in Y$. If $u \in X \otimes Y$ and $u = \sum_{i=1}^{n} x_i \otimes y_i$, then

$$\pi((S\otimes T)(u)) = \pi(\sum_{i=1}^{n}(Sx_i)\otimes(Ty_i))$$
$$\leq \sum_{i=1}^{n}\|Sx_i\|\|Ty_i\| \leq \|S\|\|T\|\sum_{i=1}^{n}\|x_i\|\|y_i\|,$$

so $\pi((S\otimes T)u) \leq \|S\|\|T\|\pi(u)$. Hence, when we take the projective norms on $X\otimes Y$ and $W\otimes Z$, $S\otimes T$ is an operator and $\|S\otimes T\|\leq\|S\|\|T\|$. However, taking norms of both sides of the equation $(S\otimes T)(x\otimes y) = (Sx)\otimes(Ty)$, we obtain the other inequality $\|S\otimes T\|\geq\|S\|\|T\|$. Therefore, $\|S\otimes T\| = \|S\|\|T\|$. Finally, by Proposition 2.8 there exists a unique bounded linear extension of $S\otimes T$ to $X\hat{\otimes}_\pi Y$ and $W\hat{\otimes}_\pi Z$. Denote this operator by $S\otimes_\pi T$. We summarize this procedure in the following result.

Proposition 3.5. *Let $S : X\to W$ and $T : Y\to Z$ be operators. Then there exists a unique operator $S\otimes_\pi T : X\hat{\otimes}_\pi Y\to W\otimes_\pi Z$ such that $(S\otimes_\pi T)(x\otimes y) = (Sx)\otimes(Ty)$ for every $x\in X, y\in Y$. Moreover, $\|S\otimes_\pi T\| = \|S\|\|T\|$.*

Example 3.6. We will construct the space $\ell_1\hat{\otimes}_\pi X$ for an arbitrary Banach space X. Given $a = (a_n)\in\ell^1$, $x\in X$, we associate the elementary tensor $a\otimes x$ with the sequence $(a_nx)\in\ell(X)$. Notice that

$$\sum_{n=1}^{\infty}\|a_nx\| \leq (\sum_{n=1}^{\infty}|a_n|)\|x\| < \infty.$$

We equip $\ell^1(X)$ with the usual norm

$$\|(x_n)\|_1 = \sum_{n=1}^{\infty}\|x_n\|.$$

Then we may extend the construction above linearly to all of $\ell^1\otimes X$ to obtain a map $J : \ell^1\otimes X\to\ell^1(X)$ given on elementary tensors by $J(a\otimes x) = (a_nx)$. Now, if $u = \sum_{i=1}^m a_i\otimes x_i\in\ell^1\otimes X$, where $a_i = (a_{i,n})_n$ for each i, then

$$\|J(u)\|_1 = \|(\sum_{i=1}^{m}a_{i,n}x_i)_n\|_1 = \sum_{n=1}^{\infty}\|(\sum_{i=1}^{m}a_{i,n}x_i)_n\|$$
$$\leq\sum_{n=1}^{\infty}\sum_{i=1}^{m}\|a_{i,n}x_i\| = \sum_{i=1}^{m}(\sum_{n=1}^{\infty}|a_{i,n}|)\|x_i\| = \sum_{i=1}^{m}\|a_i\|\|x_i\|,$$

and, since we chose an arbitrary representation of u, we have $\|J(u)\|_1\leq\pi(u)$.
Next, we will prove that $\|J(u)\|_1\geq\pi(u)$. As before, let $u = \sum_{i=1}^m a_i\otimes x_i$. Then $J(u) = (u_n)$, where $u_n = \sum_{i=1}^m a_{i,n}x_i$. We claim $\sum_{n=1}^\infty e_n\otimes u_n\to u$ in $\ell^1\otimes_\pi X$; here (e_n) is the standard basis for ℓ^1. Let Π_k be the projection onto the first k coordinates, so that $\Pi_k(a) = \sum_{n=1}^k a_ne_n\to a$ as $k\to\infty$. Then

$$\pi(u-\sum_{n=1}^{k}e_n\otimes u_n) = \pi(\sum_{i=1}^{m}a_i\otimes x_i - \sum_{n=1}^{k}\sum_{i=1}^{m}e_n\otimes a_{i,n}x_i)$$
$$= \pi(\sum_{i=1}^{m}(a_i\otimes x_i - \sum_{n=1}^{k}a_{i,n}e_n\otimes x_i)) = \pi(\sum_{i=1}^{m}(a_i-\Pi_ka_i)\otimes x_i)$$

$$\leq \sum_{i=1}^{m} \|a_i - \Pi_k a_i\| \|x_i\|.$$

Hence, $\pi(u - \sum_{n=1}^{k} e_n \otimes u_n)$ as $k \to \infty$ and

$$\pi(u) = \pi(\sum_{n=1}^{\infty} e_n \otimes u_n) \leq \sum_{n=1}^{\infty} \|u_n\| = \|J(u)\|_1,$$

as desired. This proves that J is an isometry. As $\ell^1(X)$ is complete, J extends to a unique isometry from $\ell^1 \hat{\otimes}_\pi X$ into $\ell^1(X)$. Indeed, I claim this extension is surjective. Given a sequence $(x_n) \in \ell^1(X)$, it suffices to show that $\sum_{n=1}^{\infty} e_n \otimes x_n$ converges in $\ell^1 \hat{\otimes}_\pi X$. (Recall that $u = \sum_{i=1}^{m} a_i \otimes x_i$.) It is then clear that J will map this series to u. As $\ell^1 \hat{\otimes}_\pi X$ is a Banach space, it is enough to show the series is Cauchy, that is that its tail can be made arbitrarily small. But we have

$$\pi(\sum_{n=j}^{k} e_n \otimes x_n) \leq \sum_{n=j}^{k} \|x_n\|,$$

so this holds. Therefore, $\ell^1 \hat{\otimes}_\pi X$ and $\ell^1(X)$ are isometrically isomorphic, so we may identify them.

3.2. **Duality.** The following theorem will give us a simpler characterization of the projective norm.

Theorem 3.7. *Given a bounded bilinear map $B : X \times Y \to Z$, there exists a unique operator $\tilde{B} : X \hat{\otimes}_\pi Y \to Z$ such that $\tilde{B}(x \otimes y) = B(x, y)$ for every $x \in X, y \in Y$. The correspondence between B and $\tilde{B}$ is an isometric isomorphism between Banach spaces $\mathcal{B}(X \times Y, Z)$ and $\mathcal{L}(X \hat{\otimes}_\pi Y, Z)$.*

Proof. By the universal property of tensor products, there exists a unique linear map $\tilde{B} : X \otimes Y \to Z$ such that $\tilde{B}(x \otimes y) = B(x, y)$ for every x, y. Let us show $\tilde{B}$ is bounded under the projective norm. Let $u = \sum_{i=1}^{n} x_i \otimes y_i \in X \otimes Y$. Then

$$\|\tilde{B}(u)\| = \|\sum_{i=1}^{n} B(x_i, y_i)\| \leq \|B\| \sum_{i=1}^{n} \|x_i\| \|y_i\|.$$

As the representation of u was arbitrary, it follows that $\|\tilde{B}(u)\| \leq \|B\|\pi(u)$. Thus $\tilde{B}$ is bounded and $\|\tilde{B}\| \leq \|B\|$. Also, $\|B(x, y)\| = \|\tilde{B}(x \otimes y)\| \leq \|\tilde{B}\| \|x\| \|y\|$ so $\|B\| \leq \|\tilde{B}\|$. Therefore, $\tilde{B} : X \otimes_\pi Y \to Z$ has a unique extension to $X \hat{\otimes}_\pi Y$ with the same norm; we we will also denote this map by $\tilde{B}$. It is clear the map $B \mapsto \tilde{B}$ is a linear isometry, so we just need to show it's surjective. If $L \in \mathcal{L}(X \hat{\otimes}_\pi Y, Z)$, then $B(x, y) := L(x \otimes y)$ is bounded bilinear and satisfies $\tilde{B} = L$. Therefore, we have an isometric isomorphism, so we may make the canonical identification

$$\mathcal{B}(X \times Y, Z) = \mathcal{L}(X \hat{\otimes}_\pi Y, Z).$$

□

Remark 3.8. Taking Z to be a field of scalars in the canonical identification above, we obtain $\mathcal{B}(X \times Y) = (X \hat{\otimes}_\pi Y)^*$. This means we may consider the action of a bounded bilinear form B as a bounded linear functional on $X\hat{\otimes}_\pi Y$ given by

$$\langle \sum_{i=1}^n x_i \otimes y_i, B\rangle = \sum_{i=1}^n B(x_i, y_i).$$

This immediately gives the formula

$$\pi(u) = \sup\{|\langle u, B\rangle| : B \in \mathcal{B}(X \times Y), \|B\| \leq 1\}.$$

We give an example that showcases the utility of this equivalent definition of π.

Example 3.9. We will calculate the tensor diagonal, D, of $\ell^2\hat{\otimes}_\pi\ell^2$, which is the closed subspace generated by the tensors $e_n \otimes e_n$, where (e_n) is the standard basis for ℓ^2.
Let $u \in D$ have the representation $u = \sum_{n=1}^k a_n e_n \otimes e_n$. Then $\pi(u) \leq \sum_{n=1}^k |a_n|$ by the usual definition of π. Now consider the bilinear form on $\ell^2 \times \ell^2$ given by

$$B(x, y) := \sum_{n=1}^k \mathrm{sgn}(a_n) x_n y_n,$$

where $\mathrm{sgn}(a)$ is a scalar of absolute value 1 such that $\mathrm{sgn}(a)a = |a|$. It's then clear that $\|B\| = 1$. Therefore, by our new formula for π,

$$\pi(u) \geq \langle u, B\rangle = \sum_{n=1}^k a_n B(e_n, e_n) = \sum_{n=1}^k |a_n|.$$

Therefore, $\pi(u) = \sum_{n=1}^k |a_n|$, so D is isometrically isomorphic to ℓ^1.

4. Nuclear Operators and the Approximation Property

4.1. **Nuclear Bilinear Forms.** Given a tensor of the form $u = \sum_{i=1}^n \varphi_i \otimes \psi_i \in X^\# \otimes Y^\#$, we can associate to it a bilinear form given by

$$B_u(x, y) = \sum_{i=1}^n \varphi_i(x)\psi_i(y).$$

In particular, elements of $X^* \otimes Y^*$ define bounded bilinear forms in this way, so we obtain an injective operator of unit norm from $X^* \otimes_\pi Y^*$ into $\mathcal{B}(X \times Y)$. Extending this to the completion gives an operator

$$J : X^*\hat{\otimes}_\pi Y^* \to \mathcal{B}(X \times Y)$$

of unit norm. This map is not necessarily surjective, so we have the following definition.

Definition 4.1. A bilinear form on $X \times Y$ is said to be *nuclear* if it lies in the range of J.

Thus a bilinear form B is nuclear if and only if there exist bounded sequences $(\varphi_n) \subset X^*, (\psi_n) \subset Y^*$ with $\sum_{n=1}^\infty \|\varphi_n\|\|\psi_n\| < \infty$ such that

$$B(x, y) = \sum_{n=1}^\infty \varphi_n(x)\psi_n(y)$$

for every x, y. This leads to another definition.

Definition 4.2. We call an expression of the form $\sum_{n=1}^{\infty} \varphi_n \otimes \psi_n$ a *nuclear representation* of B. We define the *nuclear norm* of B to be

$$\|B\|_N := \inf\{\sum_{n=1}^{\infty} \|\varphi_n\|\|\psi_n\| : B(x,y) = \sum_{n=1}^{\infty} \varphi_n(x)\psi_n(y)\},$$

where the infimum is taken over all nuclear representations of B.

That this is indeed a norm can be seen by analogy with the proof that π is a norm. Let $\mathcal{B}_N(X \times Y)$ denote the space of nuclear bilinear forms equipped with the nuclear norm. As $B_N(X \times Y) = \text{range } J$, it is a vector space, and it is also not difficult to see that it is a Banach space. Indeed, initially we might suspect $B_N(X \times Y)$ is simply $X^* \hat{\otimes}_\pi Y^*$, as it would be if the linear map $J : X^* \hat{\otimes}_\pi Y^*$ described earlier were injective.

However, J can fail to be injective for the following reason. Let $u \in X^* \hat{\otimes}_\pi Y^*$ be represented by $\sum_{n=1}^{\infty} \varphi_n \otimes \psi_n$. If the bilinear form B corresponding to u is 0, then $B(x,y) = \sum_{n=1}^{\infty} \varphi_n(x)\psi_n(y) = 0$ for all $x \in X, y \in Y$. But to deduce from this that $u = 0$, we must have $\langle u, A\rangle = \sum_{n=1}^{\infty} A(\varphi_n, \psi_n) = 0$ for every $A \in B(X^* \times Y^*) = (X^* \hat{\otimes}_\pi Y^*)^*$. These conditions are not equivalent in general.

Because the nuclear norm is the quotient norm of $X^* \hat{\otimes}_\pi Y^*$, we have

$$B_N(X \times Y) = X^* \hat{\otimes}_\pi Y^* / \ker J.$$

Here

$$\ker J = \{u \in X^* \hat{\otimes}_\pi Y^* | B_u(x,y) = 0 \ \forall x \in X, y \in Y\}.$$

We will return to this problem in the next subsection.

4.2. **Nuclear Operators.** We can define *nuclear operators* in a similar way to nuclear bilinear forms. We have an operator $J : X^* \hat{\otimes}_\pi Y \to \mathcal{L}(X,Y)$ that takes $u = \sum_{n=1}^{\infty} \varphi_n \otimes y_n$ as input and outputs the operator $L_u : X \to Y$ given by

$$L_u(x) = \sum_{n=1}^{\infty} \varphi_n(x) y_n.$$

Definition 4.3. A *nuclear operator* is an operator in range J. The *nuclear norm* of a nuclear operator T is given by

$$\|T\|_N := \inf\{\sum_{n=1}^{\infty} \|\varphi_n\|\|y_n\| : T(x) = \sum_{n=1}^{\infty} y_n\},$$

where the infimum is taken over all representations of T of the form $T(x) = \sum_{n=1}^{\infty} \varphi_n(x) y_n$, where $(\varphi_n) \subset X^*, (y_n) \subset Y^*$ are bounded sequences such that $\sum_{n=1}^{\infty} \|\varphi_n\|\|y_n\| < \infty$. The (Banach) space of nuclear operators is denoted $\mathcal{N}(X,Y)$ and we have

$$\mathcal{N}(X,Y) = X^* \hat{\otimes}_\pi Y^* / \ker J.$$

4.3. **The Approximation Property.** In our discussion of nuclear operators, the question arose of when, given two Banach spaces X, Y, we have the equality

$$\mathcal{N}(X,Y) = X^* \hat{\otimes}_\pi Y^*.$$

We saw that this reduced to the question of when, given $u \in X^*\hat{\otimes}_\pi Y^*$ with representation $\sum_{n=1}^\infty x_n \otimes y_n$, we have $u = 0$. For finite rank tensors we have that $\sum_{i=1}^n x_i \otimes y_i = 0$ if and only if $\sum_{i=1}^n \varphi(x_i)y_i = 0$ for every $\varphi \in X^*$, but it is not so clear in the case of the completed tensor product that the implication goes both ways. We do know that, in the case of $X^*\hat{\otimes}_\pi Y^*$, since the dual of $X\hat{\otimes}_\pi Y$ is $\mathcal{B}(X \times Y) = \mathcal{L}(X, Y^*)$ we have that $u = 0$ if and only if $\sum_{n=1}^\infty \langle y_n, Tx_n\rangle = 0$ for every $T \in \mathcal{L}(X, Y^*)$. For a finite-rank operator $T \in \mathcal{L}(X, Y^*)$, the map $\varphi(x_n) := \langle \cdot, Tx_n\rangle \in X^*$, so the condition

$$\sum_{n=1}^\infty \varphi(x_n)y_n = 0 \ \forall \varphi \in X^*$$

does imply the condition

$$\sum_{n=1}^\infty \langle y_n, Tx_n\rangle = 0 \ \forall T \in \mathcal{L}(X, Y^*).$$

But if X and Y are infinite-dimensional, we have operators that are not finite-rank. Thus we want to approximate infinite-rank operators by finite-rank ones. This motivates the following proposition.

Proposition 4.4. *For a Banach space X, the following are equivalent:*

(i) If $K \subseteq X$ is compact and $\epsilon > 0$, there exists a finite-rank operator $S : X \to X$ such that $\|x - Sx\| \leq \epsilon$ for every $x \in K$.

(ii) If Y is a Banach space, $T : X \to Y$ is an operator, $K \subseteq X$ is compact, and $\epsilon > 0$, then there exists a finite-rank operator $S : X \to Y$ such that $\|Tx - Sx\| \leq \epsilon$ for every $x \in K$.

(iii) If Y is a Banach space, $T : Y \to X$ is an operator, $K \subseteq Y$ is compact, and $\epsilon > 0$, then there exists a finite-rank operator $S : Y \to X$ such that $\|Ty - Sy\| \leq \epsilon$ for every $y \in K$.

Proof. First we will show that (i) implies (ii). Let $T : X \to Y$ be a nonzero operator, let $K \subseteq X$ be compact, and let $\epsilon > 0$. Then there exists a finite-rank operator $R : X \to X$ such that $\|x - Rx\| \leq \epsilon/\|T\|$ for every $x \in K$. Let $S := TR$. Then $S : X \to Y$ is a finite-rank operator and $\|Tx - Sx\| \leq \epsilon$ for every $x \in K$.

Now we will show that (i) implies (iii). Let $T : Y \to X$ be an operator, let $K \subseteq Y$ be compact, and let $\epsilon > 0$. We may apply (i) to $T(K)$, which is a compact subset of X, to obtain that there exists a finite-rank operator $R : X \to X$ such that $\|x - Rx\| \leq \epsilon$ for every $x \in T(K)$. Let $S := RT$. Then $S : Y \to X$ and $\|Ty - Sy\| \leq \epsilon$ for every $y \in K$.

(ii) clearly implies (i), so we are done. □

Definition 4.5. We say a Banach space X has the *approximation property* if it satisfies any of the conditions in Proposition 4.4.

Our concluding theorem shows that the approximation property is precisely what's required to make statements such as $\mathcal{N}(X, Y) = X^*\hat{\otimes}_\pi Y^*$.

Theorem 4.6. *If X is a Banach space, then the following are equivalent:*

(i) X has the approximation property.

(ii) If $u \in X^*\hat{\otimes}_\pi X$ is represented by $\sum_{n=1}^\infty \varphi_n \otimes x_n$, where $(\varphi_n) \subset X^*, (x_n) \subset X$ are bounded sequences with $\sum_{n=1}^\infty \|\varphi_n\|\|x_n\| < \infty$ and if $\sum_{n=1}^\infty \varphi_n(x)x_n = 0$ for all $x \in X$, then $u = 0$.

(iii) For every Banach space Y, if $u \in X\hat{\otimes}_\pi Y$ is represented by $\sum_{n=1}^\infty x_n \otimes y_n$, where $(x_n) \subset X, (y_n) \subset Y$ are bounded sequences with $\sum_{n=1}^\infty \|x_n\|\|y_n\| < \infty$ and if $\sum_{n=1}^\infty \varphi(x_n)y_n = 0$ for every $\varphi \in X^*$, then $u = 0$.

(iv) For every Banach space Y, if $u \in X\hat{\otimes}_\pi Y$ is represented by $\sum_{n=1}^\infty x_n \otimes y_n$, where $(x_n) \subset X, (y_n) \subset Y$ are bounded sequences with $\sum_{n=1}^\infty \|x_n\|\|y_n\| < \infty$ and if $\sum_{n=1}^\infty \psi(y_n)x_n = 0$ for every $\psi \in Y^*$, then $u = 0$.

Proof. First we show that (i) implies (iv). Let $u = \sum_{n=1}^\infty x_n \otimes y_n \in X\hat{\otimes}_\pi Y$ satisfy the conditions in the first two lines of (iv). We will prove that $u = 0$. Without loss of generality, assume that (x_n) and (y_n) are chosen so that $x_n \to 0$ and $\sum_{n=1}^\infty \|y_n\| < \infty$. Let $T : X \to Y^*$ be an operator and take $\epsilon > 0$. By assumption, X has the approximation property. Therefore, there exists a finite-rank operator $S : X \to Y^*$ such that $\|Tx_n - Sx_n\| \le \epsilon$ for every $n \in \mathbb{N}$. Then

$$Sx = \sum_{i=1}^m \varphi_i(x)\psi_i$$

for some $\varphi_i \in X^*, \psi_i \in Y^*$, so

$$\langle u, S\rangle = \sum_{n=1}^\infty \sum_{i=1}^m \varphi_i(x_n)\psi_i(y_n) = \sum_{i=1}^m \varphi_i(\sum_{n=1}^\infty \varphi_i(y_n)x_n) = 0.$$

Thus we have the bound

$$|\langle u, T\rangle| \le |\langle u, T - S\rangle| + |\langle u, S\rangle| \le \sum_{n=1}^\infty \|Tx_n - Sx_n\|\|y_n\| \le \epsilon \sum_{n=1}^\infty \|y_n\|,$$

so since ϵ was arbitrary, $\langle u, T\rangle = 0$. But T was also arbitrary, so $u = 0$.

Now we show that (iv) implies (iii). Suppose $u = \sum_{n=1}^\infty x_n \otimes y_n$ once again and that u satisfies the conditions in (iii). Then, if $\varphi \in X^*, \psi \in Y^*$ are arbitrary, we have

$$0 = \psi(\sum_{n=1}^\infty \varphi(x_n)y_n) = \sum_{n=1}^\infty \varphi(x_n)\psi(y_n) = \varphi(\sum_{n=1}^\infty \psi(y_n)x_n).$$

It follows that $\sum_{n=1}^\infty \psi(y_n)x_n = 0$, so since ψ was arbitrary we may apply (iv) to obtain $u = 0$.

Next we show that (ii) implies (i). Suppose for the sake of contradiction that X does not have the approximation property. Let $E = \mathcal{L}(X, X)$, which we know to have the topology of uniform convergence on compacta. This topology is generate by continuous seminorms of the form

$$p_K(T) = \sup\{\|Tx\| : x \in K, K \text{ compact}\}.$$

If F be the subspace of finite-rank operators in E, then stating that X does not have the approximation property is equivalent to stating that the identity operator, I, does not belong to the closure $\overline{F}$. Then, by the Hahn–Banach Separation Theorem (a good source for this theorem is [3] in the References), there exists a continuous linear functional $\Phi \in E^*$ such that $\Phi(T) = 0$ for every $T \in F$ and $\Phi(I) = 1$. There exists, by continuity of Φ, a compact subset $K \subset X$ such that for every $T \in E$, $|\Phi(T)| \leq \rho_K(T)$. Since K is compact, we may choose a sequence (x_n) such that $x_n \to 0$ and K is contained in the closed convex hull of (x_n). Then we have $|\Phi(T)| \leq \sup_n \|Tx_n\|$ for every $T \in E$. As $Tx_n \to 0$ as well, we may consider (Tx_n) as a subset of $c_0(X)$. Let Z be the subset of $c_0(X)$ consisting of all elements of this form as T ranges over E; then we may think of Φ as a bounded linear functional on Z. By the Hahn–Banach Separation Theorem again, we may extend this functional to all of $c_0(X)$. Since $c_0(X)$ is dual to $\ell^1(X^*)$, there exists $(\varphi_n) \in \ell^1(X^*)$ such that for every $T \in E$, $\Phi(T) = \sum_{n=1}^{\infty} \varphi(Tx_n)$. But this gives us

$$\Phi(I) = \sum_{n=1}^{\infty} \varphi_n(x_n) = 1,$$

even though $\Phi(T) = 0$ for every finite rank operator T and thus $\sum_{n=1}^{\infty} \varphi_n(x)x_n = 0$ for every $x \in X$. As this is a contradiction, we conclude that X must have the approximation property. $\square$

References

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[2] Reed, Michael and Simon, Barry. *Methods of Modern Mathematical Physics: I. Functional Analysis (Revised and Enlarged Edition)*. Academic Press, Inc. (1980). (The stated theorem, which appears here as Proposition 2.8, is on page 9.)

[3] Nagy, Gabriel. *Notes on Real Analysis (version 2.0)*. Kansas State University (2002). The reference for the Hahn–Banach Separation Theorem is at the following URL: https://www.math.ksu.edu/ nagy/real-an/ap-e-h-b.pdf.