University of Waterloo

Lecture Notes

Category Theory and Homological Algebra

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Contents

Abstract

This is a series of lecture notes for a class on category theory and homological algebra taught by Jason Bell.

1 Sept. 5, 2019.

Adam has eaten four bananas in the last 24 hours. Dan and Jason are smoothie brothers.

In this class, we will use the notion of a *class*, i.e., a collection of objects (usually sets) defined by some list of properties. We can't talk about the set of all sets because of Russell's paradox, so instead we talk about the class of all sets. Classes that are not sets are called *proper classes*. For example, the set of all groups is not a set, so we talk about the class of all groups. For similar reasons, we talk about the class of all vector spaces.

Definition 1.1. A *category* $\mathcal{C}$ consists of the following data:

(i) A class of *objects*, denoted $\mathrm{Ob}(\mathcal{C})$.

(ii) For all $A, B \in \mathrm{Ob}(\mathcal{C})$, a class of *morphisms* (also called *arrows* or *maps*), denoted $\mathrm{Hom}_{\mathcal{C}}(A, B)$. Each morphism has a *source* and a *target*, each of which is a single object. (The source and target objects are not necessarily distinct.) If $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$, we will write $f : A \to B$ for short.

(iii) We have a composition $\circ : \mathrm{Hom}_{\mathcal{C}}(A, B) \times \mathrm{Hom}_{\mathcal{C}}(B, C) \to \mathrm{Hom}_{\mathcal{C}}(A, C)$ whose action we denote by $(f, g) \mapsto g \circ f$.

(iv) The composition is required to be associative, so if $f : A \to B$, $g : B \to C$, and $h : C \to D$, we require that $h \circ (g \circ f) = (h \circ g) \circ f$.

(v) For every $A \in \mathrm{Ob}(\mathcal{C})$, ther eexists an *identity morphism* $\mathrm{id}_A \in \mathrm{Hom}_{\mathcal{C}}(A, A)$ such that for all $f : B \to A$, $f \circ \mathrm{id}_A = f$ and for every $g : A \to B$, $\mathrm{id}_A \circ g = g$.

Remark 1.2. We remark that the notion of the source and target of a morphism can be made rigorous by including the data of source and target maps that send a morphism to its source and target, respectively. But we will not be that rigorous here.

Remark 1.3. Often the $\mathrm{Hom}_{\mathcal{C}}(A, B)$ are sets, in which case we call them *Hom-sets* of $\mathcal{C}$. The category $\mathcal{C}$ is said to be *locally small* in this case. It is said to be *small* if in addition, $\mathrm{Ob}(\mathcal{C})$ is a set. Adina asks what we call it if the morphisms form a proper class but the objects form a set. Jason doesn't know. I don't know either. But the reason this isn't usually considered is that in category theory, you want to be able to use Yoneda's lemma (which we will encounter later), and you cannot do this if the morphisms form a proper class.

Example 1.4. The category of sets is denoted *Set*. Its objects are sets (which form a proper class), and its morphisms are functions from one set to another.

Example 1.5. The category of groups is denoted *Grp*. Its objects are groups (which form a proper class), and its morphisms are group homomorphisms.

Example 1.6. The category of abelian groups is denoted *Ab*. Its objects are abelian groups (which form a proper class), and its morphisms are group homomorphisms.

Example 1.7. The category of topological spaces is denoted *Top*. Its objects are topological spaces (which form a proper class), and its morphisms are continuous maps.

Example 1.8. The category of compact Hausdorff spaces is denoted *CHaus*. Its objects are compact Hausdorff topological spaces, and its morphisms are continuous maps.

Example 1.9. The category of pointed topological spaces is denoted Top^*. The objects are pairs (X, x) where X is a topological space and $x \in X$ is a point (which we call the *distinguished point* of X). A morphism $f : (X, x) \to (Y, y)$ is a continuous maps $f : X \to Y$ such that $f(x) = y$.

Example 1.10. Let X be a topological space. The category $\mathrm{Top}(X)$ has as its objects the open subsets $U \subseteq X$ and as its morphisms the inclusion map $i : U \to V$ if $U \subseteq V$ and the empty set otherwise (i.e., there are no morphisms in this second case). It is easy to check that composition is associative, that there is an identity morphism, and that the identity morphism behaves the way it should. This is a small category because X is a set, and so its subsets are a set by the power set axiom of ZFC, and there's at most one morphism between any two (not necessarily distinct) objects.

Example 1.11. A group can be thought of as a category with only one object. But I interrupt Jason to say this is an assignment question (A1, Q7), so he stops.

Example 1.12. Given a ring R, we have categories *R-Mod* and *Mod-R* of left and right R-modules, respectively. The morphisms are R-module homomorphisms. In particular, when k is a field, we have a category of k–vector spaces, denoted Vec_k.

We also want to have a notion of maps *between* categories, so we invent *functors*.

Definition 1.13. Given categories $\mathcal{C}$ and $\mathcal{D}$, a *functor* $F : \mathcal{C} \to \mathcal{D}$ consists of a map $F : \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D})$ and for all $A, B \in \mathrm{Ob}(\mathcal{C})$, a map which, misusing notation, we also denote by $F : \mathrm{Hom}_{\mathcal{C}(A,B)} \to \mathrm{Hom}_{\mathcal{C}}(F(A), F(B))$, such that these maps satisfy the following conditions:

(i) For every $A \in \mathrm{Ob}(\mathcal{C})$, $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$.

(ii) For every $f : A \to B$ and $g : B \to C$, $F(f \circ g) = F(f) \circ F(g)$.

Remark 1.14. Sometimes functors defined this way are called *covariant functors*, and you can also define *contravariant functors* which have exactly the same definition except they satisfy $F(f \circ g) = F(g) \circ F(f)$ instead.

Remark 1.15. This definition is not usually given in a rigorous way. A *forgetful* functor is one that "forgets" part of some algebraic structure. For example, there is a forgetful functor $F : \text{Grp} \to \text{Set}$ that forgets the group structure. There is a functor from *Ab* to *Grp* given by sending an abelian group to its corresponding group. Notice that this is a bit different because you aren't really forgetting any structure here. Rather, this second forgetful functor "forgets predicates". There is a pretty good discussion of the different types of forgetful functors on the Wikipedia page: `enwp.org/Forgetful_functor`.

Example 1.16. There is a forgetful functor $F : \text{Grp} \to \text{Set}$ that forgets the group structure. That is, it sends a group to its underlying set and sends a group homomorphism to its underlying map of sets.

Example 1.17. There is a functor from *Ab* to *Grp* given by sending an abelian group to its corresponding group; whether you consider this forgetful or not depends on whether you consider an abelian group a group that is abelian or whether you consider it a group equipped with the data of being abelian.

It takes more work to check that the next two examples form a category. I encourage the reader *not* to care about those details, unless they really want to.

Example 1.18. There is an abelianization functor $F : \text{Grp} \to \text{Ab}$ given by $G \mapsto G/G'$ where G' is the commutator subgroup of G.

Example 1.19. There is a functor $F : \text{Top}^* \to \text{Grp}$ by $(X, x) \mapsto \pi_1(X, x)$, where $\pi_1(X, x)$ is the fundamental group of X with respect to x. Similarly, there is a functor $\text{Top} \to \text{Grp}$ that gives you the *fundamental groupoid.* This is done in Assignment 1.

Example 1.20. The category of small categories is denoted Cat_0. Its morphisms are functors. (He restricted to small categories to be sure this forms a set. Note that the functors have to come from set maps, and set maps form a set, so even the Hom-classes of this category are actually Hom-sets.)

What if we also want a notion of morphisms between functors? These are called *natural transformations*, and we introduce them now.

Definition 1.21. Let $\mathcal{C}$, $\mathcal{D}$ be categories, and let $F, G : \mathcal{C} \to \mathcal{D}$ be functors. A natural transformation $\alpha : F \to G$ consists of the following data:

(i) For every $A \in \text{Ob}(\mathcal{C})$, there is a morphism $\alpha_A \in \text{Hom}_{\mathcal{D}}(F(A), G(A))$.

(ii) These maps are chosen so that for all objects $A, B \in \text{Ob}(\mathcal{C})$ and all morphisms $f : A \to B$ in $\mathcal{C}$, the following diagram commutes:

$$\begin{array}{ccc} F(A) & \xrightarrow{F(f)} & F(B) \\ \big\downarrow{\scriptstyle \alpha_A} & & \big\downarrow{\scriptstyle \alpha_B} \\ G(A) & \xrightarrow{G(f)} & G(B) \end{array}$$

Example 1.22. Retaining our previous notation, let id_F be the natural transformation whose associated morphisms are all the identity. This is called the *identity natural transformation.*

Definition 1.23. Let $\mathcal{C}$, $\mathcal{D}$ be categories, and let $F, G : \mathcal{C} \to \mathcal{D}$ be functors. If there exist natural transformations $\alpha : F \to G$ and $\beta : G \to F$ such that $\alpha \circ \beta = \mathrm{id}_G$ and $\beta \circ \alpha = \mathrm{id}_F$, we say that the functors F and G are *(naturally) isomorphic* and say we have a *natural isomorphism* from $\mathcal{C}$ to $\mathcal{D}$ (or from $\mathcal{D}$ to $\mathcal{C}$).

Example 1.24. Let $\mathcal{C} := \mathrm{FVec}_{\mathbb{C}}$ be the category of finite-dimensional $\mathbb{C}$–vector spaces, whose morphisms are just linear maps.

We have an identity functor $F : \mathcal{C} \to \mathcal{C}$ given by $F(V) := V$ for all $V \in \mathrm{Ob}(\mathcal{C})$ and $F(T) := T$ for all $T \in \mathrm{Hom}_{\mathcal{C}}(V, W)$, where $V, W \in \mathrm{Ob}(\mathcal{C})$.

We also have a functor $G : \mathcal{C} \to \mathcal{C}$ given by $G(V) := V^{**}$, where V^{**} denotes the double dual vector space of V, and, for $T : V \to W$, given by a map $G(T) : V^{**} \to W^{**}$ defined by $G(T)(e_v) := e_{T(v)}$ where $e_v : V^* \to \mathbb{C}$ is given by $e_v(f) = f(v)$ for $f : V \to \mathbb{C}$.

Are F and G isomorphic functors? They are. You can define natural transformations α and β by $\alpha_V(v) := e_V$ and $\beta_v(e_v) := v$. Then the composition gives the identity. (It actually requires a lot of detail checking to make this example work. It's probably good to go through all the details once and then never again.)

2 Sept. 10, 2019

Definition 2.1. Given a category $\mathcal{C}$, its *opposite category* $\mathcal{C}^{\mathrm{op}}$ is defined as having the same objects as $\mathcal{C}$ but with each morphism reversed. That is, $\mathrm{Ob}(\mathcal{C}^{\mathrm{op}}) := \mathrm{Ob}(\mathcal{C})$, but $\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(A, B) := \mathrm{Hom}_{\mathcal{C}}(B, A)$.

Example 2.2. Consider $\mathrm{Vec}_{\mathbb{C}}^{\mathrm{op}}$, the opposite category of the category of finite-dimensional $\mathbb{C}$–vector spaces. Consider (once again) the dual functor $F : \mathrm{Vec}_{\mathbb{C}} \to \mathrm{Vec}_{\mathbb{C}}^{\mathrm{op}}$ defined by $F(V) := V^*$ and $F(T) : W^* \to V^*$ for $T : V \to W$ where $F(T)$ is given by taking the dual of T. Then if $T_1 : X \to Y$ and $T_2 : Y \to Z$, we have $T_2 \circ T_1 : X \to Z$. Also, $F(T_1) : Y^* \to X^*$, $F(T_2) : Z^* \to Y^*$, $F(T_1) \circ F(T_2) : Z^* \to X^*$, and $F(T_2 \circ T_1) : Z^* \to X^*$. Thus, F is a contravariant functor from $\mathrm{Vec}_{\mathbb{C}}$ to $\mathrm{Vec}_{\mathbb{C}}$. Notice (and this is basically a terminological point) that it is *not* a contravariant functor from $\mathrm{Vec}_{\mathbb{C}}$ to $\mathrm{Vec}_{\mathbb{C}}^{\mathrm{op}}$.

Remark 2.3. This is true in general. The contravariant functors $F : \mathcal{C} \to \mathcal{D}$ are precisely the covariant functors $F : \mathcal{C} \to \mathcal{D}^{\mathrm{op}}$, and vice versa.

From last class, we know that if $F : \mathrm{Vec}_{\mathbb{C}} \to \mathrm{Vec}_{\mathbb{C}}^{\mathrm{op}}$ and $G : \mathrm{Vec}_{\mathbb{C}}^{\mathrm{op}} \to \mathrm{Vec}_{\mathbb{C}}$ both take the dual, then $G \circ F : \mathrm{Vec}_{\mathbb{C}} \to \mathrm{Vec}_{\mathbb{C}}$ is not equal to the identity functor but is naturally isomorphic to it. Now Jason gives an exercise that was suggested in my notes for last class.

Exercise 2.4. Prove that there exist natural transformations $\alpha : G \circ F \xrightarrow{\simeq} \mathrm{Id}_{\mathrm{Vec}_{\mathbb{C}}}$ and $\beta : \mathrm{Id}_{\mathrm{Vec}_{\mathbb{C}}} \to G \circ F$ such that $\beta \circ \alpha : G \circ F \to G \circ F$ and $\alpha \circ \beta : \mathrm{Id}_{\mathrm{Vec}_{\mathbb{C}}} \to \mathrm{Id}_{\mathrm{Vec}_{\mathbb{C}}}$ are identities. Similarly, show that $F \circ G \simeq \mathrm{Id}_{\mathrm{Vec}_{\mathbb{C}}^{\mathrm{op}}}$.

The example from this exercise motivates the following definition.

Definition 2.5. Suppose $\mathcal{C}$ and $\mathcal{D}$ are categories and $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{C}$ are functors. Suppose further that $G \circ F \simeq \mathrm{Id}_{\mathcal{C}}$ and $F \circ G \simeq \mathrm{Id}_{\mathcal{D}}$. Then we say that $\mathcal{C}$ and $\mathcal{D}$ are *equivalent* as categories or that there is an *equivalence of categories* between $\mathcal{C}$ and $\mathcal{D}$.

Example 2.6. Suppose that $\mathcal{C}$ is the category having two objects, which we call 1 and 2, with corresponding identity morphisms id_1 and id_2 as well as morphisms $f : 1 \to 2$ and $f^{-1} : 2 \to 1$ (so that 1 and 2 are isomorphic in $\mathcal{C}$). Define $F : \mathcal{C} \to \mathcal{D}$ by $1 \mapsto A$, $2 \mapsto A$, $\mathrm{id}_1 \mapsto \mathrm{id}_A$, and $\mathrm{id}_2 \mapsto \mathrm{id}_A$. Define $G : \mathcal{D} \to \mathcal{C}$ by $A \mapsto 1$ and $\mathrm{id}_A \mapsto \mathrm{id}_1$. Then one can check that $F \circ G = \mathrm{id}_{\mathcal{D}}$ and $G \circ F \simeq \mathrm{id}_{\mathcal{C}}$, although I haven't actually checked (but it should be true). Incidentally, originally Jason had tried this without having the morphisms f and f^{-1} in $\mathcal{C}$, and there it failed to be true.

In general, since equivalence of categories is analogous to homotopy equivalence, it should preserve the "connected components" of a category, which is why the maps f and f^{-1} are needed. I have not made this intuition precise. Probably you can make it precise by looking at the geometric realization of the nerve of the category, which is a way of turning it into a topological space and under which correspondence equivalence of categories literally becomes homotopy equivalence. Jason suggests taking skeletal subcategories, and I am not sure how this relates to the previous notion.

Example 2.7. Even though $\mathrm{Vec}_{\mathbb{C}} \simeq \mathrm{Vec}_{\mathbb{C}}^{\mathrm{op}}$, we now give an example of a category that is not equivalent to its opposite category. Let $\mathcal{C}$ be the following category (with the implied identity morphisms):

$$0 \longrightarrow 1 \longrightarrow 2 \longrightarrow \ldots$$

Note that this category has a unique morphism $i \to j$ if and only if $i \leq j$. Its opposite category looks like this (with the implied identity morphisms):

$$0 \longleftarrow 1 \longleftarrow 2 \longleftarrow \ldots$$

This category has a unique morphism $i \to j$ if and only if $i \geq j$. Jason sketches an argument that we cannot have functors $F : \mathcal{C} \to \mathcal{C}^{\mathrm{op}}$ and $G : \mathcal{C}^{\mathrm{op}} \to \mathcal{C}$ such that $G \circ F \simeq \mathrm{Id}_{\mathcal{C}}$ and $F \circ G \simeq \mathrm{Id}_{\mathcal{D}}$. I do not write it down, but it does not seem complicated.

Example 2.8. Let k be an algebraically closed field. Let $\mathcal{C}$ be the category of finitely-generated reduced commutative k-algebras. Let $\mathcal{D}$ be the category of affine varieties over k. Then $\mathcal{C} \simeq \mathcal{D}^{\mathrm{op}}$.

If $A \in \mathrm{Ob}(\mathcal{C})$, then we can write $A \simeq k[x_1, ..., x_n]/(p_1(x_1, ..., x_n), ..., p_d(x_1, ..., x_n))$. Define $F : \mathcal{C} \to \mathcal{D}$ by $F(A) := Z(p_1, ..., p_d) \subseteq k^n$. Similarly, writing $V = Z(p_1, ..., p_n)$, define $G : \mathcal{D} \to \mathcal{C}$ by $G(V) := k[x_1, ..., x_n]/\sqrt{(p_1, ..., p_d)}$. These two functors induce an equivalence of categories.

In algebraic geometry, we love sheaves, and therefore we love presheaves.

Definition 2.9. Let X be a topological space. Recall that we had a (small) category $\mathrm{Top}(X)$ whose objects are open subsets of X and whose morphisms are inclusions $i : U \to V$ when (and only when) $U \subseteq V$.

A *presheaf* of sets (resp. groups, abelian groups, rings, R-modules,...) on X is a contravariant functor $F : \mathrm{Top}(X) \to \mathrm{Set}$ (resp. Grp, Ab, Ring, R-Mod,...).

But what does that even mean?

Example 2.10. Let $X = S^3$. Let $F : \mathrm{Top}(X) \to \mathrm{Ring}^{\mathrm{op}}$ be defined by $F(U) := C^\infty(U)$, the ring of smooth functions $U \to \mathbb{R}$. For every inclusion $U \to V$ in $\mathrm{Top}(X)$, we have a restriction map $F(V) \to F(U)$ that just restricts each smooth function on V to one on U (which preserves smoothness). Then F is a functor.

Definition 2.11. Let $\mathcal{A}$ and $\mathcal{B}$ be categories. Suppose $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{A}$ are functors such that for all $A \in \mathrm{Ob}(\mathcal{A})$ and $B \in \mathrm{Ob}(\mathcal{B})$, there exists an isomorphism

$$\alpha_{A,B} : \mathrm{Hom}_{\mathcal{B}}(F(A), B) \xrightarrow{\simeq} \mathrm{Hom}_{\mathcal{A}}(A, G(B))$$

(and notice that if we cut this definition off here, it would be too weak, because in the category of sets, for example, this would just be saying these Hom-sets have the same cardinality) such that the following holds. Suppose that objects and maps are chosen so that the following two diagrams commute:

$$\begin{array}{ccc} A' & \xrightarrow{f} & G(B) \\ {\scriptstyle\varphi}\uparrow & & \downarrow{\scriptstyle G(\psi)} \\ A & \longrightarrow & G(B') \end{array} \qquad \begin{array}{ccc} F(A') & \xrightarrow{g} & B \\ {\scriptstyle F(\varphi)}\uparrow & & \downarrow{\scriptstyle\psi} \\ F(A) & \longrightarrow & B' \end{array}$$

Then the following diagram, which we call (*), also commutes:

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{A}}(A', G(B)) & \xleftarrow[\alpha_{A',B}]{} & \mathrm{Hom}_{\mathcal{B}}(F(A'), B) \\ \downarrow{\scriptstyle G(\psi)\circ f\circ\varphi} & & \downarrow{\scriptstyle \psi\circ g\circ F(\varphi)} \\ \mathrm{Hom}_{\mathcal{A}}(A, G(B')) & \xleftarrow[\alpha_{A,B'}]{} & \mathrm{Hom}_{\mathcal{B}}(F(A), B') \end{array}$$

Definition 2.12. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. The *product category* $\mathcal{C} \times \mathcal{D}$ is defined as follows. First, we define $\mathrm{Ob}(\mathcal{C} \times \mathcal{D}) := \mathrm{Ob}(\mathcal{C}) \times \mathrm{Ob}(\mathcal{D})$. Next, we define the morphisms from an object (C_1, D_1) to an object (C_2, D_2) to be the pairs (f, g) where $f \in \mathrm{Hom}_{\mathcal{C}}(C_1, C_2)$ and $g \in \mathrm{Hom}_{\mathcal{D}}(D_1, D_2)$. Composition is defined by $(f_1, g_1) \circ (f_2, g_2) := (f_1 \circ_{\mathcal{C}} f_2, g_1 \circ_{\mathcal{D}} g_2)$. The identity morphisms are defined by $1_{(C,D)} := (1_C, 1_D)$.

It is pretty easy to verify that product categories are categories.

Definition 2.13. A *bifunctor* is a functor whose domain is a product category.

Intuitively, we can think of a bifunctor as something that is a (co- or contravariant) functor in each of its two coordinates. Note that bifunctors can be contravariant in one coordinate and covariant in the other.

Example 2.14. Given a category $\mathcal{C}$, $\mathrm{Hom}_{\mathcal{C}}(-,-)$ is a bifunctor. It is covariant if you fix the first coordinate and contravariant if you fix the second.

Remark 2.15. Natural transformations between bifunctors are just natural transformations between the corresponding functors. Notice that the commutativity of the diagram (*) looks like the commutativity condition in the definition of natural transformations.

Indeed, suppose we have functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ and a natural isomorphism $\Phi : \mathrm{Hom}_{\mathcal{D}}(F-,-) \overset{\simeq}{\Rightarrow} \mathrm{Hom}_{\mathcal{C}}(-,G-)$. This happens if and only if F is left adjoint to G (resp. G is right adjoint to F).

3 Sept. 12, 2019

Example 3.1. We have a forgetful functor $G : \mathrm{Ab} \to \mathrm{Set}$. We also have a *free functor* $F : \mathrm{Set} \to \mathrm{Ab}$ that sends a set X to the abelian group $F(X) := \bigoplus_{x \in X} \mathbb{Z}e_x$ where $\{e_x\}_{x \in X}$ is a basis. Given a map of sets $f : X \to Y$, there is a corresponding map of bases $F(f) : \{e_x\}_{x \in X} \to \{e'_y\}_{y \in Y}$.

This example satisfies the following universal property. Suppose X is a set and $i_X : X \to F(X)$ is the inclusion map. Suppose A is an abelian group and $f : X \to A$ is some map of sets. Then there exists a unique group homomorphism $\tilde{f} : F(X) \to A$ such that the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{i_X} & F(X) \\ & \searrow^{f} & \downarrow{\scriptstyle \exists! \tilde{f}} \\ & & A \end{array}$$

The functors F and G are adjoints. Suppose A, A' are sets and B, B' are abelian groups. Suppose $\varphi : A \to A'$ is a set map and $\psi : B \to B'$ is a group homomorphism. Plugging this into the first definition we gave of adjoint functors, you can check that it is satisfied.

Remark 3.2. In general, free functors are left adjoint to forgetful functors. Suppose $\mathcal{C}$ is a category whose objects are sets with a forgetful functor $G : \mathcal{C} \to \mathrm{Set}$. If a free functor $F : \mathrm{Set} \to \mathcal{C}$ exists, then it satisfies the universal property that for every set X (with inclusion $i_X : X \to F(X)$) and every map of sets $f : X \to Y$ where $Y \in \mathrm{Ob}(\mathcal{C})$, there exists a unique $\tilde{f} \in \mathrm{Hom}_{\mathcal{C}}(F(X), Y)$ such that the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{i_X} & F(X) \\ & \searrow^{f} & \downarrow{\scriptstyle \exists! \tilde{f}} \\ & & Y \end{array}$$

Example 3.3. Let $\mathcal{C} := \mathrm{Ab}$. Given a basis $\{e_x\}_{x \in X}$, we have an inclusion map $i_X : X \to \bigoplus_{x \in X} \mathbb{Z}e_x$. Analogously, if we consider Vec_k for some field k, we have an inclusion map $i_X : X \to \bigoplus_{x \in X} ke_x$. Analogously, if we consider commutative k-algebras, we have an inclusion map $X \to k[t_i : i \in X]$ for some $\{t_i\}_{i\in X}$. Analogously, if we consider non-commutative (associative) k-algebras, we have an inclusion map $X \to k\langle t_i : i \in X\}$ where the t_i are non-commuting variables. Analogously, if we consider groups, we have an inclusion map $X \to \langle X\rangle$ where $\langle X\rangle$ is the free group generated by X.

Remark 3.4. We will sometimes use the notations FX and GX for $F(X)$ and $G(X)$, respectively, especially in the context of adjoint functors.

Example 3.5. Let $\mathcal{F}$ be the category of fields. We have a forgetful functor $G : \mathcal{F} \to \text{Set}$. However, its adjoint, which would be a free functor $F : \text{Set} \to \mathcal{F}$, does not exist. Why not? Let $X := \{1\}$. Suppose we have a left adjoint $F : \text{Set} \to \mathcal{F}$. Then we have an inclusion map $i_X : X \to FX$, where FX is some field. Let K be a field with characteristic different from that of FX. We have a set map from $X = \{1\}$ to K given by sending the only element of X to $1 \in K$. However, this does not extend to a field homomorphism $FX \to K$ making the usual diagram commute because FX and K have different characteristics.

Note that when G and F are adjoint functors, $\text{Hom}(X, GFX) \simeq \text{Hom}(FX, FX)$. This sometimes comes in handy.

Example 3.6. We have a forgetful functor $G : \text{CHaus} \to \text{Top}$. It has a left adjoint $\beta : \text{Top} \to \text{CHaus}$ given by $X \mapsto \beta X$ where βX is the Stone–Čech compactification of X. This is a problem on Assignment 1.

Example 3.7. Let Top^* be the category of pointed topological spaces. We have a functor $\pi_1 : \text{Top}^* \to \text{Grp}$ given by $(X, x) \mapsto \pi_1(X, x)$, the fundamental group of X based at x. We claim there does not exist an adjoint $F : \text{Grp} \to \text{Top}^*$ to π_1.

Let $X := \mathbb{C}$ and $Y := S^1 \subset \mathbb{C}$. Let $H := \mathbb{Z} \in \text{Ob}(\text{Grp})$. Then, if F satisfies the definition of the adjoint functor to $G := \pi_1$, the following two statements hold by definition of adjoint functors:

(i) $\text{Hom}_{\text{Grp}}(H, \pi_1(X, 1)) \simeq \text{Hom}_{\text{Top}^*}(F(H), (X, 1))$;

(ii) $\text{Hom}_{\text{Grp}}(H, \pi_1(Y, 1)) \simeq \text{Hom}_{\text{Top}^*}(F(H), (Y, 1)))$.

However,

$$\text{Hom}_{\text{Grp}}(H, \pi_1(X, 1)) = \text{Hom}_{\text{Grp}}(H, \{1\}).$$

Each has cardinality one. Also,

$$\text{Hom}_{\text{Grp}}(H, \pi_1(Y, 1)) \simeq \text{Hom}_{\text{Grp}}(H, \mathbb{Z}),$$

which has infinitely many elements. The inclusion $Y \hookrightarrow X$ induces an inclusion

$$\text{Hom}_{\text{Grp}}(H, \pi_1(Y, 1)) \hookrightarrow \text{Hom}_{\text{Grp}}(H, \pi_1(X, 1)),$$

which is impossible because the former has infinitely many elements and the latter only has one. This is the contradiction for which we were looking.

4 Sept. 17, 2019

Today's class was cancelled.

5 Sept. 19, 2019

Ehsaan teaches today's class. He explains that Jason sent him Chris Hawthorne's class notes to use for the lecture. He then gives us the following exercise (and then solves it).

Exercise 5.1. Consider the following two diagrams:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} \\ C & \xrightarrow{g} & D \end{array} \qquad \begin{array}{ccc} \mathrm{Hom}(A,X) & \xleftarrow[f^*]{} & \mathrm{Hom}(B,X) \\ {\scriptstyle u^*}\uparrow & & {\scriptstyle v^*}\uparrow \\ \mathrm{Hom}(C,X) & \xleftarrow[g^*]{} & \mathrm{Hom}(D,X) \end{array}$$

(Here if $\varphi : B \to X$ is a morphism, then $f^*(\varphi) := \varphi \circ f$.) Prove that the left diagram commutes if and only if the right diagram commutes.

Proof. The forward direction (left commutes implies right commutes) just uses the fact that $\mathrm{Hom}(-, X)$ is a contravariant functor. The other direction can be done as follows. We want to show that $v \circ f = g \circ u$ assuming the right diagram commutes. We know that for all $\psi : D \to X$,

$$f^*(v^*(\psi)) = u^*(g^*(\psi)).$$

We set $X := D$ and $\psi := \mathrm{id}_D$. Then

$$f^*(v^*(\mathrm{id})) = u^*(g^*(\mathrm{id})),$$

so

$$v \circ f = g \circ u,$$

and we are done. □

Today we want to (i) prove right adjoints are unique, (ii) discuss Yoneda's lemma, and (iii) give some examples of adjoints. First we discuss adjoints. Recall that if $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{A}$ are functors, then G is a right adjoint for F if

$$\mathrm{Hom}_{\mathcal{B}}|(FA, B) \simeq \mathrm{Hom}_{\mathcal{A}}(A, GB),$$

where $\simeq$ denotes natural isomorphism of bifunctors. Indeed, given a (covariant) functor $F : \mathcal{A} \to \mathcal{B}$. Let $\mathcal{F} := \mathrm{Hom}(F(-), -)$. Then $\mathcal{F}$ can be viewed as a functor from the product category $\mathcal{A}^{\mathrm{op}} \times \mathcal{B} \to \mathrm{Set}$ since it is contravariant in the first coordinate and covariant in the second. (There was a similar discussion earlier in these notes but not in Jason's lectures.)

Example 5.2. (i) The forgetful functor from $\mathrm{Vec}_{\mathbb{C}}$ to Set and the free functor going the other direction are adjoints. Then the free functor is left adjoint to the forgetful functor.

Example 5.3. The category $\mathbb{R}$ can be defined as the category having exactly one object for each real number and an arrow $x \to y$ if and only if $x \leq y$. Similarly, the category $\mathbb{Z}$ has exactly one object for each integer and an arrow $x \to y$ if and only if $x \leq y$. We have an inclusion functor $i : \mathbb{Z} \hookrightarrow \mathbb{R}$ and a floor functor $\lfloor \cdot \rfloor : \mathbb{R} \to \mathbb{Z}$ that sends each real number to its floor and sends a morphism $x \to y$ to the corresponding morphism $\lfloor x \rfloor \to \lfloor y \rfloor$. Then i is left adjoint to $\lfloor \cdot \rfloor$.

(This construction is an example of how to make a category out of a poset. You have one object for each element of the poset and a morphism $x \to y$ if and only if $x \leq y$.)

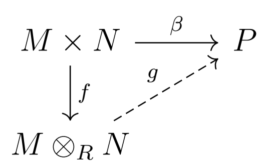
Example 5.4. We have a functor Grp $\to$ Ring given by $G \mapsto \mathbb{Z}[G]$ and a functor Ring $\to$ Grp given by $R \mapsto R^\times$. Then

$$\mathrm{Hom}(G, R^\times) \simeq \mathrm{Hom}(\mathbb{Z}[G], R^\times),$$

so these are adjoint functors.

Example 5.5. Let Vect^G denote the category of linear representations of G. If $H \leq G$, we have a restriction functor $\mathrm{res}^G_H : \mathrm{Vect}^G \to \mathrm{Vect}^H$. The left adjoint is an *induction* functor $\mathrm{ind}^G_H : \mathrm{Vect}^H \to \mathrm{Vect}^G$. The statement that these functors form an adjoint pair is called *Frobenius reciprocity.*

Example 5.6. This example is called *Hom-tensor adjunction.* Let R be a commutative ring, and let Mod_R be the category of R-modules. If $M, N \in \mathrm{Mod}_R$, then we can consider $M \otimes_R N \in \mathrm{Mod}_R$. The bilinear map $f : M \times N \to M \otimes_R N$ given by $(m, n) \mapsto m \otimes n$ satisfies the following universal property. Given a bilinear map $\beta : M \times N \to P$ there exists a unique map $g : M \otimes_R N \to P$ making the following diagram commute:



A bilinear map $\beta : M \times N \to P$ can be viewed as a map $M \to (N \to P)$ by currying. Therefore, letting $\mathrm{Bilin}(A, B)$ denote the collection of bilinear maps from A to B, we have that

$$\mathrm{Hom}(M, \mathrm{Hom}(N, P)) = \mathrm{Bilin}(M \times N, P) \simeq \mathrm{Hom}(M \otimes N, P).$$

Fixing N, the functor $(-) \otimes N$ is therefore left adjoint to the functor $\mathrm{Hom}(N, -)$.

Theorem 5.7. *Left adjonts are unique. More precisely, if $G : \mathcal{B} \to \mathcal{A}$ is a functor and $F, F' : \mathcal{A} \to \mathcal{B}$ are functors that are left adjoint to G, then $F \simeq F'$.*

The proof will be aas follows. Given suitable objects A, B, we have

$$\mathrm{Hom}(FA, B) \simeq \mathrm{Hom}(A, GB) \simeq \mathrm{Hom}(F'A, B).$$

These isomorphisms should be "natural" in A and B. Then we can use the Yoneda lemma to conclude that $FA \simeq F'A$. We now introduce some notation, followed by the lemma.

Let $\mathcal{C}$ be a category. For $A \in \mathrm{Ob}(\mathcal{C})$, let $h_A : \mathcal{C} \to \mathrm{Set}$ be defined by

$$h_A(X) := \mathrm{Hom}(A, X).$$

Given maps $A \to X$ and $X \to Y$, we have a map $A \to Y$ given by composing them. This defines the behaviour of h_A on morphisms. We know this already from the discussion of Hom functors.

Let $\mathrm{Funct}(\mathcal{C}, \mathrm{Set})$ be the category of functors $\mathcal{C} \to \mathrm{Set}$. Therefore, we have a functor $C \to \mathrm{Funct}(\mathcal{C}, \mathrm{Set})^{\mathrm{op}}$ given by $A \mapsto h_A$.

Lemma 5.8. *Let $\mathcal{C}$ be a locally small category. (We need the locally small condition so that the Hom-classes are actually Hom-sets.) Let $A, B \in Ob(\mathcal{C})$. If $h_A \simeq h_B$, then $A \simeq B$.*

We do not prove Yoneda's lemma yet.

Definition 5.9. Let $\mathcal{C}$ be a locally small category. A functor $F : \mathcal{C} \to \mathrm{Set}$ is said to be *representable* if it is naturally isomorphic to one of the Hom functors h_A. Let $\mathrm{Rep}(\mathcal{C})$ be the full subcategory of representable functors, i.e., the category whose objects are representable functors and whose morphisms are all natural transformations between those functors. (Here there is a difference of terminology. Jason calls a functor "representable" if it is equal (as opposed to naturally isomorphic) to some h_A. This issue will be brought up again later.)

We can now rephrase Yoneda's lemma as follows.

Lemma 5.10. *Let $\mathcal{C}$ be a locally small category. Then $\mathcal{C} \simeq Rep(\mathcal{C})^{op}$, where as usual $\simeq$ denotes a natural isomorphism.*

Jason will prove Yoneda's lemma. Assuming Yoneda's lemma, we can now prove that left adjoints are unique.

Proof. Suppose $G : \mathcal{B} \to \mathcal{A}$ is a functor with two left adjoints $F, F' : \mathcal{A} \to \mathcal{B}$. Then given objects A, B in the appropriate categories,

$$\mathrm{Hom}(FA, B) \simeq \mathrm{Hom}(A, GB) \simeq \mathrm{Hom}(F'A, B).$$

We want to show that $FA \simeq F'A$ naturally in A. Fix A. Then the assumption of adjointness gives $h_{FA} \simeq h_{F'A}$. By Yoneda's lemma, we have an isomorphism $\eta_A : FA \simeq F'A$. It remains to show it is natural. This means we need to show the following diagram commutes for all A, A' and $\varphi : A \to A'$:

$$\begin{array}{ccc} FA & \xrightarrow{\eta_A} & F'A \\ \downarrow{\scriptstyle F\varphi} & & \downarrow{\scriptstyle F'\varphi} \\ FA' & \xrightarrow{\eta_{A'}} & F'A' \end{array}$$

But applying $\mathrm{Hom}(-, X)$, we obtain the diagram

$$\begin{array}{ccc} \mathrm{Hom}(FA, X) & \xleftarrow[\eta_A^*]{} & \mathrm{Hom}(F'A, X) \\ {\scriptstyle (F\varphi)^*}\uparrow & & \uparrow{\scriptstyle (F'\varphi)^*} \\ \mathrm{Hom}(FA', X) & \xleftarrow[\eta_{A'}^*]{} & \mathrm{Hom}(F'A', X) \end{array}$$

and this diagram commutes by

$$\mathrm{Hom}(FA, B) \simeq \mathrm{Hom}(A, GB) \simeq \mathrm{Hom}(F'A, B)$$

that we obtained from adjointness, we are done. □

6 Sept. 24, 2019

Let $\mathcal{A}$ be a category. Recall that the functor $h_A : \mathcal{A} \to \mathrm{Set}$ is defined by $h_A(B) := \mathrm{Hom}_{\mathcal{A}}(A, B)$. Given $B_1 \xrightarrow{f} B_2$, we have a map $h_A(B_1) \xrightarrow{h_A(f)} h_A(B_2)$ since if we have maps $\phi : A \to B_1$ and $f : B_1 \to B_2$, we can send $\phi \mapsto f \circ \phi$, and $f \circ \phi : A \to B_2$.

Now suppose $\mathcal{A}$ is a locally small category. Then we can form the category $\mathrm{Funct}(\mathcal{A}, \mathrm{Set})$. (See Assignment 1 for the definition of this category.) We have an inclusion $\mathcal{A} \hookrightarrow \mathrm{Funct}(\mathcal{A}, \mathrm{Set})^{\mathrm{op}}$ given by $A \mapsto h_A$.

Example 6.1. Let $\mathrm{Ab}_{\mathrm{fin}}$ be the category of finite abelian groups. Any object A in this category can be written as $A \simeq \bigoplus_{i=1}^n \mathbb{Z}/p_i^{d_i}\mathbb{Z}$. We can then look at $|\mathrm{Hom}(A, B)|$. For example, if $A = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/9\mathbb{Z}$, then

$$|\mathrm{Hom}_{\mathrm{Ab}_{\mathrm{fin}}}(A, \mathbb{Z}/p\mathbb{Z})| = \begin{cases} 1 & \text{if } p \geq 5, \\ 2 & \text{if } p = 2, \\ 3 & \text{if } p = 3. \end{cases}$$

Also,

$$|\mathrm{Hom}_{\mathrm{Ab}_{\mathrm{fin}}}(A, \mathbb{Z}/p^2\mathbb{Z})| = \begin{cases} 1 & \text{if } p \geq 5, \\ 2 & \text{if } p = 2, \\ 9 & \text{if } p = 3. \end{cases}$$

We now go back to considering the map $A \mapsto h_A$. Note that if there is a natural isomorphism $\eta : h_A \to h_B$ with inverse $\epsilon : h_B \to h_A$, then for each object we have maps $h_A(C) \xrightarrow{\eta_C} h_B(C)$ and $\epsilon_C : h_B(C) \to h_A9C)$, so we have bijections $\eta_C : \mathrm{Hom}_{\mathcal{A}}(A, C) \to \mathrm{Hom}_{\mathcal{A}}(B, C)$ and vice versa, so

$$\mathrm{Hom}_{\mathcal{A}}(A, C) \simeq \mathrm{Hom}_{\mathcal{A}}(B, C).$$

Example 6.2. Let $G : \mathrm{Grp} \to \mathrm{Set}$ be the forgetful functor. Let $F : \mathrm{Set} \to \mathrm{Grp}$ be its left adjoint, which is the free functor. We claim that G is representable, in fact that $G \simeq h_{\mathbb{Z}}$. Given a group H, we have

$$h_{\mathbb{Z}}(H) = \mathrm{Hom}(\mathbb{Z}, H) \simeq H.$$

Thus,

$$\mathrm{Hom}_{\mathrm{Grp}}(F(\{x\}), H) \simeq \mathrm{Hom}_{\mathrm{Set}}(\{x\}, G(H)) \simeq H.$$

We then have a natural transformation given by the data of maps $\eta_H : G(H) \to h_{\mathbb{Z}}(H)$. It suffices to specify a map $H \to \mathrm{Hom}_{\mathrm{Grp}}(\mathbb{Z}, H)$, and we define this by $h \mapsto \phi_h(n) := h^n$ (so $\phi_h(1) = h$). We also define a map going the other way by giving the data of morphisms $\epsilon_H : h_{\mathbb{Z}}(H) \to G(H)$, i.e., $\mathrm{Hom}(\mathbb{Z}, H) \to H$ given by $\epsilon_H(\phi) := \phi(1) \in H$. The corresponding maps η and ϵ define a natural isomorphism.

Definition 6.3. Let k be a field, and let C be a commutative k-algebra. Let C-Mod be the category of C-modules. Then $\mathrm{Hom}_{C\text{-Mod}}(M, N)$ is the set of maps $f : M \to N$ such that $f(c_1 m_1 + m_2) = c_1 f(m_1) + f(m_2)$. Given a C-module M, a *derivation* is a k-linear map $d : C \to M$ such that $d(c_1 c_2) = c_1 d(c_2) + c_2 d(c_1)$. Let $\mathrm{Der}_k(M)$ denote the set of k-linear derivations $d : C \to M$.

We observe that $\mathrm{Der}_k(M)$ is a C-module since linear combinations $d_1 + cd_2$ where $c \in k$ of derivations are derivations. In fact, even if $c \in C$, then $\hat{d}(a) := cd(a)$ is a derivation if d is a derivation. We have a functor $\mathrm{Der}_k : C\text{-Mod} \to C\text{-Mod}$ given by $M \mapsto \mathrm{Der}_k(M)$. Given a homomorphism $M_1 \xrightarrow{f} M_2$, we have a map $\mathrm{Der}_k(M_1) \xrightarrow{\mathrm{Der}_k(f)} \mathrm{Der}_k(M_2)$ because if $d : C \to M_1$ is a derivation, then $f \circ d : C \to M_2$ is also a derivation since we have

$$f(d(c_1 c_2)) = f(c_1 d(c_2) + c_2 d(c_1)) = c_1 f \circ d(c_2) + c_2 f \circ d(c_1).$$

It turns out that Der_k is a representable functor (in a sense that we will describe). We can now define Kähler differentials.

Definition 6.4. Let k be a field and C a commutative k-algebra. Let $\Omega_{C/k}$ be the module of Kähler differentials, which we construct as follows. Let M be the free C-module on all symbols $\{d(c) \mid c \in C\}$. Let N be the submodule generated by $d(1)$, $d(c_1 + \lambda c_2) - d(c_1) - \lambda d(c_2)$, and $d(c_1 c_2) - c_1 d(c - 2) - c_2 d(c_1)$ for all $c_1, c_2 \in C$ and all $\lambda \in k$. Let $\Omega_{C/k} := M/N$.

Example 6.5. Take $C := k[T]$. Elements are polynomials $p(t) := p_0 + p_1 t_1 + ... + p_n t^n$ for $p_i \in k$. A short calculation repeatedly using the product rule gives

$$d(p_0 + p_1 t + ... + p_n t^n) = (p_1 + 2p_2 t + ... + np_n t^{n-1})d(t).$$

(For example, $d(t^2) = td(t) + td(t) = 2td(t)$.) In general, $d(p(t)) = p'(t)d(t)$, so $\Omega_{k[t]/k} = k[t]d(t)$ (often written $k[t]\,\mathrm{d}t$). This is isomorphic to C as a C-module.

More generally, we always have a map $h_{\Omega_{C/k}} : C\text{-Mod} \to C\text{-Mod}$ given by $B \mapsto \mathrm{Hom}_{C\text{-Mod}}(\Omega_{C/k}, B)$.

Example 6.6. We have a natural isomorphism $\eta : \mathrm{Der}_k \to h_{\Omega_{C/k}}$ given as a family of morphisms $\mathrm{Der}_k(M) \xrightarrow{\eta_M} \mathrm{Hom}_{C\text{-Mod}}(\Omega_{C/k}, M)$ given by $d \mapsto (\underline{d(c)} \mapsto d(c))$ where $\underline{d(c)}$ is the *formal symbol* $d(c)$, which is a generator of $\Omega_{C/k}$. (Recall that Kähler differentials were defined in terms of a bunch of generators which are formal symbols $d(c)$. To check the maps in this example are well-defined, we just use the fact that the relations satisfied by these formal symbols are the same as the relations satisfied by derivations in Der_k.) We have an inverse natural transformation given as morphisms $\mathrm{Hom}_{C\text{-Mod}}(\Omega_{C/k}, M) \xrightarrow{\epsilon_M} \mathrm{Der}_k(M)$ given by $(\epsilon_M)(f)(c) := f(\underline{d(c)})$ where $\underline{d(c)}$ is again the formal symbol $d(c)$, which is a generator of $\Omega_{C/k}$.

Recall that we defined presheaves earlier in this course. Now we define sheaves.

Definition 6.7. Let X be a topological space, and let $\mathrm{Top}(X)$ be the category whose objects are open subsets of X and with a morphism $V \to U$ if and only if $V \subseteq U$. A *sheaf* is a presheaf $\mathcal{F} : \mathrm{Top}(X)^{\mathrm{op}} \to \mathcal{C}$ (where $\mathcal{C}$ could be the category of sets, groups, rings, abelian groups, C-modules, etc.) with two additional properties.

(i) (Separatedness.) If $U = \bigcup_{i \in I} U_i$ is an open cover and $f, g \in \mathcal{F}(U)$ satisfy $f|_{U_i} = g|_{U_i}$ for every $i \in I$, then $f = g$.

(ii) (Gluing.) If $U = \bigcup_{i \in I} U_i$ is an open cover and there is $f_i \in \mathcal{F}(U_i)$ such that $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ for all $i, j \in I$, then there exists $f \in \mathcal{F}(U)$ such that $f|_{U_i} = f_i$ for all $i \in I$.

Example 6.8. Take $X := \mathbb{R}$ and $\mathcal{F} : \mathrm{Top}(X)^{\mathrm{op}} \to \mathrm{Ring}$ by letting $\mathcal{F}(U)$ be the ring of bounded continuous maps $f : U \to \mathbb{R}$. Then $\mathcal{F}$ is a presheaf. However, $\mathcal{F}$ does not have gluing! Indeed, take the open set $U_n := (-n, n)$. Take the map $f_n \in \mathcal{F}(U_n)$ given by $f_n(x) := x$. Consider an inclusion $U_i \hookrightarrow U_j$. Then $f_j|_{U_i} = f_i$. However, $\bigcup_n U_n = \mathbb{R}$ so the map $f(x)$ obtained by gluing the f_i along the inclusions is unbounded on $\mathbb{R}$.

What are the representable presheaves? In other words, when $\mathcal{F} : \mathrm{Top}(X)^{\mathrm{op}} \to \mathrm{Set}$ is a presheaf, when does $\mathcal{F} \simeq h_U$ for $h_U(V) = \mathrm{Hom}(U, V)$ (assuming $U \subseteq V$)? This question motivates Yoneda's lemma.

7 Sept. 26, 2019

Let $\mathcal{A}$ be a locally small category. Let $\mathcal{F}$ be the full subcategory of $\mathrm{Funct}(\mathcal{A}, \mathrm{Set})$ containing precisely the functors h_A. (Here there is a terminological difference. Jason calls these "representable functors", but to me that means a functor naturally isomorphic to some h_A, not necessarily equal to some h_A. Both approaches work. In Jason's approach, we are technically passing to a skeletal subcategory. In the other approach, we would have to prove that it suffices to define the functor G (to be introduced later) on representatives of isomorphism classes of functors. But in our approach we are sweeping stuff under the rug as well by assuming that the theory works out when we pass to a skeletal subcategory.) Then $\mathcal{A} \simeq \mathcal{F}^{\mathrm{op}}$, where $\simeq$ denotes natural isomorphism. Jason explains that in proving Yoneda, we pass to this subategory because it is locally small and because, as Jason puts it, "Equivalence does not care about the number of copies of isomorphic objects."

Consider $h_U : \mathrm{Top}(X)^{\mathrm{op}} \to \mathrm{Set}$ given by $V \mapsto \mathrm{Hom}(U, V)$. Jason explains that if $\eta : h_U \simeq F$, then η is determined by the map $\eta_U \to F(U)$. He also explains that there is a natural transformation $\eta : h_U \to h_V$ if and only if $U \subseteq V$. We will now prove these things in the course of proving Yoneda's lemma.

We begin by restating the lemma.

Lemma 7.1. *Let $\mathcal{A}$ be a locally small category. (We need the locally small condition so that the Hom-classes are actually Hom-sets.) Let $A, B \in Ob(\mathcal{A})$. If $h_A \simeq h_B$ (where $\simeq$ denotes natural isomorphism), then $A \simeq B$ (where $\simeq$ denotes isomorphism).*

Proof. Let $\mathcal{F}$ be the full subcategory of $\text{Funct}(\mathcal{A}, \text{Set})$ containing all representable functors. We begin by constructing functors $F : \mathcal{A} \to \mathcal{F}^{\text{op}}$ and $G : \mathcal{F}^{\text{op}} \to \mathcal{A}$.

First, we define F. Let $F(A) := h_A$. Given $f : A \to B$ we have a map $\eta = F(f) : h_B \to h_A$. For every $C \in \text{Ob}(\mathcal{A})$, $\eta_C : h_B(C) \to h_A(C)$, which can be thoguht of as a map $\text{Hom}_{\mathcal{A}}(B, C) \to \text{Hom}_{\mathcal{A}}(A, C)$, is given by $\psi \mapsto \psi \circ f$. Checking that this defines a functor is a short exercise.

Next, we define G. Let $G(h_A) := A$. (We are assuming that we can do this, as I explained in the first paragraph of today's notes.) Also, given $\eta : h_B \to h_A$, we define $G(\eta) : A \to B$ by $G(\eta) := \eta_B(\text{id}_B)$. One can check as an exercise that G is a functor.

We also need to check that $G \circ F = \text{id}_A$ and $F \circ G \simeq \text{id}_{\mathcal{F}^{\text{op}}}$. We have $G(F(A)) = G(h_A) = A$. Given $f : A \to B$, we have $\eta_f : h_B \to h_A$ and $G(\eta_f) : A \to B$. We claim that $G(\eta_f) = f$. Now, we have $(\eta_f)_B : h_B(B) \to h_A(B)$ and

$$G(\eta_f) = (\eta_f)_B(\text{id}_B).$$

We then have

$$f = G(\eta_f) = (\eta_f)_B(\text{id}_B) = \text{id}_B \circ f = f.$$

This proves that $G \circ F = \text{id}_{\mathcal{A}}$.

We also calculate $F(G(h_A)) = F(A) = h_A$. Also, given $\eta : h_B \to h_A$ we have $G(\eta) : A \to B$ and $\eta_{G(\eta)} : h_B \to h_A$. We claim that $\eta = \eta_{G(\eta)}$. We have to check that for all $C \in \text{Ob}(\mathcal{A})$, $\eta_C = (\eta_{G(\eta)})_C$. We have $G(\eta) = \eta_B(\text{id}_B)$. What is $(\eta_{G(\eta)})_C$? It is the map $h_B(C) \to h_A(C)$ given by $\phi \mapsto \phi \circ G(\eta)$. We want to show that

$$\eta_C = (\eta_{G(\eta)})_C.$$

We have

$$(\eta_{G(\eta)})_C(\phi) = \phi \circ (\eta_B(\text{id}_B)).$$

We therefore want to show that $\eta_C(\phi) = \phi \circ (\eta_B(\text{id}_B))$. We want to use the fact that η is a natural transformation to show this. We have $\phi \in \text{Hom}(B, C)$. Thus, we can make the following commutative diagram:

$$\begin{array}{ccc} h_B(B) & \xrightarrow{\eta_B} & h_A(B) \\ \downarrow{\scriptstyle h_B(\phi)} & & \downarrow{\scriptstyle h_A(\phi)} \\ h_B(C) & \xrightarrow{\eta_C} & h_A(C) \end{array}$$

Take $\text{id}_B \in h_B(B)$. Mapping it to the right and then down, we get $\phi \circ \eta_B(\text{id}_B)$. Mapping it down and then to the right instead, we get $\eta_C(\phi)$. Therefore, $\eta_C(\phi) = \phi \circ \eta_B(\text{id}_B)$ by naturality of η, which was what we wanted. $\square$

Definition 7.2. A category is *concrete* if each object in that category is a set. (Sometimes you see "concrete" meaning that it has a faithful functor to Set. Jason would call that "concretizable". We will stick to his terminology.)

Corollary 7.3. *Let $\mathcal{A}$ be a small category. Then $\mathcal{A} \simeq \hat{\mathcal{A}}$, where $\hat{\mathcal{A}}$ is some concrete category, i.e., $\mathcal{A}$ is concretizable.*

We sketch a proof.

Proof. The idea is that given $A \in \mathrm{Ob}(\mathcal{A})$, we can consider $(h_A(B))_{B \in \mathrm{Ob}(\mathcal{A})}$, which is a set. We define

$$\hat{A} := \bigsqcup_{B \in \mathrm{Ob}(\mathcal{A})} \mathrm{Hom}_{\mathcal{A}}(A, B).$$

Given $f : A \to B$, we define $\hat{f} : \hat{B} \to \hat{A}$ as follows. If $\phi \in \hat{B}$, then $\phi \in \mathrm{Hom}_{\mathcal{A}}(B, C)$ for some C. We want $\hat{f}(\phi) \in \hat{A}$. Thus, we can simply define $\hat{f}(\phi) := \phi \circ f \in \mathrm{Hom}_{\mathcal{A}}(A, C)$. Let $\hat{\mathcal{A}}$ be the category whose objects are $\hat{A}$ for A an object of A and whose morphisms are $\hat{f} : \hat{B} \to \hat{A}$ for $f : A \to B$. Then $\hat{\mathcal{A}}$ is indeed a category by the Yoneda lemma. Moreover, $\mathcal{A} \simeq \hat{\mathcal{A}}^{\mathrm{op}}$ since $\hat{\mathcal{A}} \simeq \mathcal{F}$. $\square$

We will now discuss the following topics: limits and colimits, initial and final objects, constructions and examples of the former, products and equalizers, and RAPL (pronounced "rapple", stands for "right adjoints preserve limits").

Colimits are denoted $\varinjlim$. Limits are denoted $\varprojlim$. Jason says he remembers this because it's exactly the opposite of what he would expect.

Definition 7.4. Let $\mathcal{C}$ be a category. An object $I \in \mathrm{Ob}(\mathcal{C})$ is *initial* if there exists a unique morphism from it to any other object in $\mathcal{C}$. It is *final* or *terminal* if there exists a unique morphism to it from any other object in $\mathcal{C}$.

Example 7.5. The initial object in Set is $\emptyset$. The terminal object is any singleton $\{x\}$.

Example 7.6. The initial object in Ring is $\mathbb{Z}$. The terminal object is the ring with one element.

Example 7.7. Let $\mathcal{C} := \mathrm{Field}^*$ be the category of non-zero fields. This has no initial object by the following argument. A homomorphism of non-zero fields is injective. So the kernel is trivial. So if F is an initial object, then using the map $\phi : F \to \mathbb{F}_2$ we see that $F \simeq \mathbb{F}_2$. But we also have a map $F \to \mathbb{Q}$, and homomorphisms preserve field characteristic, which is a contradiction.

The category $\mathcal{C}$ also has no terminal objects by the following argument. Suppose that K is terminal. We have a map $\phi : \mathbb{Q} \to K$, so $\mathrm{char}(K) = 0$. We also have a map $\mathbb{F}_2 \to K$, so $\mathrm{char}(K) = 2$. This is a contradiction.

The moral of this story is that whenever you have some invariant preserved by the morphisms of a category such that any object has a unique value of that invariant and such that there exist two objects with different values of that invariant, then there cannot exist initial or terminal objects.

Proposition 7.8. *If initial objects exist, they are unique up to unique isomorphism.*

Proof. Suppose that I_1 and I_2 are initial. Then there exists a unique $f : I_1 \to I_2$ and a unique $g : I_2 \to I_1$. Then $g \circ f : I_1 \to I_1$ and $f \circ g : I_2 \to I_2$. But id_{I_1} is the unique map $I_1 \to I_1$ since I_1 is initial, and id_{I_2} is the unique map $I_2 \to I_2$ since I_2 is initial. Therefore, f and g are isomorphic, and since the isomorphism $f : I_1 \to I_2$ is unique, they are unique up to unique isomorphism. $\square$

We now begin our discussion of limits and colimits.

Definition 7.9. Let $\mathcal{B}$ be a category (generally small). Let $\mathcal{C}$ be a category (generally $\mathcal{B}$ is a subcategory of $\mathcal{C}$). Let $T : \mathcal{B} \to \mathcal{C}$ be a functor (usually the inclusion functor). The data $(\mathcal{C}, \mathcal{B}, T)$ is called a *diagram* based on $\mathcal{B}$. (Sometimes we will just give the functor $T : \mathcal{B} \to \mathcal{C}$.) If $\mathcal{B}$ is small, it is called a *small diagram*.

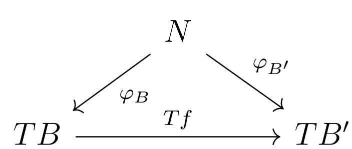
8 Oct. 1, 2019

We will use the notation $\varinjlim$ for colimits and $\varprojlim$ for limits.

Definition 8.1. Let $(\mathcal{C}, \mathcal{B}, T)$ be a diagram. A *cone* over that diagram is an object $N \in \mathrm{Ob}(\mathcal{C})$ along with maps

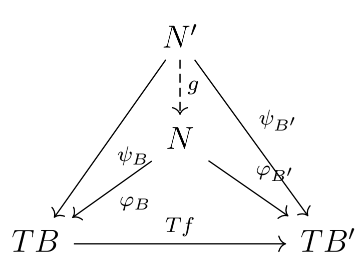
$$\varphi_B \in \mathrm{Hom}_{\mathcal{C}}(N, TB)$$

for all $B \in \mathrm{Ob}(\mathcal{B})$ such that for all $B, B' \in \mathrm{Ob}(\mathcal{B})$ and all $f \in \mathrm{Hom}_{\mathcal{B}}(B, B')$, the following diagram commutes:



We write such a cone as $(N, \{\varphi_B\}_{B \in \mathcal{B}})$.

Note that we can define a category of cones as follows. The objects are cones $(N, \{\varphi_B\})$ (over a diagram $(\mathcal{C}, \mathcal{B}, T)$). The morphisms $(N', \{\psi_B\}) \to (N, \{\varphi_B\})$ are morphisms $g \in \mathrm{Hom}_{\mathcal{C}}(N', N)$ such that the following diagram commutes (for all choices of the appropriate variables):

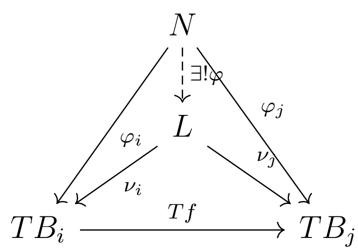


One can check that this satisfies the definition of a category.

Definition 8.2. A *limit* of the diagram $T : \mathcal{B} \to \mathcal{C}$ is a terminal object in the corresponding category of cones. We denote it by $\varprojlim T$.

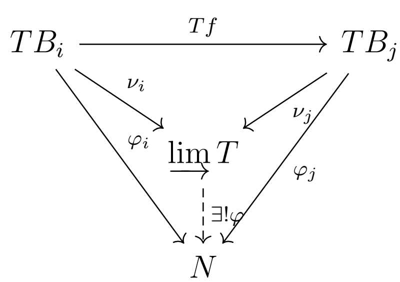
Remark 8.3. Since terminal objects are unique up to unique isomorphism, so is $\varprojlim T$ (when it exists).

Suppose $T : \mathcal{B} \to \mathcal{C}$ is a diagram and that $L := \varprojlim T$ exists. If $(N, \{\varphi_B\})$ is another cone, then there exists a unique map φ such that the following diagram commutes (for arbitrary choices of the appropriate variables):



(We have abbreviated maps φ_{B_i} to φ_i, and likewise for ν.)

Reversing the arrows gives the dual notion of a *colimit* (an initial object in the category of cocones):



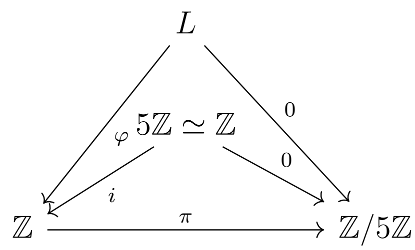
Jason writes the following table.

TABLE		
Limits	Colimits	Diagram
$\varprojlim$	$\varinjlim$	$T : \mathcal{B} \to \mathcal{C}$
Terminal object of $\mathcal{C}$	Initial object of $\mathcal{C}$	$\mathcal{B} = \emptyset$
Equalizer (kernel)	Coequalizer (cokernel)	$1_A \circlearrowright A \rightrightarrows B \circlearrowleft 1_B$
Product	Coproduct	$\mathcal{B}$ = a collection of objects with only identity maps as morphisms
Inverse limit	Direct limit	$\mathcal{B}$ = directed set
Pullback	Pushout	$\mathcal{B} = X \leftarrow Z \to Y$ (for pullbacks) or $X \to Z \leftarrow Y$ (for pullbacks)

Example 8.4. Let $\mathcal{B}$ be a subcategory of Ab. Consider the diagram

$$\mathbb{Z} \rightrightarrows \mathbb{Z}/5\mathbb{Z}$$

where the top arrow is the map π and the bottom arrow is the map 0 given by $\pi(n) := n+5\mathbb{Z}$ and $0(n) := 0$. What is the limit if $\mathcal{B} := \{\mathbb{Z}, \mathbb{Z}/5\mathbb{Z}\}$ (with $T : \mathcal{B} \to \text{Ab}$ given by inclusion). We have the following diagram:



[The bottom horizontal arrow should be an equalizer with a second 0 arrow below the π arrow. Fix this later. Probably also missing some arrows...] We claim that $\text{im}(\varphi) \subseteq \ker(\pi)$. This is just the statement that $\text{im}(\varphi) \subseteq 5\mathbb{Z}$, which is true.

Example 8.5. Let $\mathbb{Z}_p$ be the p-adic integers. Recall that a *directed set* is a set I with a preorder $\leq$ (so $\leq$ is transitive and reflexive) such that if $a, b \in I$, there exists $c \in I$ such that $a \leq c$ and $b \leq c$. We will see how these can be defined as a limit.

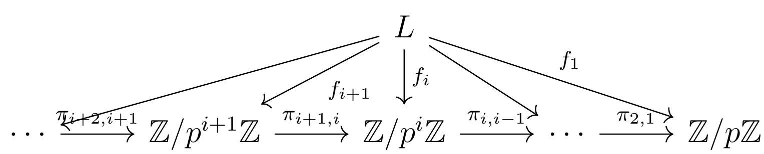
Take $I := (\mathbb{N}, \leq)$. Then I can be thought of as a category with a unique morphism $i \to j$ if and only if $i \geq j$. We have a functor $T : I \to \text{Ab}$ (or Ring) given on objects by $T(i) := \mathbb{Z}/p^i\mathbb{Z}$ and, for morphisms $\theta_{ij} : i \to j$, by morphisms $T(\theta_{ij}) =: \pi_{ij} : \mathbb{Z}/p^i\mathbb{Z} \to \mathbb{Z}/p^j\mathbb{Z}$ defined by $a + p^i\mathbb{Z} \mapsto a + p^J\mathbb{Z}$.

We claim that

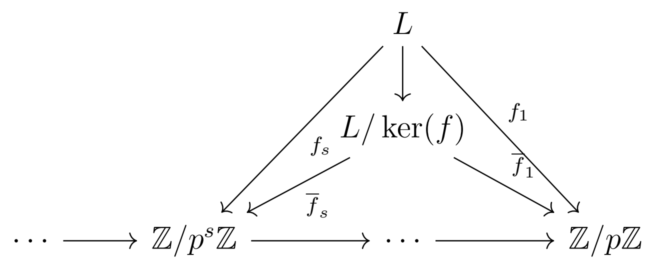
$$\mathbb{Z}_p := \{(a_1, a_2, ...) \in \prod_{i=1}^{\infty} \mathbb{Z}/p^i\mathbb{Z} \mid \pi_{i+1,i}(a_{i+1}) = a_i \text{ for all } i \geq 1\}$$

is $\varprojlim T$. Sequences satisfying the condition $\pi_{i+1,i}(a_{i+1}) = a_i$ for all $i \geq 1$ are called *admissible*. For example, if $p = 5$, then $(1 + 5\mathbb{Z}, 6 + 25\mathbb{Z}, 31 + 125\mathbb{Z}, \dots)$ is admissible, but $(1 + 5\mathbb{Z}, 7 + 25\mathbb{Z}, \dots)$ is not.)

Suppose that we have a cone

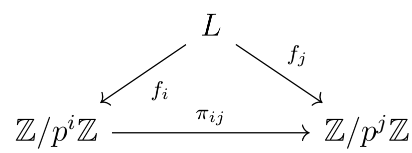


We claim that without loss of generality, we may assume L is a subset of $\prod_{i=1}^{\infty} \mathbb{Z}/p^i\mathbb{Z}$. We have a map $f : L \to \prod_{i=1}^{\infty} \mathbb{Z}/p^i\mathbb{Z}$ given by $a \mapsto (f_1(a), f_2(a), f_3(a), \dots)$. If f is not injective, then we can make the following diagram:



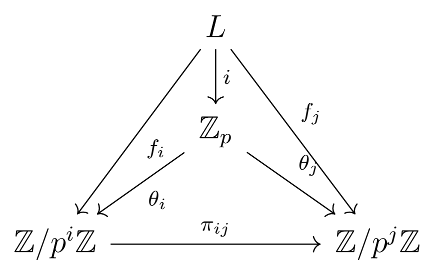
We may replace L with $L/\ker(f)$ since L is a terminal object. So we may assume that f is the inclusion of L into $\prod_{i=1}^{\infty} \mathbb{Z}/p^i\mathbb{Z}$.

We also claim that $L \subseteq \mathbb{Z}_p$. Indeed, let $(a_1, a_2, \dots) \in L$, and consider the following diagram:

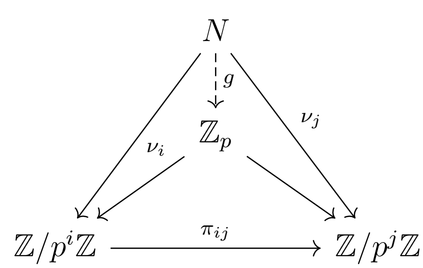


Because $\pi_{ij}(f_i(a_1, a_2, \dots)) = \pi_{ij}(a_i) = a_j = f_j(a_1, a_2, \dots)$, the diagram commutes, so $L \subseteq \mathbb{Z}_p$ as claimed.

Finally, we claim that $L \simeq \mathbb{Z}_p$. This is proved by considering the following diagram:

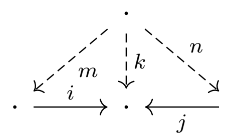


Indeed, commutativity of the triangle with vertices L, $\mathbb{Z}_p$, and $\mathbb{Z}/p^i\mathbb{Z}$ gives that i must be the identity.
Why is $\mathbb{Z}_p$ a limit? Consider the following diagram:



We have $g(n) := (\nu_1(n), \nu_2(n), \dots)$ and it satisfies the admissibility condition so lies in $\mathbb{Z}_p$. This shows that $\mathbb{Z}_p$ is a limit, completing this example.

Example 8.6. Think of $\mathbb{N}$ as a category with one object $\cdot$ and an arrow $i : \cdot \to \cdot$ for each natural number i. Consider the following diagram:



This is a cone when $mi = k = nj$. Translating back to number-theoretic language, this says that $i \mid k$ and $j \mid k$. Reasoning this way, one can show that the pullback of m and n is the object $\cdot$ together with maps $\operatorname{lcm}(m,n)/m$ and $\operatorname{lcm}(m,n)/n$. (This is also the pushout in this category.)

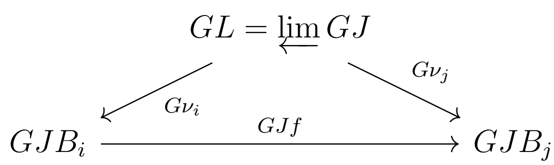
Example 8.7. The product of a collection $\{G_i\}$ of groups in Grp is the usual product $\prod G_i$.

9 Oct. 3, 2019

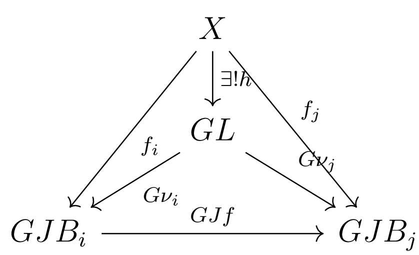
We now prove that right adjoints preserve limits.

Theorem 9.1. *Let $\mathcal{B}, \mathcal{C}, \mathcal{D}$ be categories. Let $J : \mathcal{B} \to \mathcal{D}$ be a diagram. Let $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{C}$ be an adjoint pair (so G is right adjoint to F). If $\varprojlim J$ exists, then $\varprojlim GJ$ exists in $\mathcal{C}$ and $\varprojlim GJ = G(\varprojlim J)$.*

Proof. First, observe that the following diagram forms a cone in $\mathcal{C}$:



Next, we need to show that $\varprojlim GJ = GL$. We write down the following diagram:

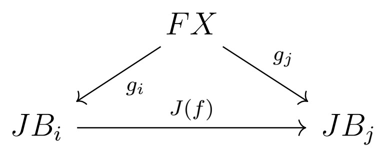


We want to show we have such a diagram. By adjointness, we have the following diagram where the horizontal arrows are isomorphisms:

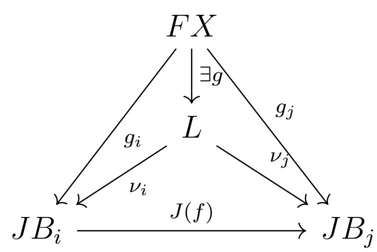
$$\begin{array}{ccc} \operatorname{Hom}_{\mathcal{D}}(FX, JB_i) & \longrightarrow & \operatorname{Hom}_{\mathcal{C}}(X, GJB_i) \\ \downarrow & & \downarrow \\ \operatorname{Hom}_{\mathcal{D}}(FX, JB_j) & \longrightarrow & \operatorname{Hom}_{\mathcal{C}}(X, GJB_j) \end{array}$$

Given g_i in the top-left corner, it gets sent to $J(f) \circ g_i = g_j$ in the bottom-left corner and then to f_j in the bottom-right corner. On the other hand, g_i gets sent to f_i in the top-right corner and then $GJ(f) \circ f_i$ in the bottom-right corner. Since the diagram commutes by adjointness, $GJ(f) \circ f_i = f_j$.

Now, for all $B_i \in \mathrm{Ob}(\mathcal{B})$, we have a map $\mathrm{Hom}_{\mathcal{D}}(FX, JB_i) \to \mathrm{Hom}_{\mathcal{C}}(X, GJB_i)$ which is an isomorphism. Let g_i be the morphism corresponding to $f_i : X \to GJB_i$ along this correspondence. Then $J(f) \circ g_i = g_j$. We thus have the following cone in $\mathcal{D}$:



Since this is a cone, there exists $g : FX \to L$. Thus, we have the following diagram:



Now, suppose there exists $g' \neq g$ making that commute. Then we have a correspondence between $\nu_i \circ g'$ and $G(\nu_i) \circ h'$, so a correspondence between g_i and f_i. This means $g = g'$, so applying G, we get the result. $\square$

Corollary 9.2. *Left adjoints preserve colimits.*

We now give a bunch of examples of products.

Suppose we have a set $\mathcal{S}$, which we regard as a category with one object (with an identity morphism) for each element and no other objects or morphisms. Suppose we have a functor $J : \mathcal{S} \to \mathcal{C}$ where $\mathcal{C}$ is some other category, and let B_i be the image of $i \in \mathrm{Ob}(\mathcal{S})$ along J. Then $\varinjlim J$ is the coproduct of $\{B_i \mid i \in \mathcal{S}\}$, i.e.,

$$\varinjlim J = \bigsqcup_i B_i.$$

Also,

$$\varprojlim J = \prod_i B_i.$$

The product in the category of pointed topological spaces is $\prod_i (X_i, x_i) = (\prod_i X_i, (x_i)_i)$. The product in the category of topological spaces is just their usual product as topological spaces. Since the Stone–Cech compactification functor β is left adjoint to the forgetful functor by Assignment 1, and since left adjoints preserve colimits,

$$\beta\left(\bigsqcup X_i\right) = \bigsqcup \beta(X_i)$$

when everything exists.

Given a singleton $*$ that's sent to x in the topological space X and y in the topologial space Y, the pushout along the inclusions in the category of topological spaces is $X \bigsqcup Y/\sim$ where $x \sim y$. The pushout in the category of pointed topological spaces is similar except you want $*$ to be sent to x in X and y in Y (where the pointed spaces are (X, x) and (Y, y)), and then your distinguished point in $X \bigsqcup Y$ is $x \sim y$.

In the categories Ab and Grp, the product is just the product of abelian groups (resp. groups). The coproduct in Ab is the direct sum. The coproduct in Grp is the free product. More generally, if we have homomorphisms $i : A \to G$ and $j : A \to H$ of groups, then the colimit along these maps is the amalgamated free product $G *_A H$, defined as the group generated by G and H where we set $j(a) = i(a)$ for all $a \in A$.

10 Oct. 8, 2019

Definition 10.1. We say a functor $F : \mathcal{C} \to \mathcal{D}$ is *continuous* if $F(\varprojlim J) = \varprojlim FJ$ for all diagrams J. Similarly, if F preserves colimits, it is *cocontinuous*.

Definition 10.2. If all small limits (resp. small colimits) exist in a category $\mathcal{C}$, we say $\mathcal{C}$ is *complete* (resp. *cocomplete*). (Note that saying small limits exist just means that for all small diagrams $F : \mathcal{B} \to \mathcal{C}$, $\varprojlim F$ exists.) If a category is both complete and cocomplete, we say it is *bicomplete*.

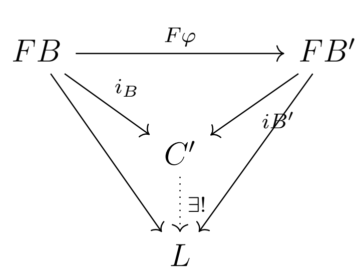
Theorem 10.3. *Let $\mathcal{C}$ be a category. Then $\mathcal{C}$ is complete if and only if all small products exist in $\mathcal{C}$ and equalizers exist in $\mathcal{C}$.*

Proof. We will only prove that a category is cocomplete if and only if (small) coproducts and coequalizers exist.

The forward direction is immediate. It remains to prove that if small coproducts and coequalizers exist, then the category is cocomplete. Let $\mathcal{B}$ be a small category. Let $F : \mathcal{B} \to \mathcal{C}$ be a diagram. We need to show $\varinjlim F$ exists. Let

$$C' := \bigsqcup_{B \in \mathrm{Ob}(\mathcal{B})} F(B). \quad (*)$$

This exists because $\mathrm{Ob}(\mathcal{B})$ is a set and coproducts exist. Given a morphism $\varphi : B \to B'$ in $\mathcal{B}$, we want to have a picture like this in $\mathcal{C}$:



Note that

$$\mathrm{Mor}(\mathcal{B}) = \bigsqcup_{B,B'\in \mathrm{Ob}(\mathcal{B})} \mathrm{Hom}_{\mathcal{B}}(B, B').$$

All of the Hom-sets are indeed sets because $\mathcal{B}$ is small. Therefore, $\mathrm{Mor}(\mathcal{B})$ is a set. Given a morphism $\varphi : B \to B'$, we can define *source* and *range* maps by $s(\varphi) := B$ and $r(\varphi) := B'$. Let

$$C := \bigsqcup_{\varphi\in \mathrm{Mor}(\mathcal{B})} F(s(\varphi)). \quad (**)$$

We do this because we are about to use it to make a coequalizer. We wish to construct two maps $\phi, \psi : C \to C'$. Let $\varphi_i : B \to B_i$ be maps for $i = 1, 2, 3$ where the $\{B_i\}_{i=1,2,3}$ are some objects in $\mathcal{B}$. Let $\varphi_4 : B' \to B_4$ be some other map between two objects of $\mathcal{B}$. Then these induce maps $\alpha_{B,\varphi_i} : FB \to C$ for $i = 1, 2, 3$ and $\alpha_{B,\varphi_4} : FB' \to C$.

Given $\varphi : B \to B'$, we have a diagram

$$\begin{array}{ccc} FB & \xrightarrow{\alpha_{B,\varphi}} & C \\ \Big\downarrow i_B & \swarrow{\scriptstyle \exists!\Phi} & \\ C' & & \end{array}$$

Since C is defined as a coproduct, in this new picture C' will be a colimit so will satisfy the universal property, which means there exists a unique Φ satisfying $\Phi \circ \alpha_{B,\varphi} = i_B$. Similarly, we have the picture

$$\begin{array}{ccc} FB & \xrightarrow{\alpha_{B,\varphi}} & C \\ \Big\downarrow F\varphi & & \Big\downarrow \exists!\Psi \\ FB' & \xrightarrow{i_{B'}} & C' \end{array}$$

where there exists a unique Ψ satisfying $\Psi \circ \alpha_{B,\varphi} = i_{B'} \circ F\varphi$.

We now claim that $\varinjlim F = \varinjlim(C \xrightarrow{\Phi,\Psi} C')$. Here this notation just means the coequalizer along those two maps. Because the RHS is a limit of a coequalizer, it exists by assumption. We should have a map $h : C' \to L$ where L is the limit such that $h \circ \Phi = h \circ \Psi$. Consider the following diagram:

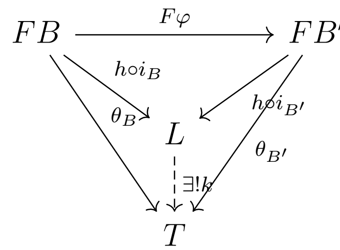
$$\begin{array}{ccc} FB & \xrightarrow{F\varphi} & FB' \\ \Big\downarrow h\circ i_B & \swarrow{\scriptstyle h\circ i_{B'}} & \\ L & & \end{array}$$

We need to check that this commutes, i.e., that $h \circ i_{B'} \circ F\varphi = h \circ i_B$. We know that $h \circ \Phi = h \circ \Psi$. Thus,

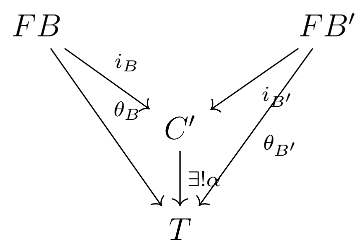
$$h \circ \Phi \circ \alpha_{B,\varphi} = h \circ \Psi \circ \alpha_{B,\varphi},$$

so $h \circ i_B = h \circ i_{B'} \circ F\varphi$ as desired.

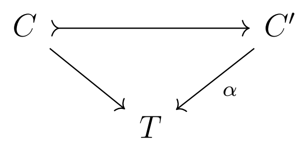
Next, we want to show that L is initial in the category of cocones. Suppose T is another cocone equipped with maps $\theta_B : FB \to T$ and $\theta_{B'} : FB' \to T$. We want to show there exists a unique $k : L \to T$ such that the following diagram commutes:



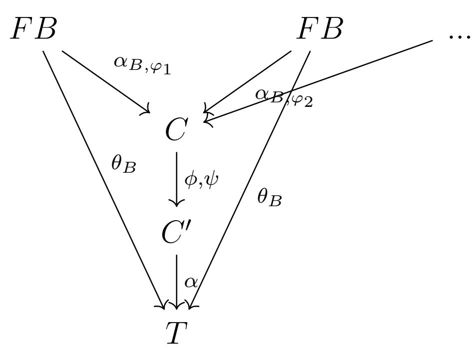
We know that the following diagram commutes:



Therefore, we claim the following diagram also commutes:



Here I am using the swallowtail arrow to represent the two maps $\Phi : C \to C'$ and $\Psi : C \to C'$ of the coequalizer. We need to show that $\alpha \circ \Phi = \alpha \circ \Psi$. Consider the following diagram:



Since C is an initial object for this cocone category and since $\theta_B = \alpha \circ \Phi \circ \alpha_{B,\varphi} = \alpha \circ \Psi \circ \alpha_{B,\varphi}$, we must have $\alpha \circ \Phi = \alpha \circ \Psi$ for this diagram to commute (which it does because C is initial). This implies commutativity of the second-last diagram as well, which completes the proof. □

Corollary 10.4. *The following categories are bicomplete:*

(i) Ab. The products are $\prod_{i \in I} A_i$ *and the coproducts are* $\bigoplus_{i \in I} A_i$.
(ii) Grp. The products are $\prod_{i \in I} G_i$ *and the coproducts are* $*_{i \in I} G_i$. *(Note: If someone knows*

a good way to write big asterisks in LaTeX, let me know!)
(iii) R-Mod, R a ring. The products are $\prod M_i$ *and the coproducts are* $\bigoplus' M_i$.
(iv) Comm-rings (i.e., the category of commutative rings). The products are $\prod R_i$ *and the coproducts are* $\bigotimes_{\mathbb{Z}}^{Res} R_i$. *This notation indicates that the tensor products are taken over* $\mathbb{Z}$ *and that the elementary tensors are finite sums of the form* $r_{i_1} \otimes r_{i_2} \otimes \cdots$ *where all but finitely many* r_{i_j}*'s are equal to 1. Indeed, if*

$$R_i = \mathbb{Z}[x_s^{(i)} : s \in S]/\langle J_i \rangle,$$

then

$$\bigotimes_{\mathbb{Z}}^{Res} R_i \simeq \mathbb{Z}\left[\bigcup_i x_s^{(i)}\right] / \langle J_i : i \in I \rangle.$$

(v) Set. The products are $\prod X_i$ *and the coproducts are* $\bigsqcup X_i$ *(disjoint union).*
(vi) Top.

One can show that the equalizer of maps f, g in R-Mod is $\ker(f-g)$ and the coequalizer is $\operatorname{coker}(f-g)$. In the categories Set and Top, the equalizer of maps f and g is $\{x \in X \mid f(x) = g(x)\}$.

11 Oct. 10, 2019

ASSIGNMENT 2 IS NOW DUE OCT. 24, NOT OCT. 22.

In a concrete category, we expect that the equalizer of maps $f, g : X \to Y$ is $\{x \in X \mid f(x) = g(x)\}$, and we expect that the coequalizer is $Y/\sim$ where $\sim$ is the smallest equivalence relation under which $f(x) \sim g(x)$ for all $x \in X$ and $Y/\sim$ is an object in the category.

Example 11.1. In Grp, given homomorphisms $f, g : G \to H$, the coequalizer is H/N where N is the smallest normal subgroup of H containing $f(x)g(x)^{-1}$ for all $x \in G$.

Let R be a commutative ring. In R-Mod, we have notions like "injective", "projective", "flat", "faithfully flat", and "free". Given $A \in \mathrm{Ob}(R\text{-Mod})$, we have functors F, G given by

$$F(M) := M \otimes_R A$$

and for $f : M \to N$, $F(f) := f \otimes \mathrm{id}_A : M \otimes_R A \to N \otimes_R A$, and $G := \mathrm{Hom}_{R\text{-Mod}}(A, -)$. Then F is left adjoint to G, so F preserves colimits.

Definition 11.2. A sequence of modules and module homomorphisms

$$M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3$$

is *exact* at M_2 if $\mathrm{im}(f) = \ker(g)$ (which implies $g \circ f = 0$). A sequence of modules and homomorphisms is said to be an *exact sequence* if it is exact at every module.

Example 11.3. The sequence

$$\mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/2\mathbb{Z}$$

is exact, where the first map is multiplication by 2 and the second is reduction modulo 2.

In general, $M \xrightarrow{f} N \to 0$ is exact if and only if f is onto, and $0 \to M \xrightarrow{g} N$ is exact if and only if g is one-to-one.

Definition 11.4. A sequence of the form

$$0 \to M \to N \to Q \to 0$$

that is exact is called a *short exact sequence.*

The following result follows from tensor-Hom adjunction.

Corollary 11.5. *If $M \xrightarrow{f} N \xrightarrow{g} Q \to 0$ is exact in R-Mod, then so is*

$$M \otimes_R A \xrightarrow{f \otimes id_A} N \otimes_R A \xrightarrow{g \otimes id_A} Q \otimes_R A \to 0.$$

Does F preserve limits? No. If F preserved limits, it would preserve kernels. But the following example shows it does not.

Example 11.6. Consider the exact sequence

$$0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z}.$$

(We consider these as R-modules over the ring $R := \mathbb{Z}$, and $\times 2$ denotes the multiplication by 2 map.) Take $A := \mathbb{Z}/2\mathbb{Z}$ and tensor on the right with A. Then we obtain the sequence

$$0 \to \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \xrightarrow{(\times 2) \otimes \mathrm{id}} \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}.$$

But this sequence is no longer exact because the kernel of $(\times 2) \otimes \mathrm{id}$ is non-trivial. Indeed, $1 \otimes 1 \neq 0$ in $\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$, but $1 \otimes 1$ gets sent to

$$2 \otimes 1 = 2(1 \otimes 1) = 1 \otimes 2 = 1 \otimes 0 = 0$$

in $\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$.

Definition 11.7. An R-module A is *flat* if any of the following equivalent conditions holds:

(i) If $M \xrightarrow{f} N$ is injective, then

$$M \otimes_R A \xrightarrow{f \otimes \mathrm{id}_A} N \otimes_R A$$

is injective.

(ii) The functor $F = - \otimes_R A$ preserves exact sequences.

Definition 11.8. A module A is *faithfully flat* if

$$0 \to M \to N \to Q \to 0$$

is exact if and only if

$$0 \to M \otimes_R A \to N \otimes_R A \to Q \otimes_R A \to 0$$

is exact.

Remark 11.9. Dan asks where the term "flat" comes from. Jason doesn't know. I google it and find out that Serre came up with this term in his GAGA paper. There is an MO question about this where Brian Conrad comments that he asked Serre where the term came from and Serre did not remember. However, Serre emphasized that the importance of flatness was first understood by Grothendieck.

Example 11.10. We have $\mathbb{Z}/n\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q} = (0)$ because

$$1 \otimes 1 = 1 \otimes (n(1/n)) = n(1 \otimes (1/n)) = n \otimes (1/n) = 0.$$

We put this fact to the following use. Consider the following short sequence (which is not exact at $\mathbb{Z}/4\mathbb{Z}$). (We add the extra 0 on the right just so it matches the definition of a short exact sequence.)

$$0 \to \mathbb{Z}/2\mathbb{Z} \hookrightarrow \mathbb{Z}/4\mathbb{Z} \to 0 \to 0.$$

Then applying $- \otimes_{\mathbb{Z}} \mathbb{Q}$ gives the sequence

$$0 \to 0 \to 0 \to 0 \to 0$$

which is exact. Therefore, $\mathbb{Q}$ is not faithfully flat as a $\mathbb{Z}$-module. However, $\mathbb{Q}$ is flat as a $\mathbb{Z}$-module.

Remark 11.11. If A is flat, then $- \otimes A$ preserves kernels but not necessarily products. Also, $- \otimes A$ is a left adjoint, so it preserves coproducts.

Remark 11.12. Someone asks whether tensoring (on the right) with flat modules preserves products (or more generally limits). Jason says that it does not and comes up with the following example. As a $\mathbb{Z}$-module, $\mathbb{Q}$ is flat. Let $A := \mathbb{Q}$ and $R := \mathbb{Z}$. Let $M_i := \mathbb{Z}/i\mathbb{Z}$. Then

$$\left(\prod_i M_i\right) \otimes_R A$$

is non-trivial, but

$$\prod_i (M_i \otimes_R A)$$

vanishes because each tensor product $\mathbb{Z}/i\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q}$ vanishes. (To define the tensor product $(\prod_i M_i) \otimes_R A$, note that we have a map $\mathbb{Z} \to \prod_i M_i$ given by $1 \mapsto (1 + i\mathbb{Z})_i$.

Similarly, $\operatorname{Hom}(A,-)$ preserves kernels, so if

$$0 \to M \xrightarrow{f} N \xrightarrow{g} Q$$

is exact, then so is

$$0 \to \operatorname{Hom}(A,M) \xrightarrow{f\circ -} \operatorname{Hom}(A,N) \xrightarrow{g\circ -} \operatorname{Hom}(A,Q),$$

i.e., $\operatorname{Hom}(A,-)$ is *left exact*.

Definition 11.13. A module P is called *projective* if any of the following equivalent conditions holds:

(i) If $M \xrightarrow{f} N \to 0$ is exact, then so is

$$\operatorname{Hom}(P,M) \xrightarrow{f\circ -} \operatorname{Hom}(P,N) \to 0.$$

(ii) For every surjective module homomorphism $f : N \to M$ and every module homomorphism $g : P \to M$, there exists a module homomorphism $h : P \to N$ such that $f \circ h = g$. (We do not require uniqueness.) This can be summarized in the following diagram:

$$\begin{array}{ccc} & & N \\ & \overset{\exists h}{\nearrow} & \big\downarrow f \\ P & \xrightarrow{g} & M \end{array}$$

(iii) The functor $\operatorname{Hom}(P,-)$ preserves exact sequences.

Remark 11.14. Adina asks whether there's a notion of "faithfully projective" that is analogous to being faithfully flat. Jason says that in R-Mod, being faithfully projective is equivalent to being projective and faithfully flat. This should all generalize to abelian categories.

Proposition 11.15. *Let R be a commutative ring. Then every free R-module is faithfully flat.*

Proof. We can write a free R-module in the form $\bigoplus_{i\in I} R$. Then

$$M \otimes \left(\bigoplus_{i\in I} R\right) \simeq \bigoplus_{i\in I}(M \otimes_R R) \simeq \bigoplus_{i\in I} M.$$

Therefore,

$$0 \to M \xrightarrow{f} N \xrightarrow{g} Q \to 0$$

is exact if and only if

$$0 \to \bigoplus_i M \xrightarrow{(f,f,\dots)} \bigoplus_i N \overset{(g,g,\dots)}{\twoheadrightarrow} \bigoplus_i Q \to 0.$$

But this latter sequence is just

$$0 \xrightarrow{M} \otimes \left(\bigoplus_i R \right) \to N \otimes \left(\bigoplus_i R \right) \to Q \otimes \left(\bigoplus_i R \right) \to 0.$$

□

The notion of projective objects extends to an arbitrary category.

Definition 11.16. Given a category $\mathcal{C}$ and $M \xrightarrow{f} N$ in $\mathcal{C}$, we say that f is an *epimorphism* if whenever $g \circ f = h \circ f$ for some other morphisms g, h, then $g = f$.

In Set, the epimorphisms are precisely the onto maps. However, epimorphisms are not necessarily onto in every category, as the following example shows.

Example 11.17. Work in Haus, the category of Hausdorff topological spaces. The map $f : (0,1) \to S^1$ given by $x \mapsto e^{2\pi i x}$ is not onto because $\text{im}(f) = S^1 \setminus \{1\}$. However, suppose we have maps $g, h : S^1 \to Z$ for some other Hausdorff topological space Z such that $f \circ f = h \circ f$. Then we can form a map $\Psi : S^1 \to Z \times Z$ given by $x \mapsto (g(x), h(x))$. Consider the diagonal $\Delta(Z) := \{(z,z) \mid z \in Z\} \subseteq Z \times Z$. Since Z is Hausdorff, $\Delta(Z)$ is closed. Since Ψ is continuous, $\Psi^{-1}(\Delta(Z))$ is closed. But $S^1 \setminus \{1\} \subseteq \Psi^{-1}(\Delta(Z))$, so $S^1 = \overline{S^1 \setminus \{1\}} \subseteq \Psi^{-1}(\Delta(Z))$. It follows that $g(x) = h(x)$ for all $x \in S^1$, so $g = h$. Therefore, f is an epimorphism.

Note that $\text{Hom}(A, -)$ is covariant and $\text{Hom}(-, A)$ is contravariant. In particular, if

$$M \xrightarrow{f} N \xrightarrow{g} Q \to 0$$

is exact, then so is

$$0 \to \text{Hom}(Q, A) \xrightarrow{-\circ g} \text{Hom}(N, A) \xrightarrow{-\circ f} \text{Hom}(M, A).$$

We can use this duality to define the notion of an "injective" module by analogy with the projective case.

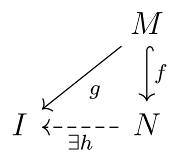
Definition 11.18. An R-module I is *injective* if any of the following equivalent conditions holds:

(i) If $0 \to M \xrightarrow{f} N$ is exact, then

$$\text{Hom}(N, I) \xrightarrow{-\circ f} \text{Hom}(M, I) \to 0$$

is exact.

(ii) For every injective module homomorphism $f : M \to N$ and every module homomorphism $g : M \to I$, there exists a module homomorphism $h : N \to I$ such that $h \circ f = g$. (We do not require uniqueness.) This can be summarized in the following diagram:



(iii) The functor $\mathrm{Hom}(-, I)$ reverses short exact sequences and preserves exactness.

12 Oct. 22, 2019

Last week was reading week.

Last time we discussed the following concepts: flatness, injectivity, faithful flatness, projectivity. We recall what these mean.

(i) The R-module M is *flat* if the functor $F := - \otimes_R M : R\text{-Mod} \to R\text{-Mod}$ is exact. (Note that since this functor is always right exact, we need only prove left exactness.)

(ii) The R-module M is *faithfully flat* if the sequence of R-modules

$$0 \to M \to M' \to M'' \to 0$$

is exact if and only if

$$0 \to F(M) \to F(M') \to F(M'') \to 0$$

is exact.

(iii) The R-module P is *projective* if $\mathrm{Hom}(P, -)$ is exact. (Note that since this functor is always left exact, we need only prove right exactness.)

(iv) The R-module I is *injective* if $\mathrm{Hom}(-, I)$ is exact. (Note that since this functor is always left exact, we need only prove right exactness.)

Suppose P is projective. Then if

$$M \xrightarrow{g} M' \to 0$$

is exact, so is

$$\mathrm{Hom}(P, M) \to \mathrm{Hom}(P, M') \to 0$$

where the first map is given by $\psi \mapsto g \circ \psi$. In R-Mod, P is projective if and only if P is a direct summand of a free module. Why? Suppose P is projective. Then:

(i) There exists a free module $R^{\oplus I}$ that surjects onto P (just take all the generators), so we get an exact sequence

$$R^{\oplus I} \xrightarrow{g} P \to 0.$$

Thus, we have an exact sequence

$$\mathrm{Hom}(P, R^{\oplus I}) \to \mathrm{Hom}(P, P) \to 0.$$

Take the identity map id : $\mathrm{Hom}(P, P)$. Then there exists $\bar{g} : P \to R^{\oplus I}$ such that the following diagram commutes:

$$\begin{array}{ccc} P & \xrightarrow{\bar{g}} & R^{\oplus I} \\ & \overset{\mathrm{id}}{\searrow} & \downarrow g \\ & & P \end{array}$$

Let $Q := \ker(g)$. Then we have an exact sequence

$$0 \to Q \hookrightarrow R^{\oplus I} \xrightarrow{g} P \to 0.$$

Definition 12.1. We say a short exact sequence

$$0 \to A \to B \to C \to 0$$

splits if $B \simeq A \oplus C$.

Example 12.2. The short exact sequence

$$0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/2\mathbb{Z} \to 0$$

does not split.

Definition 12.3. Suppose we have a short exact sequence

$$0 \to A \to B \xrightarrow{g} C \to 0.$$

A map $h : C \to B$ is called a *section* if $g \circ h = \mathrm{id}_C : C \to C$.

Lemma 12.4. *If we have a section, then the short exact sequence splits.*

Proof. We do not prove this in class. Basically, given a short exact sequence

$$0 \to A \to B \to C \to 0,$$

we define $\psi : A \oplus C \to B$ by $(a, c) \mapsto f(a) + h(c)$, and then we prove this is an isomorphism using the existence of a section. □

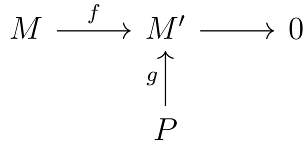
Now, returning to our earlier setting, since the diagram

$$\begin{array}{ccc} P & \xrightarrow{\bar{g}} & R^{\oplus I} \\ & \overset{\mathrm{id}}{\searrow} & \downarrow g \\ & & P \end{array}$$

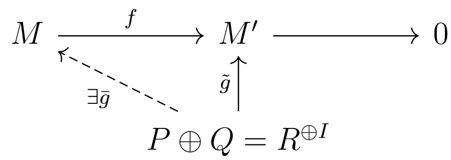
commutes and since

$$0 \to Q \hookrightarrow R^{\oplus I} \xrightarrow{g} P \to 0$$

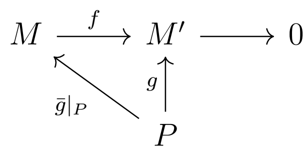
is a short exact sequence, $\bar{g} : P \to R^{\oplus I}$ is a section. Therefore, the short exact sequence splits. Therefore, $R^{\oplus I} \simeq Q \oplus P$.
Conversely, suppose P is a direct summand of $R^{\oplus I}$. Then $R^{\oplus I} = P \oplus Q$. We want to show P is projective. We have the following diagram in which all sequences are exact:



We define a map $\tilde{g} : P \oplus Q = \mathbb{R}^{\oplus I} \to M'$ by $\tilde{g}(p, q) := g(p)$. Then we also have the following diagram where all sequences are exact:



(I will explain the "$\exists \bar{g}$" part shortly.) We remark that free modules are projective. Indeed, a map $g : R^{\oplus I} \to M'$ is determined by where it sends the basis elements $x \in I$. Since f is onto, for every $x \in I$, there exists $a_x \in M$ such that $f(a_x) = g(x)$. So since $R^{\oplus I}$ is free, there exists $\bar{g} : R^{\oplus I} \to M$ making the diagram commute. Now, $\bar{g}|_P$ gives the following commutative diagram in which all sequences are exact:



We have proved the following result.

Proposition 12.5. *An R-module P is projective if and only if P is a direct summand of $R^{\oplus I}$ for some generating set I.*

Definition 12.6. A projective module P is *stably free* if there exists $n \in \mathbb{N}$ such that $P \oplus R^n \simeq R^{\oplus I}$.

Proposition 12.7. *Let P be a projective module. Then there exists some free module $R^{\oplus J}$ (with J not necessarily finite) such that $P \oplus R^{\oplus J}$ is free.*

Proof. The following trick is known as *Eilenberg's swindle*. Since P is projective, there exists Q such that $P \oplus Q \simeq R^{\oplus I}$ for some set of generators I. Then

$$(P \oplus Q) \oplus (P \oplus Q) \oplus \cdots = R^{\oplus I} \oplus R^{\oplus I} \oplus \cdots ,$$

which is free. But we also have

$$(P \oplus Q) \oplus (P \oplus Q) \oplus \cdots = P \oplus (Q \oplus P) \oplus (Q \oplus P) \oplus \cdots$$

$$= P \oplus (R^{\oplus I} \oplus R^{\oplus I} \oplus \cdots),$$

where the parenthetical summand is free. □

Proposition 12.8. *There exists a commutative ring R and a projective R-module P such that $P \oplus R \simeq R^3$ but $P \not\simeq R^2$.*

Proof. This is known as *Swan's counterexample.* Let

$$R := \mathbb{R}[x, y, z]/(x^2 + y^2 + z^2 - 1).$$

(These are just functions vanishing on the 2-sphere.) We have an exact sequence

$$R^3 \xrightarrow{f} R \to 0$$

where $f : R^3 \to R$ is given by $f(a, b, c) := ax + by + cz$. We claim that f is onto. Indeed, let $u \in R$, so $f(ux, uy, uz) = ux^2 + uy^2 + uz^2 = u$ in R. In particular, $f(x, y, z) = 1$.

Now, let $P := \ker(f)$. Then we have a short exact sequence

$$0 \to P \hookrightarrow R^3 \xrightarrow{f} R \to 0.$$

Here $f(x, y, z) = 1$, and since 1 generates R, we can define a homomoprhism $\bar{f} : R \to R^3$ by $\bar{f}(1) := (x, y, z)$. This is a section, so our short exact sequence splits and we have $R^3 \simeq P \oplus R$. We claim that $P \not\simeq R^2$.

Suppose for the sake of contradiction that $P \simeq R^2$. Then $P \subseteq R^3$ has an R-basis of size 2, say

$$v = (a(x, y, z), b(x, y, z), c(x, y, z)) \text{ and } w = (d(x, y, z), e(x, y, z), f(x, y, z)).$$

Recall that we have an isomorphism $R^3 \simeq P \oplus R$. Thus, we have a map $P \oplus R \to R^3$ given by $(v, 0) \mapsto v$, $(w, 0) \mapsto w$, and $(0, 1) \xrightarrow{\bar{f}} (x, y, z)$. So $\{v, w, (x, y, z)\}$ is a basis for R^3. Therefore, letting $\{e_1, e_2, e_3\}$ be the standard basis for R^3, there exist $r_{ij} \in R$ for $1 \leq i, j \leq 3$ such that

$$r_{11}v + r_{12}w + r_{13}u = e_1,$$

$$r_{21}v + r_{22}w + r_{23}u = e_2,$$

$$r_{31}v + r_{32}w + r_{33}u = e_3.$$

Then there exists an invertible matrix $A \in \mathrm{GL}_3(R)$ such that

$$A = \begin{pmatrix} - & v & - \\ - & w & - \\ - & u & - \end{pmatrix} = I.$$

It follows that

$$\det \begin{pmatrix} - & v & - \\ - & w & - \\ - & u & - \end{pmatrix}$$

is non-zero whenever we evaluate (x, y, z) at a point in S^2. Swan showed that

$$\det \begin{pmatrix} - & v & - \\ - & w & - \end{pmatrix}$$

vanishes at some point $(x, y, z) = (\alpha, \beta, \gamma) \in S^2$. He used topology to do this, as follows. No non-topological proof of this theorem is known (afaik).

Recall that if $\psi : S^2 \to S^2$ is continuous, then either $\psi(\alpha, \beta, \gamma) = (\alpha, \beta, \gamma)$ or $(\psi(\alpha, \beta, \gamma) = (-\alpha, -\beta, -\gamma)$ for some $(\alpha, \beta, \gamma) \in S^2$. We can view v as a map

$$v = (a(x, y, z), b(x, y, z), c(x, y, z)) : \mathbb{R}^3 \to \mathbb{R}^3.$$

If $v(\alpha, \beta, \gamma) = 0$ for some $(\alpha, \beta, \gamma) \in S^2$, we have a contradiction, since then the determinant vanishes at (α, β, γ). We may thus assume that $v(\alpha, \beta, \gamma) \neq 0$ for all $(\alpha, \beta, \gamma) \in S^2$. Then the map

$$v \mapsto v/||v|| \in S^2$$

is a continuous map $\psi : S^2 \to S^2$. So there exists (α, β, γ) such that

$$v(\alpha, \beta, \gamma) = ||v(\alpha, \beta, \gamma)||(\alpha, \beta, \gamma).$$

But now the first row of our 3×3 matrix from earlier is a scalar multiple of the third row. $\square$

Definition 12.9. A map $f : A \to B$ is a *monomorphism* (like an injection) if whenever $h_1, h_2 : C \to A$ are such that $f \circ h_1 = f \circ h_2$, we have $h_1 = h_2$.

Remark 12.10. Monomorphisms do not have to be one-to-one, even when the objects are sets.

Example 12.11. An abelian group $(A, +)$ is *divisible* if for all $a \in A$ and $n \in \mathbb{Z} \setminus \{0\}$, there exists $b \in A$ such that $nb = a$. For example, $\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$ are divisible. Let $\mathcal{C}$ be the category of divisible abelian groups. Then $\pi : \mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$ given by $a \mapsto a + \mathbb{Z}$ is not one-to-one but is a monomorphism.

Indeed, suppose $h_1, h_2 : A \to \mathbb{Q}$, $h_1 \neq h_2$, and $\pi \circ h_1 = \pi \circ h_2$. Then there exists $a \in A$ such that $h_1(a) \neq h_2(a)$. Let $\alpha := h_1(a)$ and $\beta := h_2(a)$. We have $\pi(\alpha) = \pi(\beta)$ if and only if $\alpha = \beta + n$ for some $n \in \mathbb{Z}$. But $n \neq 0$ since $\alpha \neq \beta$ by assumption. Since A is divisible, there exists $a' \in A$ such that $(2n)a' = a$.

We have $h_1(2na') = h_1(a) = \alpha$, so $h_1(a') = \alpha/(2n)$ in $\mathbb{Q}$. Similarly, $h_2(a') = \beta/(2n)$. But $\alpha/(2n) - \beta/(2n) = n/(2n) = 1/2$, so $\pi \circ h_1(a') \neq \pi \circ h_2(a')$, which is a contradiction.

We can now say that I is an injective object in $\mathcal{C}$ if whenever $A \xrightarrow{f} B$ is a monomorphism, the map $\text{Hom}_{\mathcal{C}}(B, I) \to \text{Hom}_{\mathcal{C}}(A, I)$ given by $\psi \mapsto \psi \circ f$ is onto.

We will now discuss the Govorov–Lazard theorem.

Definition 12.12. We say $\mathcal{B}$ is a *filtered* category if every finite subcategory of $\mathcal{B}$ has a cocone.

The same proof as before shows that it suffices to show that we have a cocone for coproduct and coequalizer diagrams.

Theorem 12.13. *Suppose $\{M_i\}$ is a family of objects in some small filtered subcategory of R-Mod. Then $\varinjlim M_i \simeq \bigsqcup M_i/\sim$ where the equivalence relation $\sim$ is given by $M_i \ni m \sim m' \in M_j$ if there exists some M_k and maps $f : M_i \to M_k$, $g : M_j \to M_k$ such that $f(m) = g(m')$. (Note that the R-module structure on $\bigsqcup M_i/\sim$ is given by $[m_1] + [m_2] := [f(m_1) + g(m_2)]$, which is well-defined.*

We are allowed to assume this last result for Assignment 2.

Definition 12.14. If $F : \mathcal{B} \to \mathcal{A}$ is a diagram (i.e., a functor) and $\mathcal{B}$ is filtered, then $\varinjlim F$ is said to be a *filtered colimit*.

13 Oct. 24, 2019

I missed today's class. Below, I have typeset Wilson's notes combined with parts of Chris's notes. It seems they just stated and proved the *Govorov–Lazard theorem*, which I state now.

Theorem 13.1. *Let R be a commutative ring. Then $M \in Ob(R\text{-}Mod)$ is flat if and only if $M = \varinjlim \mathcal{B}$ where $\mathcal{B}$ is a filtered subcategory of the subcategory of R-Mod consisting of free R-modules with the usual module homomorphisms.*

Proof. We will prove that if the M_i in the colimit come from a a filtered subcategory $\mathcal{B}$ of R-Mod whose objects are free, then $\varinjlim M_i$ is flat. The idea is to suppose that we have an exact sequence

$$0 \to N' \xrightarrow{f} N.$$

We wish to show that in this case,

$$0 \to M \otimes N' \xrightarrow{\mathrm{id} \otimes f} M \otimes N$$

is exact. Let $F', F : R\text{-Mod} \to R\text{-Mod}$ be given on objects by $F'(L) := L \otimes N'$ and $F(L) := L \otimes N$. With this notation, we need to show that

$$0 \to F'(M) \to F(M)$$

is exact. For each $M_0 \in \mathrm{Ob}(\mathcal{B})$, we have a map $M_0 \otimes N' \xrightarrow{\mathrm{id} \otimes f} M_0 \otimes N$, and this is a map $F(M_0) \to F'(M_0)$, so we claim it defines a natural transformation. (Call this map α_{M_0} and the corresponding natural transformation α.) Suppose $h : M_0 \to M_0'$ is a morphism for some M_0', and consider the following diagram:

$$\begin{array}{ccc} F(M_0) & \xrightarrow{\alpha_{M_0}} & F'(M_0) \\ \big\downarrow{\scriptstyle F(h)} & & \big\downarrow{\scriptstyle F'(h)} \\ F(M_0') & \xrightarrow{\alpha_{M_0'}} & F'(M_0') \end{array}$$

Suppose $m \otimes n' \in F(M_0)$. Following it right and down, we get

$$m \otimes n' \mapsto m \otimes f(n') \mapsto h(m) \otimes f(n').$$

Following it down and right instead, we get

$$m \otimes n' \mapsto h(m) \otimes n' \mapsto h(m) \otimes f(n').$$

Since $M = \varinjlim \mathcal{B}$, we claim we have the exact sequence

$$M \otimes N' = (\varinjlim \mathcal{B}) \otimes N' \simeq \varinjlim(M_i \otimes N') \xhookrightarrow{\varinjlim f} \varinjlim(M_i \otimes N) \simeq (\varinjlim M_i) \otimes N = M \otimes N.$$

The two isomorphisms follow from LAPC (left adjoints preserve colimits) and the fact that tensor product is a left adjoint. It remains to prove that the map

$$g : \varinjlim(M_i \otimes N') \xrightarrow{\varinjlim f} \varinjlim(M_i \otimes N)$$

is actually injective. This map is given by

$$\bigsqcup M_i \otimes N' \xrightarrow{\mathrm{id} \otimes f} \bigsqcup M_i \otimes N' / \sim$$

where $\sim$ is the usual equivalence relation. Suppose $x \in \bigsqcup M_i \otimes N' / \sim$ satisfies $g(x) \sim 0$. Then there exists some module M_j and some morphism $\theta = F'(\psi) : M_i \otimes N \to M_j \otimes N$ satisfying $\theta(g(x)) = 0$. Then by naturality of α, the following diagram commutes:

$$\begin{array}{ccc} M_i \otimes N' & \xrightarrow{\alpha_{M_i}} & M_i \otimes N \\ \downarrow{\scriptstyle F(\psi)} & & \downarrow{\scriptstyle \theta} \\ M_j \otimes N' & \xrightarrow{\alpha_{M_j}} & M_j \otimes N \end{array}$$

Since α_{M_j} is injective, we have $F(\psi)(x) = 0$, so $x \sim 0$. It follows that g is injective, which completes the proof. $\square$

14 Oct. 29, 2019

Preadditive categories are more general than additive categories, which are more general than pre-abelian categories, which are more general than abelian categories.

In a *preadditive* category, the Hom sets have an abelian group structure and the composition maps are bilinear. (In particular, there is a unique *zero morphism* in each Hom set that is its identity element as an abelian group.) Thus, the composition operation satisfies $g \circ (f_1 + f_2) = g \circ f_1 + g \circ f_2$ and $(g_1 + g_2) \circ f = g_1 \circ f + g_2 \circ f$. In particular, $\mathrm{End}_{\mathcal{C}}(A) := \mathrm{Hom}_{\mathcal{C}}(A, A)$ is a ring when $\mathcal{C}$ is a preadditive category. The multiplication operation is composition.

Remark 14.1. The functor $\mathrm{Hom}(P, -)$ is *additive* because the map

$$\mathrm{Hom}(M, M') \to \mathrm{Hom}(\mathrm{Hom}(P, M), \mathrm{Hom}(P, M'))$$

is given by $f \mapsto (\psi \mapsto f \circ \psi)$, and $(f_1 + f_2) \circ \psi = f_1 \circ \psi + f_2 \circ \psi$.

We say a preadditive category is *additive* if any set of finitely many objects has a product and a coproduct. (We will show these are isomorphic.)

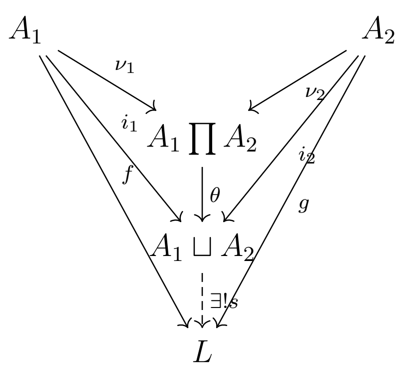
Proposition 14.2. *Let $\mathcal{C}$ be an additive category. Then:*

(i) $\mathcal{C}$ has an initial object I, a terminal object T, and $I \simeq T$.

(ii) If $A_1, \ldots, A_n \in Ob(\mathcal{C})$, then $\prod A_i \simeq \coprod A_i$.

Proof. (i) Taking the empty coproduct gives an object L with a unique map to any other object C. Taking the empty product similarly gives a terminal object T. We then have maps $L \to T$ and $T \to L$ by the universal property of each. Composing gives a map $L \to L$, which must be the identity. Similarly, we get a map $T \to T$ that is the identity. It follows that the maps $L \to T$ and $T \to L$ are left and right inverses, so $L \simeq T$. We will call $O := T = L$ the *initial/terminal object* of $\mathcal{C}$.

(ii) It suffices to do the $n = 2$ case; the rest follows by induction. We have projections $\pi_j : A_1 \prod A_2 \to A_j$ and inclusions $i_j : A_j \to A_1 \sqcup A_2$ for $j = 1, 2$. We thus have a map $\theta : A_1 \prod A_2 \to A_1 \sqcup A_2$ by the universal property of product. We want to show $A_1 \prod A_2$ is the coproduct of A_1 and A_2. We have maps $\nu_j : A_j \to A_1 \prod A_2$ for $j = 1, 2$. These satisfy $\pi_j \circ \nu_j = \mathrm{id}_{A_j}$ and $\pi_j \circ \nu_{j+1} = 0_{A_{j+1}, A_j}$ where addition in the subscript is taken mod 2, $O_{A,B}$ is the zero map from A to B, and $j = 1, 2$. We have the following commutative diagram:



Here we know the map s exists and is unique by universal properties. It satisfies $s \circ i_{A_1} = f$ and $s \circ i_{A_2} = g$.

We claim θ is the unique map making the diagram commute. Suppose instead that there exists $\theta' : A_1 \prod A_2 \to A_1 \sqcup A_2$ that makes the diagram commute, and let $\psi : \theta - \theta'$. We claim that $\psi = 0$. Note that $\psi : A_1 \prod A_2 \to A_1 \sqcup A_2$. What is $\psi \circ \nu_1$? We have

$$\psi \circ \nu_1 = \theta \circ \nu_1 - \theta' \circ \nu_1$$

$$= i_1 - i_1 = 0.$$

Similarly, $\psi \circ \nu_2 = 0$. We claim that $\psi \circ \mathrm{id}_{A_1 \prod A_2} = 0$. Note that $\mathrm{id}_{A_1 \prod A_2} = \nu_1 \circ \pi_1 + \nu_2 \circ \pi_2$, so we calculate

$$\psi \circ (\nu_1 \circ \pi_1 + \nu_2 \circ \pi_2)$$
$$= (\psi \circ \nu_1) \circ \pi_1 + (\psi \circ \nu_2) \circ \pi_2$$
$$= 0.$$

Thus, $\psi = 0$. But how do we know that $\mathrm{id}_{A_1 \prod A_2} = \nu_1 \circ \pi_1 + \nu_2 \circ \pi_2$? We have

$$\pi \circ (\nu_1 \circ \pi_1 + \nu_2 \circ \pi_2)$$
$$= (\pi_1 \circ \nu_1) \circ \pi_1 + (\pi_1 \circ \nu_2) \circ \pi_2$$
$$= \mathrm{id}_{A_1} \circ \pi_1 + 0 \circ \pi_2$$
$$= \pi_1,$$

and similarly $\pi_2 \circ (\nu_1 \circ \pi_1 + \nu_2 \circ \pi_2) = \pi_2$. But $\mathrm{id}_{A_1 \prod A_2}$ also has these properties, so by the universal property, $\mathrm{id}_{A_1 \prod A_2} = \nu_1 \circ \pi_1 + \nu_2 \circ \pi_2$.

Thus, ψ really does equal 0. Now, we have a map $\xi : A_1 \prod A_2 \to L$ by the universal property, and we want to show this is equal to $s \circ \theta$. We calculate

$$(\xi - s \circ \theta) \circ \nu_1$$
$$= f - s \circ \theta \circ \nu_1$$
$$= f - s \circ i_1$$
$$= f - f = 0.$$

Similarly, $(\xi - s \circ \theta) \circ \nu_2 = 0$, so $\xi = s\theta$ from before. This completes the proof. □

We say a category is *preabelian* if it is additive and kernels and cokernels exist. The kernel is the equalizer of a map with the zero map, and the cokernel is defined analogously.

To be precise, let $\mathcal{C}$ be a preadditive category, and let $f : X \to Y$ be a morphism in $\mathcal{C}$. A *kernel* of f is an object K with a morphism $k : K \to X$ such that (i) $f \circ k$ is the zero morphism from K to Y, and (ii) given any morphism $k' : K \to X$ such that $f \circ k'$ is the zero morphism from K to Y, there is a unique morphism $u : K' \to K$ such that $k \circ u = k'$. Cokernels can be defined analogously.

Note that the cokernel of a morphism $f : A \to B$ is often thought of as $B/\mathrm{im}(f)$. We think of it instead as the morphism $g : B \to B/\mathrm{im}(f)$.

Definition 14.3. A monomorphism is *normal* if it is the kernel of some morphism. Similarly, an epimorphism is *normal* if it is the cokernel of some morphism.

A preabelian category is *abelian* if all its monomorphisms and epimorphisms are normal.

15 Oct. 31, 2019

I begin by summarizing the definitions from last class. Preadditive = Hom-sets are abelian groups (so in particular there's a zero morphism) and composition is bilinear (which just means it distributes over addition in this case). Additive = preadditive + finite products and coproducts exist (and we proved they are isomorphic). Preabelian = additive + kernels and cokernels exist (where kernels and cokernels are special types of equalizer and coequalizer, respectively). Abelian = preabelian + monomorphisms and epimorphisms are normal.

Note that we can define the image of a morphism by $\text{im}(f) := \ker(\text{coker}(f))$. On the next assignment, we will prove that given a morphism $f : A \to B$ in an abelian category, there exists a monomorphism $u : A \to \text{im}(f)$ and a monomorphism $v : \text{im}(f) \to B$.

Example 15.1. Let R be a ring. Then the category R-Mod of left R-modules and the category Mod-R of right R-modules are both abelian categories. We have an addition map $\text{Hom}_R(M, N)$. What is a monomorphism in R-Mod. We claim that $i : A \to B$ is mono if and only if it is one-to-one. Why? If i is not one-to-one, then there exists $C := \ker(i) \subseteq A$ and C is non-zero. Thus, we have a non-zero inclusion map $j : C \hookrightarrow A$, which gives rise to the following equalizer:

$$C \xhookrightarrow{j \neq 0, 0} A \xrightarrow{i} B$$

Thus, $i \circ j = i \circ 0 = 0$. So i is not a monomorphism.

If i is one-to-one, then we claim i is mono. Why? If i is not mono, then we have the following equalizer:

$$C \xrightarrow{h,k} A \xrightarrow{i} B$$

Applying the forgetful functor to Set, we get that i is not mono in Set. But in Set, this implies i is not one-to-one, so i is not one-to-one in R-Mod either.

The following result is known as *Mitchell's embedding theorem.*

Theorem 15.2. *Let $\mathcal{A}$ be a small abelian category. Then there exists a ring R (not necessarily commutative, but unital and associative) and a fully faithful and exact functor $F : \mathcal{A} \to R$-Mod.*

Jason talks about a piece of advice he would have given his undergrad self: when he sees a famous theorem in his field, he'll try to prove it himself. If he gets stuck, he reads the beginning of the proof and tries again, etc. He usually tries for about two days before reading the proof. So now he explains his thought process for trying to prove this theorem.

First, he asked himself, "Can you recover R?" No, you can't because $R\text{-Mod} \simeq M_n(R)\text{-Mod}$. Then he made "Attempt 1": look at endomorphisms. If $M \in \text{Ob}(R\text{-Mod})$, then $\text{End}_R(M) = \text{Hom}_R(M, M)$. In particular, if $M = (0)$, then $\text{End}_R((0))$ is the zero ring, which is unfortunate. But consider $\text{End}_R(R) = \{\varphi_r : R \to R \mid r \in R\}$ where $\varphi_r(a) := ra$. Then

$\mathrm{End}_R(R^n) \simeq M_n(R)$. So this also doesn't work.

What if we try $\mathrm{End}_R(P)$ for P projective? Then $\mathrm{Hom}(P, -)$ is exact. Suppose $R := \mathbb{Q}$ and $S := \mathbb{Q}^2$. Is R-Mod $\simeq S$-Mod? No! Why? Take $M := \mathbb{Q} \times \{0\}$ and $N := \{0\} \times \mathbb{Q}$. Suppose $\psi \in \mathrm{Hom}_S(M, N)$. Define f by $\psi(a, 0) =: (0, f(a))$. Then

$$\psi((u, v) \cdot (a, 0)) = (u, v) \cdot (0, f(a)) = (0, vf(a)) = \psi((ua, 0)) = (0, f(ua)).$$

So $f(ua) = vf(a)$ for all u, v, and setting $u := 1$, $v := 2$ gives $f(a) = 2f(a)$, so $f(a) = 0$. So $\mathrm{Hom}_S(\mathbb{Q} \times \{0\}, \{0\} \times \mathbb{Q}) = 0$.

Now, suppose we had an equivalence of categories given by functors $F : R\text{-Mod} \to S\text{-Mod}$ and G going the other way. Then

$$\mathrm{Hom}_R(G(\mathbb{Q} \times \{0\}), G(\{0\} \times \mathbb{Q}))$$

is something non-trivial because the left term in the Hom is $\mathbb{Q}^a$ for some a and the second is $\mathbb{Q}^b$ for some b. But M, N are projective and we have $M \oplus N = S$, which is free, and $\mathrm{Hom}_S(M, N) \simeq \mathbb{Q}$. This means that we don't necessarily recover the right ring this way.

At this point, Jason got stuck. It turns out Mitchell continues the proof as follows.

Build some really big projective module P instead. Take $\bigoplus_{P \text{ projective}} P$. This is a coproduct of projective modules, so it is projective. We say an object P in an abelian category $\mathcal{A}$ is a *projective generator* if $\mathrm{Hom}(P, -) : \mathcal{A} \to \mathrm{Ab}$ is exact and faithful.

(i) Show $\mathcal{A}$ embeds in a cocomplete small abelian category $\mathcal{B}$ with a projective generator $P \in \mathrm{Ob}(\mathcal{B})$. Then we have an exact, fullly faithful functor $I : \mathcal{A} \to \mathcal{B}$. Then build a functor $J : \mathcal{B} \to R$-Mod where $R = \mathrm{End}_{\mathcal{B}}(P)$ given on objects by $B \mapsto \mathrm{Hom}(P, B)$, and show it is exact and fully faithful.

We don't give the actual proof (it's long!), but I think this sketch is a lot more interesting.

Remark 15.3. If $F : \mathcal{A} \to \mathcal{B}$ is an additive functor between additive categories, we claim that $F(0_{\mathcal{A}}) = 0_{\mathcal{B}}$. Why? We claim that if $C = 0_{\mathcal{B}}$, then

$$\mathrm{End}_{\mathcal{B}}(C) = 0.$$

If $C \not\simeq 0_{\mathcal{B}}$, then $\mathrm{End}_{\mathcal{B}}(C) \neq 0$ since $\mathrm{id}_C \neq 0_{C,C}$. Why? Because if it were, we would have the following equalizer:

$$C \xrightarrow{\mathrm{id}_C, 0_{C,C}} C \xrightarrow{f} D$$

Since $f \circ 0_{C,C} = 0$ (by which I mean $0_{C,D}$), $f = 0_{C,D}$.

Notice also that if F is a full functor $F : \mathrm{Hom}_{\mathcal{A}}(0_{\mathcal{A}}, 0_{\mathcal{A}} \to \mathrm{Hom}_{\mathcal{B}}(F0, F0)$, then $F(0) = 0$. Thus, $\mathrm{id}_{F(0)} = 0_{F0,F0}$.

We now consider short exact sequences in an abelian category $\mathcal{A}$. We say

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

is *exact* if (i) f is mono, (ii) g is epi, (iii) $g \circ f = 0_{A,C}$, and (iv) $\mathrm{im}(f) \simeq \ker(g)$, i.e., $\ker(\mathrm{coker}(f)) \simeq \ker(g)$.

Note that if $f : A \to B$, we can factor it as $v \circ u$ where $u : A \to \mathrm{im}(f)$ is epi and $v : \mathrm{im}(f) \to B$ is mono. So if $g \circ f = 0$, then $g \circ v = 0$ since $0 = g \circ f = g \circ (v \circ u) = (g \circ v) \circ u$. Thus, if we give condition (iv) and specify the isomorphism works this way, we can recover (iii). Instead, we have given four separate conditions.

Remark 15.4. If $F : \mathcal{A} \to \mathcal{B}$ is exact, then

$$0 \xrightarrow{0} A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

is exact implies that

$$0 = F(0) \xrightarrow{0} FA \xrightarrow{Ff} FB \xrightarrow{Fg} FC \to 0$$

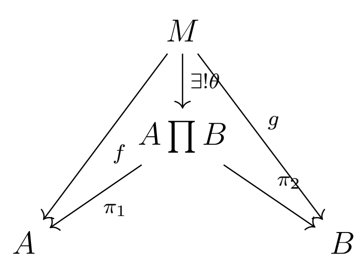
is exact. Indeed, we have $Fg \circ Ff = F(g \circ f) = F(0_{A,C}) = 0_{FA,FC}$. Thus, if F is exact, then we have a commutative diagram:

$$F(\mathrm{im}(f)) \simeq \mathrm{im}(F(f)) \xrightarrow{Fv} FB \xrightarrow{Fg,0} FC$$

Remark 15.5. If $\mathcal{A}$ is an abelian category and $M \in \mathrm{Ob}(\mathcal{A})$, then $\mathrm{Hom}(M, -) : \mathcal{A} \to \mathrm{Ab}$ is left exact. It suffices to show that $\mathrm{Hom}(M, -)$ preserves kernels and

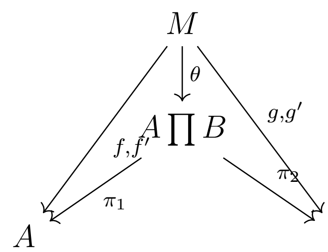
$$\mathrm{Hom}(M, A \prod B) \simeq \mathrm{Hom}(M, A) \times \mathrm{Hom}(M, B).$$

Let's show the second part first. Consider the following commutative diagram:

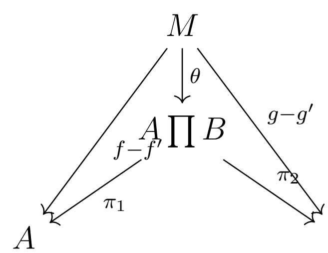


If $(f, g) \in \mathrm{Hom}_{\mathcal{A}}(M, A) \times \mathrm{Hom}_{\mathcal{A}}(M, B)$, we have a map α defined by $\alpha(f, g) := \theta$. We need to check that (i) α is a group homomorphism, (ii) α is one-to-one, and (iii) α is onto.

Let's check (i) first. If $(f, g), (f', g') \mapsto \theta$, then we have the following commutative diagram:



(Here the double arrows are *not* necessarily equalizers.) We get the following diagram from this:



Thus, $f - f' = g - g' = 0$, so α is one-to-one. Surjectivity and the fact it is a group homomorphism are left to the reader.

16 Nov. 5, 2019

We talk about the equivalence between vector bundles and projective modules. Let R be a unital but not necessarily commutative ring. Recall that the following are equivalent for an R-module P:

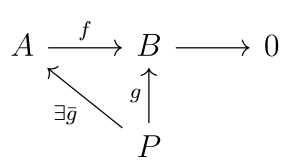
(i) P is projective.

(ii) There exists an R-module Q such that $P \oplus Q$ is a free module, i.e., $P \oplus Q \simeq R^I$ for some indexing set I.

(iii) $\operatorname{Hom}(P, -) : R\text{-Mod} \to R\text{-Mod}$ is exact.

(iv) If $A \xrightarrow{f} B$ is an epimorphism, then the induced map $\operatorname{Hom}(P, A) \to \operatorname{Hom}(P, B)$ given by $\psi \mapsto f \circ \psi$ is an epimorphism.

(v) We have the following commutative diagram where the top sequence is exact:



(vi) If $0 \to A \xrightarrow{f} B \xrightarrow{g} P \to 0$ is exact, then there exists a section $s : P \to B$, i.e., a map such that $g \circ s : P \to P$ is equal to id_P. (We can obtain this from (vi) by taking $B := P$

and $g := \mathrm{id}_P$.) This implies that the exact sequence splits, i.e., $B \simeq A \oplus P$.

Jason now gives us some wisdom about projectives. In "nice" rings, projectives are free (see the Quillen–Suslin theorem). Polynomial rings, for example. The rings where projectives are not free are those where something goes wrong. For example, they might have singularities, like $R := \mathbb{C}[x, y]/(x^2 - y^2)$. (The curve $x^2 - y^2 = 0$ has a cusp at 0.) Jason also says, "Non-finitely-generated projective modules long to be free."

Indeed, there is the following theorem, due to Bass.

Theorem 16.1. *Let R be a commutative noetherian ring, and suppose that 0 and 1 are the only idempotents of R. Then every non-finitely-generated projective module is free.*

The condition about idempotents is equivalent to connectedness of $\mathrm{Spec}(R)$. If you suspend it, it becomes false. For example, let $P := \mathbb{Z} \times \{0\}$, $Q := \{0\} \times \mathbb{Z}$, $R := \mathbb{Z} \times \mathbb{Z} \simeq P \oplus Q$. Let

$$\tilde{P} := \bigoplus_{i=1}^{\infty} P, \tilde{Q} := \bigoplus_{i=1}^{\infty} Q.$$

Then

$$\tilde{P} \oplus \tilde{Q} \simeq \bigoplus_{i=1}^{\infty} R.$$

However, $\tilde{P}$ is not free because it contains idempotents (e.g., $(0, 1) \cdot \tilde{P} = (0)$). Indeed, if we did have $\tilde{P} \simeq R^I$ for some indexing set I, then we would have $(0, 1) \cdot R^I = (0)$, which is impossible.

The following theorem is due to Kaplansky (who was Canadian).

Theorem 16.2. *Let R be a commutative local ring. Then if P is a projective R-module, P is free.*

Proof of the finitely-generated case. Let $\mathfrak{m}$ be the unique maximal (left) ideal of R. Let $K := R/\mathfrak{m}$ be the residue field. Recall Nakayama's lemma.

Lemma 16.3. *Let R be a ring, and let M be a finitely-generated R-module. Let $J(R)$ denote the Jacobson radical of R. Suppose*

$$J(R)M = M.$$

Then $M = (0)$.

Why do we need M to be finitely-generated here? If we did not have that condition, we could take $R := k[[t]]$ (which implies $J(R) = tk[[t]]$) and $M := k((t))$. Then M is an R-module and $J(R)M = M$. However, $M \neq (0)$.

We now continue with the proof of Kaplansky's theorem. Let $p_1, \dots, p_n$ be a *minimal* set of generators for P. We then have an epimorphism $R^n \to P$ given by $e_i \mapsto p_i$ where $\{e_1, \dots, e_n\}$ is a generating set for R^n. Call this map π. We then have an exact sequence

$$0 \to \ker(\pi) \to R^n \xrightarrow{\pi} P \to 0,$$

which implies that $R^n \simeq P \oplus \ker(\pi)$ since P is projective. Let $Q := \ker(\pi)$. We now tensor with $K = R/\mathfrak{m}$ to obtain

$$R^n \otimes_R K \simeq (P \oplus Q) \otimes_R K$$

$$\implies (R \otimes_R K)^n \simeq (P \otimes_R K) \oplus (Q \otimes_R K)$$

$$\implies K^n \simeq (P/\mathfrak{m}P) \oplus (Q/\mathfrak{m}Q) \text{ as } K\text{-modules.}$$

This means that $P/\mathfrak{m}P \simeq K^t$ and $Q/\mathfrak{m}Q \simeq K^{n-t}$ for some $0 \leq t \leq n$.

We claim that $t = n$. Suppose not. Then $t < n$. By relabelling, we may assume $\bar{p}_1, \dots, \bar{p}_n \in P/\mathfrak{m}P$ form a basis for $P/\mathfrak{m}P$ as a K–vector space. Let

$$P_0 := Rp_1 + \cdots + Rp_t \subseteq P.$$

Then $P = P_0 + \mathfrak{m}P$. Let $M := P/P_0$, which is a finitely-generated R-module. We have

$$\mathfrak{m}M = \mathfrak{m}(P/P_0) \simeq (\mathfrak{m}P + P_0)/P_0 = P/P_0 = M.$$

Since R is a local ring, $J(R) = \mathfrak{m}$, so by Nakayama's lemma $M = (0)$. Thus, $P = P_0$. But our choice of n was minimal, and now we have generated P with t generators where $t < n$.

This contradiction implies that $t = n$, so $Q/\mathfrak{m}Q = (0)$. Thus, $Q = \mathfrak{m}Q$. So if Q is finitely-generated, then by Nakayama's lemma $Q = (0)$, which implies

$$R^n \simeq P \oplus Q \simeq P \oplus (0) \simeq P,$$

so P is free. Why is Q finitely-generated? Fix an R-module isomorphism $\psi : R^n \to P \oplus Q$ given by $\psi(e_i) := (p_i, q_i)$. We claim that $Q = Rq_1 + \cdots + Rq_n$. Why? If not, there exists $q \in Q \setminus (Rq_1 + \cdots + Rq_n)$. Since ψ is onto, there exists $u := (a_1, \dots, a_n) = a_1e_1 + \cdots + a_ne_n$ such that $\psi(u) = (0, q)$. But

$$\psi(u) = a_1\psi(e_1) + \cdots + a_n\psi(e_n)$$

$$= (a_1p_1 + \cdots + a_np_n, a_1q_1 + \cdots + a_nq_n).$$

Thus, $q = a_1q_1 + \cdots + a_nq_n$, which completes the proof. $\square$

Let R be a (not necessarily commutative) ring. We say a proper ideal P of R is *prime* if whenever I, J are two-sided ideals of R such that $IJ \subseteq P$, then $I \subseteq P$ or $J \subseteq P$. Note that R/P does not have to be a domain in the non-commutative case, which can be seen by taking $R := M_2(\mathbb{C})$ and $P := (0)$.

Think of $\mathrm{Spec}(R)$ as the set of prime ideals of R. For every two-sided ideal I of R, define

$$\mathcal{C}(I) := \{P \in \mathrm{Spec}(R) \mid P \supseteq I\}.$$

Now put a topology on $\mathrm{Spec}(R)$ by taking the closed sets to be precisely those of the form

$$\{\mathcal{C}(I) \mid I \trianglelefteq R\}.$$

We check this defines a topology. Note that $\mathcal{C}(R) = \emptyset$ and $\mathcal{C}((0)) = \mathrm{Spec}(R)$. Also,

$$\mathcal{C}(I) \cup \mathcal{C}(J) = \mathcal{C}(IJ).$$

Indeed, if $\mathfrak{p} \in \mathcal{C}(I) \cup \mathcal{C}(J)$ is prime, then $\mathfrak{p} \supseteq I$ or $\mathfrak{p} \supseteq J$, so $\mathfrak{p} \supseteq IJ$. Also, if $\mathfrak{p} \in \mathcal{C}(IJ)$ is prime, then $\mathfrak{p} \supseteq IJ$, so $\mathfrak{p} \supseteq I$ or $\mathfrak{p} \supseteq J$, so $\mathfrak{p} \in \mathcal{C}(I) \cup \mathcal{C}(J)$.

Also,

$$\bigcap_{\alpha} \mathcal{C}(I_\alpha) = \mathcal{C}(\sum_{\alpha} I_\alpha)$$

for any family of prime ideals $\{I_\alpha\}$. Indeed, if there is a prime ideal $\mathfrak{p}$ such that $\mathfrak{p} \supseteq I_\alpha$ for every α, then $\mathfrak{p} \supseteq \sum_\alpha I_\alpha$. Also, if there is a prime ideal $\mathfrak{p}$ such that $\mathfrak{p} \supseteq \sum_\alpha I_\alpha$, then $\mathfrak{p} \supseteq I_\alpha$ for every α, so $\mathfrak{p} \in \bigcap_\alpha \mathcal{C}(I_\alpha)$. This completes the proof that the topology on $\mathrm{Spec}(R)$ (called the *Zariski topology*) is actually a topology.

We now shift back to the case where R is a commutative ring.

Theorem 16.4. *Suppose R is a commutative ring. Then the following are equivalent:*

(i) $\mathrm{Spec}(R)$ is disconnected.
(ii) $R \simeq R_1 \times R_2$ where R_1 and R_2 are non-zero rings.
(iii) There exists an idempotent $e \in R$ such that $e \neq 0, 1$.

Proof. (ii) $\implies$ (iii): Take $e = (1, 0)$.

(iii) $\implies$ (i): Let $e = e^2$, $e \neq 0, 1$. Let $I := (e)$, $J := (1 - e)$. Then

$$\mathcal{C}(I) \cup \mathcal{C}(J) = \mathcal{C}(IJ) = \mathcal{C}(0) = \mathrm{Spec}(R)$$

and

$$\mathcal{C}(I) \cap \mathcal{C}(J) = \mathcal{C}(I + J) = \mathcal{C}(R) = \emptyset.$$

(i) $\implies$ (ii): Suppose $\mathrm{Spec}(R)$ is disconnected. Then there exist disjoint $\mathcal{C}(I)$ and $\mathcal{C}(J)$ for some prime ideals I, J of R. We have

$$\mathcal{C}(I) \cap \mathcal{C}(J) = \emptyset$$

$$\iff \mathcal{C}(I + J) = \emptyset$$

$$\iff I + J = R.$$

Also,

$$\mathcal{C}(I) \cup \mathcal{C}(J) = \operatorname{Spec}(R)$$
$$\iff \mathcal{C}(IJ) = \operatorname{Spec}(R)$$
$$\iff \text{ every } \mathfrak{p} \supseteq IJ$$
$$\iff IJ \subseteq \bigcap_{\mathfrak{p} \text{ prime}} \mathfrak{p} = \sqrt{(0)}.$$

So every element of IJ is nilpotent and $I + J = R$. This implies there exist $x, y \in J$ such that $x + y = 1$, so $xy \in IJ$, so there exists n such that $(xy)^n = 0$. Now, we write

$$1 = x + y,$$

so

$$1 = (x+y)^{2n}$$
$$= x^{2n} + \binom{2n}{1} x^{2n-1} y + \cdots + \binom{2n}{2n-1} x y^{2n-1} + y^{2n},$$

and set $e := x^{2n} + \binom{2n}{1} x^{2n-1} y + \cdots + \binom{2n}{n} x^n y^n$ and the rest of the terms form $1 - e$. Then $e(1-e)$ is a sum of terms of the form $\mathbb{Z}x^i y^j$ for $i, j \geq n$, so it vanishes, so e is an idempotent. Also $e \neq 0$ because if we had $e = 0$, we would have $1 - e = 1$, but $1 - e \in (y) \subseteq J$ which is a proper ideal, a contradiction. □

Let S be a multiplicatively closed subset of R. Let $S^{-1}R := \{s^{-1}r \mid s \in S, r \in R\}/\sim$ where $s_1^{-1}r - 1 \sim s_2^{-1}r_2$ if there exists $s_3 \in S$ such that $s_3 s_2 r_1 = s_3 s_1 r_2$. If P is a prime ideal, then set $S := R - P$ (which is multiplicatively closed), and let $R_P := S^{-1}R$. Then PR_P is the unique maximal ideal of R_P. If M is an R-module, define $M_P := M \otimes_R R_P$. Then M_P is an R_P-module, and R_P itself is flat as an R_P-module.

17 Nov. 7, 2019

Theorem 17.1. *Let R be a commutative noetherian ring and P a finitely-generated R-module. The following are equivalent:*

(i) P is projective.
(ii) $P_{\mathfrak{p}} := R_{\mathfrak{p}} \otimes_R P$ is free for all $\mathfrak{p} \in \operatorname{Spec}(R)$.
(iii) $P_{\mathfrak{m}} := R_{\mathfrak{m}} \otimes_R P$ is free for all maximal ideals $\mathfrak{m}$ of R.
(iv) There exist $f_1, \ldots, f_s$, $s \geq 1$, in R such that $(f_1, \ldots, f_s) = R$ and $P_{f_i} := P \otimes_R R[\frac{1}{f_i}]$ is free for all $1 \leq i \leq s$.

"Proof". Recall that if $\mathfrak{p}$ is a prime ideal of R, then $R_{\mathfrak{p}}$ is a local ring with maximal ideal $\mathfrak{p}R_{\mathfrak{p}}$, and the ideals of $R_{\mathfrak{p}}$ are in bijection with the ideals of R contained in $\mathfrak{p}$.

(ii) $\implies$ (iv): We claim that $P \otimes_R R_\mathfrak{p} \simeq R_\mathfrak{p}^n$ for some n. (This n is called the *rank* and exists because P is finitely-generated.) We use the map

$$\Psi : R_\mathfrak{p}^n \to P \otimes_R R_\mathfrak{p}$$

defined by writing $P =: \langle p_1, \ldots, p_k \rangle$ and sending

$$(1, 0, \ldots, 0) \mapsto p_1 \otimes r_{11} s_{11}^{-1} + \cdots + p_{1k} \otimes r_{1k} s_{1k}^{-1},$$

$$\ldots$$

$$(0, \ldots, 1) \mapsto p_1 \otimes r_{n1} s_{n1}^{-1} + \cdots + p_k \otimes r_{nk} s_{nk}^{-1}$$

where $s_{ij} \notin \mathfrak{p}$ for all i, j. Jason writes, "To see the isomorphism, it is sufficient to invert $s := \prod s_{ij} \notin \mathfrak{p}$." We claim that

$$P_s := P \otimes R[\frac{1}{s}] \simeq R[\frac{1}{s}]^n$$

is free. Recall that for all $\mathfrak{p} \in \operatorname{Spec}(R)$, there exists $s_\mathfrak{p} \notin \mathfrak{p}$ such that

$$P_{s_\mathfrak{p}} := P \otimes_R R[\frac{1}{s_\mathfrak{p}}$$

is free. Take

$$I := \langle s_\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Spec}(R) \rangle \subseteq R.$$

We claim that $I = R$. If not, there exists a maximal ideal $\mathfrak{m}$ such that $I \subseteq \mathfrak{m}$. But $\mathfrak{m} \in \operatorname{Spec}(R)$ and $s_\mathfrak{m} \in I \setminus \mathfrak{m}$. For all primes $\mathfrak{p}$,

$$U_\mathfrak{p} := \{Q \in \operatorname{Spec}(R) \mid s_\mathfrak{p} \notin Q\}$$

is an open neighbourhood of $\mathfrak{p}$ in Spec. Also,

$$U_\mathfrak{p}^\complement = \mathcal{C}((s_\mathfrak{p})).$$

Recall that $\operatorname{Spec}(R)$ is quasi-compact. This means that if $\operatorname{Spec}(R) = \bigcup_\alpha U_\alpha$, then there exist $U_1, \ldots, U_n \in \{U_\alpha \mid \alpha\}$ such that $\operatorname{Spec}(R) = U_1 \cup \cdots \cup U_n$. Indeed, if $U_\alpha = \mathcal{C}(I_\alpha)^\complement$ for some ideals $\{I_\alpha\}$, then if

$$\bigcup_\alpha U_\alpha = \operatorname{Spec}(R),$$

it follows that

$$\bigcap \mathcal{C}(I_\alpha) = \emptyset$$

$$\implies \mathcal{C}(\sum I_\alpha) = \emptyset$$

$$\implies \sum I_\alpha \in R.$$

Thus, we can write $1 = x_1 + \cdots + x_s$ where $x_i \in I_{\alpha_i}$, so

$$U_{\alpha_1} \cup \ldots U_{\alpha_s} = \mathrm{Spec}(R).$$

The implication follows.

(iv) $\implies$ (i): Jason rushes through this proof, and I don't understand what he does. He also doesn't prove the other directions. I'll write down what I see anyway.

We want to show $M \to N \to 0$ is exact, so $M \to N$ is an epimorphism. This implies $\mathrm{Hom}(P, M) \to \mathrm{Hom}(P, N) \to 0$ is exact. It apparently suffices to do it for finitely-generated R-modules M, N. In this case we have a surjetion $h : R^n \to N$ and an identity map $N \to N$ which lifts to a section $\bar{h} : N \to R$. We have an exact sequence

$$0 \to \ker(h) \to R^n \xrightarrow{h} N \to 0.$$

Because of our section, $R^n \simeq N \oplus \ker(h)$, so N is projective. Jason writes $P := N$.

We now recall some facts from Assignment 3.

Localization is flat, i.e., $S^{-1}R$ is a flat R-module. This apparently has to do with the map $\psi \otimes \mathrm{id} \to \psi$ giving an isomorphism

$$\mathrm{Hom}_R(P, M) \otimes_R R[\frac{1}{f}] \simeq \mathrm{Hom}_{R[1/f]}(P_f, M_f).$$

Now P_f is free and we have an epimorphism $M \to N$ between finitely-generated modules, so we get an epimorphism

$$\mathrm{Hom}_{R_f}(P_f, M_f) \to \mathrm{Hom}_{R_f}(P_f, N_f),$$

and $\mathrm{Hom}_{R_f}(P_f, M_f) \simeq \mathrm{Hom}(P, M) \otimes_R R_f$, so we get an epimorphism

$$\mathrm{Hom}(P, M) \otimes_R R_f \to \mathrm{Hom}(P, N) \otimes_R R_f$$

where $f \in (f_1, \ldots, f_s)$.

Another fact: If $f_1, \ldots, f_s$ generate the unit ideal and $B \subseteq A$ are finitely-generated R-modules, then

$$(A/B) \otimes_R R_f = (0)$$

if and only if there exists n such that $f^n A \subseteq B$. Why? If $B \subseteq A = Ra_1 + \cdots + Ra_r$, then we have a chain

$$B = B_0 \subseteq B_1 \subseteq \cdots \subseteq B_r = A.$$

Here $B_0 := B$, $B_1 := B + Ra - 1$, etc. Then $B_{i+1}/B_i = B_i + Ra_{i+1}/B_i$. We have a surjection

$\psi : R \to B_{i+1}/B_i$ given by $\psi(r) := ra_{i+1} + B_i$. By the first isomorphism theorem,

$$B_{i+1}/B_i \simeq R/\ker(\psi) \simeq R/I_i.$$

If $I \to R \to R/J \to 0$ is exact, then $JM \to R \otimes_R M \to (R/J) \otimes M$ is exact. Notice that $(A/B) \otimes_R R_f = (0)$, so $(B_1/B_0) \otimes_R R_f = (0)$. This implies that

$$R_f/(I_0)_f \simeq (0).$$

Now $(I_0)[1/f] = R[1/f]$, so there exists $f^n \in I_0$. Jason somehow ends the proof here. More precisely, he writes "Sketch continued", then says:

(i) Show $\mathrm{Hom}_R(P, M)$ and $\mathrm{Hom}_R(P, N)$ are f.g.

(ii) Show $M \xrightarrow{g} N \to 0$ is an epimorphism, thus so is $\mathrm{Hom}(P, M) \xrightarrow{g^*} \mathrm{Hom}(P.N)$. Let $A := \mathrm{Hom}(P, N)$, $B := g^*(\mathrm{Hom}(P, M))$. Then by assumption

$$(A/B) \otimes_R R_{f_i} = (0)$$

for $i = 1, \ldots, s$. So there exist $n_1, \ldots, n_s$ such that $f_i^{n_i} A \subseteq B$. Then

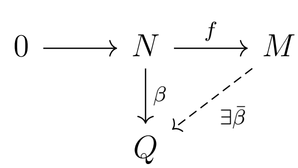
$$(1) = (f_1, \ldots, f_s) = (f_1^n, \ldots, f_s^n),$$

so $A \subseteq B$, so g^* onto, so P is projective. "□"

Recall that if I is an injective R-module, then $\mathrm{Hom}(-, I)$ is exact. The following result is called *Baer's criterion.*

Theorem 17.2. *Suppose R is a ring and Q is a left R-module. Suppose that for all left ideals I of R and R-module homomorphisms $h : I \to Q$, there exists a map $\bar{h} : R \to Q$ such that if $\iota : I \to R$ is the inclusion, then $\bar{h} \circ \iota = h$. Then Q is injective. (Note that the converse just follows from the definition of injectivity.)*

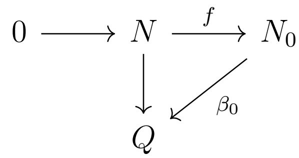
Proof. We need to show that we have the following commutative diagram:



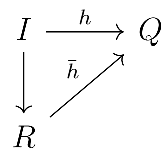
Consider

$$\mathcal{S} := \{(N', \beta') \mid N \subseteq N' \subseteq M, \beta' : N' \to Q\}.$$

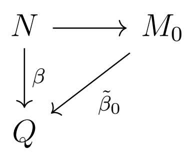
We define $(N', \beta') \leq (N'', \beta'')$ if $N' \subseteq N''$ and $\beta''|_{N'} = \beta'$. By Zorn's lemma, there exists a maximal element of $\mathcal{S}$, say (N_0, β_0). Then we have a commutative diagram:



We claim that $N_0 = M$. If not, there exists $m \in M \setminus N_0$. Let $M_0 := N_0 + Rm$. Let $I := \{x \in R \mid xm \in N_0\}$. Now define $h : I \to Q$ by $h(x) := \beta_0(xm)$. We have a commutative diagram:



Define $\tilde{\beta}_0 : M_0 \to Q$ by $\tilde{\beta}_0(rm + n_0) := \beta_0(n_0) + \bar{h}(r)$ for $r \in R$ and $n_0 \in N_0$. One can check that this is well-defined, that the following diagram commutes:

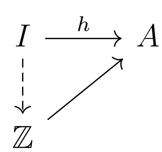


and $\tilde{\beta}_0|_N = \beta_0$. Thus $(M_0, \tilde{\beta}_0) > (M_0, \beta_0)$, contradicting maximality of the latter. The result follows. □

Corollary 17.3. *Let A be an abelian group. Then A is injective in $Ab = \mathbb{Z} - Mod$ if and only if A is divisible.*

Proof. We proved injective implies divisible on Assignment 2. Now we prove that divisible implies injective.

Let A be divisible. Let $I \trianglelefteq \mathbb{Z}$, so we have a commutative diagram:



Without loss of generality, $I \neq (0)$. Then $I = n\mathbb{Z}$ for $n \neq 0$. Then we have a map $\bar{h} : \mathbb{Z} \to A$ in the diagram above. Pick some element of $n\mathbb{Z}$. Let its image in A along h be a. Let b be the image in A of 1 along the map $\bar{h}$. Then $nb = a$ by commutativity of the diagram. This completes the proof! □

Definition 17.4. An abelian category $\mathcal{A}$ has *enough projectives* if for all $A \in \mathrm{Ob}(\mathcal{A})$, there exists a projective object P and an epimorphism $h : P \to A$. It has *enough injectives* if for all $A \in \mathrm{Ob}(\mathcal{A})$, there exists an injective object I and a monomorphism $g : A \to I$.

Easy fact: R-Mod has enough projectives. Hard fact: R-Mod has enough injectives. Sketch of hard fact:

(i) Show $\mathbb{Z}$-Mod = Ab has enough injectives.
(ii) Use the "injective production lemma" to lift to the general case.

Proof of (i). Let A be an abelian group. Then we have an injection $i : A \to D$ into some divisible (i.e., injective) abelian group. For some indexing set I, we have a surjection $f : \mathbb{Z}^I \to A$. Let $K := \ker(f) \subseteq \mathbb{Z}^I$. Then $A \simeq \mathbb{Z}^I/K$. We have a chain

$$K \subseteq \mathbb{Z}^I \subseteq \mathbb{Q}^I,$$

and $\mathbb{Q}^I$ is injective (i.e., divisible). Thus, we have an inclusion

$$\mathbb{Z}^I/K \hookrightarrow \mathbb{Q}^I/K =: D,$$

and the latter is injective. We claim that if B is divisible and $C \leq B$, then B/C is divisible. Jason actually has no proof in his notes of this. He jokes, "Well then I guess it's easy." Someone points out that if $x + C \in B/C$, then there is $y \in C$ such that $ny = x$ for some $n \neq 0$, and then $n(y + C) = ny + C = x + C$. This is the missing proof. □

Proof of (ii). Let M be an R-module. Then M is a $\mathbb{Z}$-module under addition as well. Define

$$Q := \mathrm{Hom}_{\mathbb{Z}}(R, I),$$

where $M \hookrightarrow I$ and I is an injective abelian group, i.e., an injective $\mathbb{Z}$-module. We claim that Q is an R-module. Indeed, if we take $\psi : R \to I$, then we want to say what $r \cdot \psi = \psi_r : R \to I$ is. We define $r\psi(x) := \psi(rx)$ This gives Q an R-module structure.

Also, we claim that Q is injective. Why? We have an exact sequence $0 \to N \hookrightarrow M$ and a map $\beta : N \to Q$ and we want to produce $\beta' : Q \to M$. How do we do it? Well, we have a map $h : Q \to I$. We can compose to get a map $\gamma : N \to I$, then use injectivity of I to get a map $\gamma' : M \to I$. Now, we can define $\beta' : M \to Q$. Given $m \in M$, we have $\gamma'(m) \in I$. Then we define $\beta'(m) : R \to I$ by $1 \mapsto \gamma'(m)$. This lifts uniquely to an R-module homomorphism. We want to show that $\beta'|_N = \beta$. At this point, we run out of time. We will not be completing this proof. Oh well. Jason says, "Homological algebra is like a drug."

18 Nov. 12, 2019

Let $\mathcal{A}$ be an abelian category. If $\mathcal{C}$ is a small subcategory of $\mathcal{A}$, then there exists a full small abelian subcategory $\mathcal{A}'$ of $\mathcal{A}$ containing $\mathcal{C}$. Then by Mitchell's embedding theorem, we can embed $\mathcal{A}'$ into R-Mod.

We now describe homology and cohomology. Let $\mathcal{A}$ be an abelian category. A *chain complex* is a collection $(C_n)_{n\in\mathbb{N}} =: C.$ of objects of $\mathcal{A}$ with a sequence of morphisms

$$\cdots \to C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \to \ldots$$

such that $d_n \circ d_{n+1} = 0_{C_{n+1},C_{n-1}}$ for all n. The map d_n is known as the n^{th} *differential*. Since $d_n \circ d_{n+1} = 0$, we have $\operatorname{im}(d_{n+1}) \subseteq \ker(d_n)$. If $\operatorname{im}(d_{n+1}) = \ker(d_n)$, we say the sequence is *exact at* C_n. If it is exact at every C_n, we say it is *exact*.

Let $Z_n(C.) := \ker(d_n) \subseteq C_n$. We call this the n^{th} *cycle*. Let $B_n(C.) := \operatorname{im}(d_{n+1}) \subseteq C_n$. We call this the n^{th} *boundary*. Then

$$(0) \subseteq B_n(C.) \subseteq Z_n(C.) \subseteq C_n.$$

We define

$$H_n(C.) := Z_n(C.)/B_n(C.),$$

which is in $\operatorname{Ob}(\mathcal{A})$ since $\mathcal{A}$ is abelian. We call H_n the n^{th} *homology object* of $C.$.

Note that $H_n(C.) = 0$ if and only if the chain complex is exact at C_n. Thus, homology is a measure of how far we are from being exact at C_n.

Cochain complexes are defined similarly. We consider instead sequences of the form

$$\cdots \to C^{n_1} \xrightarrow{d^{n-1}} C^n \xrightarrow{d^n} C^{n+1} \to \dots$$

such that $d^n \circ d^{n-1} = 0$ for all n. This implies $B^n(C^\cdot) := \operatorname{im}(d^{n-1}) \subseteq \ker(d^n) =: Z^n(C^\cdot)$. We then define

$$H^n(C^\cdot) := Z^n(C^\cdot)/B^n(C^\cdot),$$

the n^{th} *cohomology object*.

Remark 18.1. Note that given a chain complex (C_n), we can make a cochain complex by letting $(C')^n := C_{-n}$. Then $d^n = d_{-n}$, and the maps satisfy the right condition.

We can make a category whose objects are cochain complexes. Similarly, we can make a category whose objects are chain complexes. We will show how it works for chain complexes. A morphism $f : C. \to B.$ is defined to be a collection of morphisms $f_n : C_n \to B_n$ such that if c_i and b_i denote the differentials of $C.$ and $B.$, respectively, then the following diagram commutes:

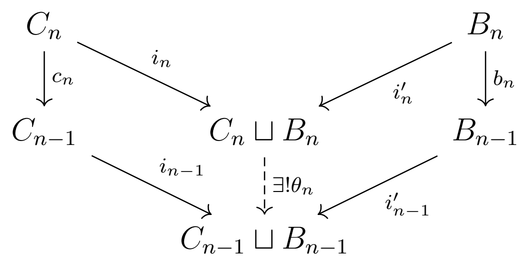
$$\begin{array}{ccccccccc} \dots & \longrightarrow & C_n & \xrightarrow{c_n} & C_{n-1} & \xrightarrow{c_{n-1}} & C_{n-2} & \longrightarrow & \dots \\ & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n-1}} & & \downarrow{\scriptstyle f_{n-2}} & & \\ \dots & \longrightarrow & B_n & \xrightarrow[b_n]{} & B_{n-1} & \xrightarrow{b_{n-1}} & B_{n-2} & \longrightarrow & \dots \end{array}$$

One can show that this has all the properties a category should have (i.e., existence of identity objects, associativity of composition).

We denote the resulting category by $\operatorname{Ch}(\mathcal{C})$. We say $C. \in \operatorname{Ch}(\mathcal{C})$ is *bounded above* if $C_n = 0$ for all sufficiently large n and *bounded below* if $C_{-n} = 0$ for all sufficiently large n. We say it is *bounded* if $C_n = 0$ for all sufficiently large $|n|$, or equivalently if it is bounded above and below.

We have a subcategory of bounded chain complexes, denoted $\operatorname{Ch}_b(\mathcal{C})$, of $\operatorname{Ch}(\mathcal{C})$.

Remark 18.2. The category $\mathcal{C}\langle(\mathcal{C})$ is abelian. Given chain complexes $C.$ and $B.$, by the universal property of coproducts, we have a commutative diagram



Then we can prove

$$\dots \to C_n \sqcup B_n \xrightarrow{\theta_n} C_{n-1} \sqcup B_{n-1} \to \dots$$

is exact. Indeed, by embedding into R-Mod, we can write $\theta_n = (c_n, b_n)$ as a map between the direct sums $C_n \oplus B_n$ and $C_{n-1} \oplus B_{n-1}$ (because the coproduct is the direct sum in R-Mod). Then one can check that $\theta_n \circ \theta_{n+1} = 0$ for all n.

For kernels, given a morphism $f : C. \to B.$, we can produce the kernel $\ker(f_n)$, and the differential $c_n : C_n \to C_{n-1}$ restricts to a morphism $c_n|_{\ker(f_n)} : \ker(f_n) \to \ker(f_{n-1})$. One can show this gives a collection of differentials on the chain complex $\ker(f). := (\ker(f_n))_{n\in\mathbb{N}}$. Cokernels can be defined similarly.

We have a zero object given by

$$\dots \to 0 \xrightarrow{0} 0 \xrightarrow{0} 0 \to dots.$$

We do not check the other properties of abelian categories, but they work out. This means we can talk about exact sequences of chain complexes.

We will soon discuss the long exact sequence in homology (or cohomology). To prove it, we will need the *snake lemma.*

Lemma 18.3. *Suppose that in an abelian category we have the following commutative diagram, where the rows are exact sequences:*

$$\begin{array}{ccccccc} & & M' & \xrightarrow{f} & M & \xrightarrow{g} & M'' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle d''} & & \\ 0 & \longrightarrow & N' & \xrightarrow{\bar{f}} & N & \xrightarrow{\bar{g}} & N'' & & \end{array}$$

Then there is an exact sequence

$$\ker(d') \to \ker(d) \to \ker(d'') \xrightarrow{\delta} coker(d') \to coker(d) \to coker(d'').$$

Proof. Prove it yourself. The hard part is constructing δ, so I will describe how this is done.

Given $x \in \ker(d'')$, view it as an element of M''. (We can do this by using the Mitchell embedding theorem to reduce to the case of modules.) Since g is surjective, there exists $y \in M$ with $g(y) = x$. Because the diagram commutes, we have

$$\bar{g}(d(y)) = d''(g(y)) = d''(x) = 0$$

(since $x \in \ker(d'')$), from which it follows that $d(y) \in \ker(\bar{g})$. Since the bottom row is exact, there exists $z \in N'$ with $\bar{f}(z) = d(y)$, and z is unique by injectivity of $\bar{f}$. We then define $\delta(x) := z + \mathrm{im}(d')$. One can check that δ is a well-defined homomorphism. One can also check by diagram chasing that the resulting long exact sequence is in fact exact. $\square$

We now show how you can get the long exact sequence in homology using the snake lemma. Start with an exact sequence of chain complexes

$$0 \to A. \xrightarrow{f.} B. \xrightarrow{g.} C. \to 0.$$

Taking kernels and cokernels, we get the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
 & & Z_n(A.) & \xrightarrow{f_n|_{Z_n(A.)}} & Z_n(B.) & \xrightarrow{g_n|_{Z_n(B.)}} & Z_n(C.) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A_n & \xrightarrow{f_n} & B_n & \xrightarrow{g_n} & C_n & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle a_n} & & \downarrow{\scriptstyle b_n} & & \downarrow{\scriptstyle c_n} & & \\
0 & \longrightarrow & A_{n-1} & \xrightarrow{f_{n-1}} & B_{n-1} & \xrightarrow{g_{n-1}} & C_{n-1} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \bar{a}_n} & & \downarrow{\scriptstyle \bar{b}_n} & & \downarrow{\scriptstyle \bar{c}_n} & & \\
 & & A_{n-1}/\mathrm{im}(a_n) & \xrightarrow{\bar{f}_{n-1}} & B_{n-1}/\mathrm{im}(b_n) & \xrightarrow{\bar{g}_{n-1}} & C_{n-1}/\mathrm{im}(c_n) & &
\end{array}$$

Note that the snake lemma gives us a long exact sequence

$$Z_n(A.) \xrightarrow{f_n|} Z_n(B.) \xrightarrow{g_n|} Z_n(C.) \xrightarrow{\delta_n} A_{n-1}/\mathrm{im}(a_n) \xrightarrow{\bar{f}_{n-1}} B_{n-1}/\mathrm{im}(b_n) \xrightarrow{\bar{g}_{n-1}} C_{n-1}/\mathrm{im}(c_n).$$

Note also that the maps $\bar{a}_{n-1}, \bar{b}_{n-1}, \bar{c}_{n-1}$ are well-defined basically because $a_{n-1}(\mathrm{im}(a_n)) = 0$.

Now, we also have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
 & & \ker(\bar{a}_n) & \longrightarrow & \ker(\bar{b}_n) & \longrightarrow & \ker(\bar{c}_n) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & A_n/\mathrm{im}(a_{n+1}) & \xrightarrow{\bar{f}_n} & B_n/\mathrm{im}(b_{n+1}) & \xrightarrow{\bar{g}_n} & C_n/\mathrm{im}(c_{n+1}) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \bar{a}_n} & & \downarrow{\scriptstyle \bar{b}_n} & & \downarrow{\scriptstyle \bar{c}_n} & & \\
0 & \longrightarrow & Z_{n-1}(A.) & \xrightarrow{f_{n-1}} & Z_{n-1}(B.) & \xrightarrow{g_{n-1}} & Z_{n-1}(C.) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \mathrm{coker}(\bar{a}_n) & \longrightarrow & \mathrm{coker}(\bar{b}_n) & \longrightarrow & \mathrm{coker}(\bar{c}_n) & &
\end{array}$$

Applying the snake lemma again, we get a long exact sequence

$$\ker(\bar{a}_n) \to \ker(\bar{b}_n) \to \ker(\bar{c}_n) \to \text{coker}(\bar{a}_n) \to \text{coker}(\bar{b}_n) \to \text{coker}(\bar{c}_n).$$

But $H_n(A.) = \ker(\bar{a}_n)$ (etc.) and $\text{coker}(\bar{a}_n) = H_{n-1}(A.)$ (etc.), so we are done.

19 Nov. 14, 2019

Given a map $\alpha : A. \to B.$ of complexes, one can check that we get an induced map $\alpha_n : Z_n(A.) \to Z_n(B.)$. We have a map $Z_n(B.) \to H_n(B.) = Z_n(B.)/\text{im}(b_{n+1})$; we call the composed map α_n as well, abusing notation. If $y \in \text{im}(a_{n+1}) = B_n(A.)$, then $\alpha_n(y) \in \text{im}(b_{n+1})$, so the map $\alpha_n : Z_n(A.) \to H_n(B.)$ factors through $B_n(A.)$. Thus, we get a map $\bar{\alpha}_n : H_n(A.) \to H_n(B.)$.

We now look at homotopy equivalence. Given maps $\alpha, \beta : A. \to B.$, we say that $\alpha \sim \beta$ (α is *homotopy equivalent* to β) if for every $n \in \mathbb{Z}$, there exists $h_n : A_n \to B_{n+1}$ such that $\alpha_n - \beta_n = h_{n-1} \circ a_n + b_{n+1} \circ h_n$ for all n. One can check that this is an equivalence relation.

Theorem 19.1. *If $\alpha, \beta : A. \to B.$ are homotopy equivalent, then α, β induce the same maps $H_n(A) \to H_n(B)$ for all n.*

Proof. Let $\gamma := \alpha - \beta$. Then $\gamma \sim 0$. It is enough to show that $\gamma : H_n(A.) \to H_n(B.)$ is 0 for all n. Note that $\gamma = h_{n-1} \circ a_n + b_{n+1} \circ h_n$. Take $x \in Z_n(A.)$. Then

$$h_{n-1} \circ a_n(x) = h_{n-1}(0) = 0,$$

and

$$\begin{aligned} \gamma_n(x) &= b_{n-1} \circ h_n(x) + h_{n-1} \circ a_n(x) \\ &= b_{n+1}(h_n(x)) \in B_n(B.). \end{aligned}$$

Thus we get a map

$$\bar{\gamma}_n : H_n(A.) \to H_n(B.)$$

such that $\bar{x} \mapsto 0$ for all $\bar{x} \in H_n(A.)$. Since $\gamma = \alpha - \beta$, this completes the proof. □

Proposition 19.2. *(We state this for the category R-Mod, but it holds in any abstract abelian category by Mitchell's embedding theorem.) Suppose we have chain complexes*

$$\to F_i \xrightarrow{\varphi_i} F_{i-1} \to \cdots \xrightarrow{\varphi_1} F_1 \to 0$$

and

$$\to G_i \xrightarrow{\psi_i} G_{i-1} \to \cdots \xrightarrow{\psi_1} G_1 \to 0,$$

where the F_i's and G_i's are projective objects. Let

$$M := H_0(F.) = F_0/\varphi_1(F_1) = \mathit{coker}(\varphi_1)$$

and

$$N := H_0(G.) = \mathit{coker}(\psi_1).$$

Then each morphism $\beta : M \to N$ is induced by a chain map $\alpha : F. \to G.$. Moreover, α is unique up to homotopy equivalence.

Proof. We can choose a map $\alpha_0 : F_0 \to G_0$. Then we get a map $\alpha_0 \circ \varphi_1 : F_1 \to G_0$, so we get a map $\alpha_1 : F_1 \to G_1$. Continuing in this way, we get maps $\alpha_n : F_n \to G_n$ for every n. Moreover, we have quotient maps $\pi_F : F_0 \to M$ and $\pi_G : G_0 \to N$, and $\beta \circ \pi_F = \pi_G \circ \alpha_0$ because this is precisely the map in homology. So extending the exact sequences with M and N, respectively, we still get exactness and everything commutes.

Suppose $x \in F_1$. Where does $\alpha_0(F_0)$ go? We have

$$\alpha_0 \circ \varphi_1(F_1) \subseteq \operatorname{im}(\psi_1).$$

But $\operatorname{im}(\psi_1) = \ker(\pi_F)$, so $\alpha_0 \circ \varphi_1 \subseteq \operatorname{im}(\psi_1)$. By a similar argument, $\alpha_1 \circ \varphi_2(F_2) \subseteq \operatorname{im}(\psi_2)$. By repeating this idea, the first claim follows by induction.

Now, suppose β is induced by $\alpha, \alpha' : F. \to G.$. We need to show $\alpha \sim \alpha'$. The two maps $\alpha_0, \alpha_0' : F_0 \to G_0$ give us a map $h_0 : F_0 \to G_1$, so $\alpha_0' - \alpha_0 = \psi_1 \circ h_0$. By commutativity and exactness, we get

$$h(\alpha_0' - \alpha_0) \subseteq \ker(\pi_G) = \operatorname{im}(\psi_1).$$

Let $\gamma_i := \alpha_i' - \alpha_i$. We want to show that

$$\psi_2 \circ h_1 + h_0 \circ \varphi_1 = \gamma_1.$$

We need $\operatorname{im}(\gamma_1 - h_0 \circ \varphi - 1) \subseteq \operatorname{im}(\varphi_2) = \ker(\psi_1)$. This holds because

$$(\psi_1 \circ \gamma_1 - \psi_1 \circ h_0 \circ \varphi_1)(x) = \varphi_1 \circ \gamma_1(x) - \gamma_0 \circ \varphi_1(x) = 0$$

by commutativity. This proves the result. □

We now discuss projective and injective resolutions.

Definition 19.3. Let $\mathcal{A}$ be an abelian category with enough projectives and injectives (e.g., R-Mod). A *projective resolution* of an object C in $\mathcal{A}$ is an exact sequence

$$\to P_2 \xrightarrow{\varphi_2} P_1 \xrightarrow{\varphi_1} P_0 \xrightarrow{\varphi_0} C \to 0$$

where each P_i is projective. An *injective resolution* of C is an exact sequence

$$0 \to C \to I_0 \to I_1 \to I_2 \to \cdots$$

where each I_k is injective.

Proposition 19.4. *If $\mathcal{C}$ has enough projectives, then projective resolutions exist in $\mathcal{C}$.*

Proof. Pick $P_0 \xrightarrow{\varphi_0} C$ where φ_0 is an epimorphism. Then letting $K := \ker(\varphi_0)$, we have an exact sequence

$$0 \to K \xrightarrow{f} P_0 \xrightarrow{\varphi_0} C \to 0.$$

There exists an epimorphism $g : P_1 \to K$ such that $f \circ g = \psi_1 : P_1 \to P_0$. Then we get an exact sequence. Let $K_1 := \ker(\psi_1)$. Then we get an exact sequence

$$K \xrightarrow{f_1} P_1 \xrightarrow{\psi_1} P_0 \xrightarrow{\psi_0} C \to 0.$$

We have an epimorphism $g_1 : P_2 \to K_1$ such that $f - 1 \circ g_1 = \psi_2 : P_2 \to P_1$. Thus, we get an exact sequence

$$P_2 \xrightarrow{\psi_2} P_1 \xrightarrow{\psi_1} P_0 \xrightarrow{\psi_0} C \to 0.$$

Continuing in this way, we get the result. □

Theorem 19.5. *Suppose that*

$$\cdots \to P_2 \xrightarrow{\varphi_2} P_1 \xrightarrow{\varphi_1} P_0 \xrightarrow{\varphi_0} C \to 0$$

and

$$0 \to Q_2 \xrightarrow{\psi_2} Q_1 \xrightarrow{\psi_1} Q_0 \xrightarrow{\psi_0} C \to 0$$

are two projective resolutions of C. Let

$$P. := \cdots \to P_2 \xrightarrow{\varphi_2} P_1 \xrightarrow{\varphi_1} P_0 \to 0$$

and

$$Q. := \cdots \to Q_2 \xrightarrow{\psi_2} Q_1 \xrightarrow{\psi_1} Q_0 \to 0.$$

Then there exist $\alpha : P. \to Q.$ and $\beta : Q. \to P.$ such that

$$\beta \circ \alpha \sim id_{P.}$$

and

$$\alpha \circ \beta \sim id_{Q.}.$$

Proof. If $F : \mathcal{C} \to \mathcal{D}$ is an additive functor between abelian categories, then $H_i(FP.) \simeq H_i(FQ.)$. Given $\alpha : P. \to Q.$, we get isomorphisms $F\alpha : H_i(FP.) \simeq H_i(FQ.)$ for each i. Using the earlier proposition, id_C is induced by $(\beta_n \circ \alpha_n)_n$ but also induced by $(\mathrm{id}_{P_i})_i$. So $\beta \circ \alpha \sim \mathrm{id}_{P.}$. Similarly, $\alpha \circ \beta \sim \mathrm{id}_Q$.

Since $\beta \circ \alpha \sim \mathrm{id}_{P.}$, we apply F to our homotopy maps h_n that witness this, and we get $F\beta \circ F\alpha \sim \mathrm{id}_{FP.}$ and $F\alpha \circ F\beta \sim \mathrm{id}_{FQ.}$. □

20 Nov. 19, 2019

I wasn't in class today; these notes are Zhihao's.

Suppose we have an exact sequence

$$0 \to A \to B \to C \to 0 \quad (*)$$

in an abelian category $\mathcal{C}$ having enough projectives. (We assume $\mathcal{C} := R$-Mod here.) Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of abelian categories, and suppose F is right-exact and additive.

Example 20.1. Suppose R is a (not necessarily commutative) ring. Let M be a left R-module and N a right R-module. Then we have a tensor product $N \otimes_R M$ that is a quotient of the free abelian group (free $\mathbb{Z}$-module) on N by the subgroup generated by the relations

$$(n + n') \otimes m = n \otimes m + n' \otimes m,$$

$$n \otimes (m + m') = n \otimes m + n \otimes m',$$

$$(nr) \otimes m = n \otimes (rm).$$

Remark 20.2. If N is an (S, R)-bimodule, then $N \otimes_R M$ is also a left S-module with $s(n \otimes m) := (sn) \otimes m$.

Then $F := - \otimes_R M : R$-Mod $\to$ Ab (or R-Mod $\to$ R-Mod if R is commutative) is right exact. We will associate a homology theory to F.

Here's the idea. If we apply F to (*) to get

$$FA \to FB \to FC \to 0$$

(noting that we don't necessarily get a short exact sequence anymore), then for all $n \geq 0$, there exist left-derived functors

$$L_n : \mathcal{C} \to \mathcal{D}$$

such that $L_0 F = F$, and (*) induces a long exact sequence

$$\begin{array}{l} \dots \longrightarrow L_2A \xrightarrow{L_2Ff} L_2B \xrightarrow{L_2Fg} L_2C \\ \xrightarrow{f_2} L_1A \xrightarrow{L_1Ff} L_1B \xrightarrow{L_1Fg} L_1C \\ \xrightarrow{f_1} L_0A = FA \longrightarrow L_0B = FB \longrightarrow L_0C = FC \longrightarrow 0 \end{array}$$

Now we describe the construction as a series of steps.

(i) Start with some $A \in \mathrm{Ob}(\mathcal{C})$.

(ii) There exists a projective resolution

$$P. = \cdots \to P_2 \xrightarrow{\varphi_2} P_1 \xrightarrow{varphi_1} P_0 \xrightarrow{\varphi_0} A \to 0.$$

(iii) Truncate the projective resolution (by removing A) to get the exact sequence

$$\cdots \to P_2 \xrightarrow{\varphi_2} P_1 \xrightarrow{\varphi_1} P_0 \to 0.$$

Observe that $H_0(P.) = A$ since $\mathrm{im}(\varphi_1) = \ker(\varphi_0)$ implies

$$H_0(P.) = P_0/\mathrm{im}(\varphi_1) = P_0/\ker(\varphi_0) \simeq A.$$

(iv) Apply F to the truncated sequence to get

$$FP. = \cdots \to FP_2 \xrightarrow{F\varphi_2} FP_1 \xrightarrow{F\varphi_1} FP_0 \to 0.$$

Note that $F(\varphi_2 \circ \varphi_3) = 0$.

(v) Compute homology $L_iFA = H_i(FP_0)$ for all $i \geq 0$. Questions: Is this well-defined? Is this a functor?

Suppose $P.$ and $Q.$ are two complexes. Put one above the other, and notice that the identity $\mathrm{id} : A \to A$ induces a chain map, so we get an isomorphism of homologies

$$H_i(FQ.) \simeq H_i(FP.).$$

Now we need to define L_iF on maps $f : A \to B$. Fix projective resolutions $P. \to A$ and $Q. \to B$, so we have a commutative diagram

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & P_2 & \xrightarrow{\varphi_2} & P_1 & \xrightarrow{\varphi_1} & P_0 & \xrightarrow{\varphi_0} & A \\
 & & \big\downarrow{\scriptstyle \beta_2} & & \big\downarrow{\scriptstyle \beta_1} & & \big\downarrow{\scriptstyle \beta_0} & & \big\downarrow{\scriptstyle f} \\
\cdots & \longrightarrow & Q_2 & \xrightarrow{\psi_2} & Q_1 & \xrightarrow{\psi_1} & Q_0 & \xrightarrow{\psi_0} & B
\end{array}$$

with exact rows where we know that f induces a chain map $\beta : P. \to Q.$. Now apply F:

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & FP_1 & \xrightarrow{F\varphi_1} & FP_0 & \xrightarrow{F\varphi_0} & FA \\
 & & \big\downarrow{\scriptstyle F\beta_1} & & \big\downarrow{\scriptstyle F\beta_0} & & \big\downarrow{\scriptstyle Ff} \\
\cdots & \longrightarrow & FQ_1 & \xrightarrow{F\psi_1} & FQ_0 & \xrightarrow{F\psi_0} & FB
\end{array}$$

Then $F\beta_i$ induces a map $L_iFA \simeq H_i(FP.) \to H_i(FQ.) \simeq L_iFB$ that is independent of the choice of β.

If β and β' both induce f, then we get a chain homotopy $h : P_{i-1} \to Q_i$ with $\gamma : \beta - \beta'$ satisfying

$$\gamma_i = \psi_{i+1} \circ h_i + h_{i-1} \circ \varphi_i.$$

Indeed, we have a commutative diagram

$$\begin{array}{ccccccc}
\cdots & \longrightarrow & P_1 & \xrightarrow{\varphi_1} & P_0 & \xrightarrow{\varphi_0} & A \\
 & \swarrow & & \swarrow{\scriptstyle h_0} & \big\downarrow{\scriptstyle \gamma_0} & \swarrow{\scriptstyle 0} & \big\downarrow{\scriptstyle f} \\
\cdots & \longrightarrow & Q_1 & \xrightarrow{\psi_1} & Q_0 & \xrightarrow{\psi_0} & B
\end{array}$$

Then since F is additive, we get

$$F(\gamma_i) = F\beta_i - F\beta_i'$$
$$= F(\psi_{i+1}) \circ F(h_i) + F(h_{i-1}) \circ F(\varphi_i).$$

Then $F(h)$ gives a chain homotopy equivalence between $F(\beta)$ and $F(\beta')$, so they induce the same homology.

Here are some nice facts.

(i) $L_0F = F$.
(ii) If A is projective, then $L_iFA = 0$ for all $i \geq 1$.

Proof of (ii). Fix a projective resolution of A:

$$\cdots \to P_2 \xrightarrow{\varphi_2} P_1 \xrightarrow{\varphi_1} P_0 \xrightarrow{\varphi_0} A \to 0.$$

Since F is right-exact, we get

$$FP_1 \xrightarrow{F\varphi_1} FP_0 \xrightarrow{F\varphi_0} FA \to 0,$$

which is still exact. We truncate to get

$$\cdots \to FP_1 \xrightarrow{F\varphi_1} FP_0 \to 0.$$

Then $L_0FA = \ker(0)/\text{im}(F\varphi_1) = FP_0/\ker(F\varphi_1) \simeq FA$. We have the following commutative diagram:

$$\begin{array}{ccccccc} P_1 & \longrightarrow & P_0 & \longrightarrow & A & \longrightarrow & 0 \\ \downarrow{\scriptstyle \beta_1} & & \downarrow{\scriptstyle \beta_0} & & \downarrow{\scriptstyle f} & & \\ Q_1 & \longrightarrow & Q_0 & \longrightarrow & B & \longrightarrow & 0 \end{array}$$

Here $A \simeq H_0(P.)$ and $B \simeq H_0(Q.)$, so the map $f : A \to B$ is induced by $\beta.$.

The computation of L_iF is independent of our choice of projective resolution if A is projective. We can write

$$P. = \cdots \to 0 \to 0 \to P_0 = A \to A \to 0.$$

We truncate to get

$$\cdots \to 0 \to 0 \to P_0 = A \to 0.$$

We apply F to get

$$\cdots \to 0 \to 0 \to FA \to 0.$$

So $H_i(FP.) = 0$ for all $i \geq 1$, and thus $L_iFA = 0$ for all $i \geq 0$. $\square$

Example 20.3. Let $F : \text{Ab} \to \text{Ab}$ be given by $F(A) := A \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$. Then we say that

$$\text{Tor}_i(A, \mathbb{Z}/2\mathbb{Z}) := L_i F(A).$$

Then $\text{Tor}_i(-, B) = L_i(- \otimes B)$. Say we wish to find $\text{Tor}_i(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z})$ for all $i \geq 0$.

We have the exact sequence

$$0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/2\mathbb{Z} \to 0.$$

We truncate to get

$$0 \to 0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \to 0.$$

We apply F to get

$$P. := 0 \to 0 \to 0 \to \mathbb{Z} \otimes_{\mathbb{Z}} (\mathbb{Z}/2\mathbb{Z})(\simeq \mathbb{Z}/2\mathbb{Z}) \to \mathbb{Z} \otimes_{\mathbb{Z}} (\mathbb{Z}/2\mathbb{Z})(\simeq \mathbb{Z}/2\mathbb{Z}) \to 0.$$

Thus, $L_i F(\mathbb{Z}/2\mathbb{Z}) = 0$ for all $i \geq 2$. Also,

$$L_0 F(\mathbb{Z}/2\mathbb{Z}) = F(\mathbb{Z}/2\mathbb{Z}) = \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \simeq \mathbb{Z}/2\mathbb{Z}.$$

For R a commutative ring, we have an isomorphism $f : R/I \otimes_R R/J \to R/(I+J)$ for ideals I, J of R. This map is given by $(y+I) \otimes (y+J) \mapsto (y) + (I+J)$.

Thus, $L_1 F(\mathbb{Z}/2\mathbb{Z}) = H_1(P.) = \mathbb{Z}/2\mathbb{Z}$.

Theorem 20.4. *Suppose we have an exact sequence*

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0,$$

and suppose F is an additive right-exact functor. We then have a long exact sequence

$$\begin{array}{ccccccc} \cdots \longrightarrow & L_2FA & \longrightarrow & L_2FB & \longrightarrow & L_2FC & \\ & & & \delta_2 & & & \\ & L_1FA & \longrightarrow & L_1FB & \longrightarrow & L_1FC & \\ & & & \delta_1 & & & \\ & FA & \longrightarrow & FB & \longrightarrow & FC & \longrightarrow 0 \end{array}$$

Example 20.5. Let $F := - \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$. We have a short exact sequence

$$0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0.$$

Note that $L_1 F\mathbb{Z} = 0$ since $\mathbb{Z}$ is projective, so we get an exact sequence

$$\begin{array}{ccccccc} 0 = L_1 F\mathbb{Z} & \longrightarrow & \mathbb{Z}/2\mathbb{Z} & & & & \\ & \swarrow_{\delta_1} & & & & & \\ \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} & \longrightarrow & \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} & \longrightarrow & \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} & \longrightarrow & 0 \end{array}$$

Proof strategy. Consider the following commutative diagram:

$$\begin{array}{ccccccccccc} & & & & 0 & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \downarrow & & \\ P. := & & \cdots & \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & A & \longrightarrow & 0 \\ & & & & \downarrow & & \downarrow & & \downarrow & & \\ U. := & & \cdots & \longrightarrow & U_1 & \longrightarrow & U_0 & \longrightarrow & B & \longrightarrow & 0 \\ & & & & \downarrow & & \downarrow & & \downarrow & & \\ Q. := & & \cdots & \longrightarrow & Q_1 & \longrightarrow & Q_0 & \longrightarrow & C & \longrightarrow & 0 \\ & & & & \downarrow & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0 & & 0 & & \end{array}$$

Fix projective resolutions:

$$P. = \cdots \to P_2 \xrightarrow{\varphi_2} P_1 \xrightarrow{\varphi_1} P_0 \to A \to 0,$$

$$Q. = \cdots \to Q_2 \xrightarrow{\psi_2} Q_1 \xrightarrow{\psi_1} Q_0 \to C \to 0.$$

Our goal is to construct

$$U_0 = \cdots \to U_2 \xrightarrow{\chi_2} U_1 \xrightarrow{\chi_1} U_0 \xrightarrow{\chi_0} B.$$

Define $U_0 := P_0 \oplus Q_0$, so the exact sequence

$$0 \to P_0 \to P_0 \oplus Q_0 \to Q_0 \to 0$$

is split. Then we have the following commutative diagram:

$$\begin{array}{ccc} 0 & & 0 \\ \downarrow & & \downarrow \\ P_0 & \xrightarrow{\varphi_0} & A \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle f} \\ P_0 \oplus Q_0 & \xrightarrow{\chi_0} & B \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle g} \\ Q_0 & \xrightarrow{\psi_0} & C \\ \downarrow & & \downarrow \\ 0 & & 0 \end{array}$$

Here the map i is given by $i(p) := (p, 0)$ and the map π is given by $\pi(p, q) := q$. We need to find χ_0. We calculate

$$\begin{aligned} \chi_0(p, q) &= \chi_0(p, 0) + \chi_0(0, q) \\ &= f \circ \varphi_0(p) + \hat{\psi}_0(q) \end{aligned}$$

where $\hat{\psi}_0$ is defined using the projectivity of Q_0 as the map that makes the following diagram commute:

$$\begin{array}{ccc} & & B \\ & \overset{\exists\hat{\psi}_0}{\nearrow} & \downarrow g \\ Q_0 & \xrightarrow{\psi_0} & C \\ & & \downarrow \\ & & 0 \end{array}$$

Then we have

$$\begin{aligned} g \circ \chi_0(p, q) &= g \circ (f \circ \varphi_0(p)) + g \circ \hat{\psi}_0(q) \\ &= 0 + \psi_0(q). \end{aligned}$$

Now that we have χ_0, we can find χ_1. We have the following commutative diagram:

$$\begin{array}{ccccccc} 0 & & & & & & \\ \downarrow & & & & & & \\ P_1 & \xrightarrow{\varphi_1} & P_0 & \xrightarrow{\varphi_0} & A & \longrightarrow & 0 \\ \downarrow i_1 & & \downarrow i_0 & & \downarrow f & & \\ P_1 \oplus Q_1 & \xrightarrow{\chi_1} & P_0 \oplus Q_0 & \xrightarrow{\chi_0} & B & & \\ \downarrow \pi_1 & & \downarrow \pi_0 & & \downarrow g & & \\ Q_1 & \xrightarrow{\psi_1} & Q_0 & \xrightarrow{\psi_0} & C & & \end{array}$$

Let $U_1 := P_1 \oplus Q_1$. We need

$$\chi_1(p, 0) = i_0 \circ \varphi_1(p).$$

Thus,

$$\pi_0(\chi_1(0, q)) = \psi_1(q).$$

Since Q_1 is projective, we get a commutative diagram

$$\begin{array}{ccc} & & P_0 \oplus Q_0 \\ & \overset{\exists\hat{\psi}_1}{\nearrow} & \downarrow \pi_0 \\ Q & \xrightarrow{\psi_1} & Q_0 \end{array}$$

Define $\chi_1(0, q) := \hat{\psi}_1(q)$. Then

$$\chi_1(p, q) = \chi_1(p, 0) + \chi_1(0, q)$$

$$= i_0 \circ \varphi_1(p) + \hat{\psi}_1(q)$$

and

$$\pi_0 \circ \chi_1(p, q) = \pi_0 \circ \hat{\psi}_1(q)$$
$$= \psi_1(q)$$

as desired. We now need to check that

$$\operatorname{im}(\chi_1) = \ker(\pi_0).$$

We will check that $\operatorname{im}(\chi_1) \subseteq \ker(\chi_0)$. We have

$$\chi_0 \circ \chi_1(p, 0) = \chi_0 \circ i_0 \circ \varphi_1(p)$$
$$= f \circ \varphi_0 \circ \varphi_1(p)$$
$$= 0.$$

We also calculate

$$\chi_0 \circ \chi_1(0, q) = \chi_0 \circ \hat{\psi}_1(q)$$
$$= (f \circ \varphi_0 + \hat{\psi}_0) \circ \hat{\psi}_1(q).$$

This proves the claim. □

21 Nov. 21, 2019

Continuing from last time, Jason gives us the following commutative diagram with exact columns:

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
P. & \longrightarrow & A \\
\downarrow & & \downarrow \\
U. & \longrightarrow & B \\
\downarrow & & \downarrow \\
Q. & \longrightarrow & C \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

From this we get the following commutative diagram with exact rows and second and third columns (from the left):

$$
\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \ker(\varphi_0) & \longrightarrow & P_0 & \xrightarrow{\varphi_0} & A \\
 & & \Big\downarrow{\scriptstyle i_0|} & & \Big\downarrow{\scriptstyle i_0} & & \Big\downarrow{\scriptstyle f} \\
0 & \longrightarrow & \ker(\chi_0) & \longrightarrow & P_0 \otimes Q_0 & \xrightarrow{\chi_0} & B \\
 & & \Big\downarrow{\scriptstyle \pi_0|} & & \Big\downarrow{\scriptstyle \pi_0} & & \Big\downarrow{\scriptstyle g} \\
0 & \longrightarrow & \ker(\psi_0) & \longrightarrow & Q_0 & \xrightarrow{\psi_0} & C \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
\end{array}
$$

The following result is called the *nine lemma*.

Lemma 21.1. *In a picture like the above where the rows are exact and the second and third columns from the left are exact, the first column is also exact.*

We apply this lemma to get exactness of the leftmost column. Fix $P_1 \xrightarrow{\varphi_1} \ker(\varphi_0) \subseteq P_0$ and $Q_1 \xrightarrow{\psi_1} \ker(\psi_0) \subseteq Q_0$. We then get a commutative diagram

$$
\begin{array}{ccccc}
P_1 & \xrightarrow{\varphi_1} & P_0 & \xrightarrow{\varphi_0} & A \\
\Big\downarrow{\scriptstyle i_1} & & \Big\downarrow{\scriptstyle i_0} & & \Big\downarrow{\scriptstyle f} \\
P_1 \oplus Q_1 & \overset{\chi_1}{\dashrightarrow} & P_0 \oplus Q_0 & \xrightarrow{\chi_0} & B \\
\Big\downarrow{\scriptstyle \pi_1} & & \Big\downarrow{\scriptstyle \pi_0} & & \Big\downarrow{\scriptstyle g} \\
Q_1 & \xrightarrow{\psi_1} & Q_0 & \xrightarrow{\psi_0} & A \\
 & & \downarrow & & \downarrow \\
 & & 0 & & 0
\end{array}
$$

Jason explains the projection maps π_i. For example, $(0, q_1) \in P_1 \oplus Q_1$ gets sent to q_1 in Q_1, and $(p_1, 0) \in P_1 \oplus Q_1$ gets sent to $0 \in Q_1$. The map π_0 is defined similarly, so $(P_0, 0) = \ker(\pi_0)$. Starting with $(p_1, q_1) \in P_1 \oplus Q_1$, we want to define $\chi_1(p_1, q_1)$ so that stuff commutes. We have

$$
\begin{aligned}
\chi_1(p_1, 0) &= \chi_1 \circ i_1(p_1) \\
&= i_0 \circ \varphi_1(p_1) \\
&= (\varphi_1(p_1), 0)
\end{aligned}
$$

and

$$\chi_1(0, q_1) = (f(q_1), \psi_1(q_1)).$$

We need to construct $h : Q_1 \to P_0$ that makes everything work. Regardless of how we choose h though, the diagram commutes. We need to choose h so that $P_1 \oplus Q_1 \xrightarrow{\chi_1} P_0 \oplus Q_0 \xrightarrow{\chi_0} B$ is exact.

Note that $\psi_1 1 : Q_1 \to \ker(\psi_0)$ is a surjection, so we can lift it along the projection $\pi_0|$ to get $\tilde{h} : Q_1 \to \ker(\chi_0) \subseteq P_0 \oplus Q_0$. We set $\tilde{h}(q_1) := (p, \psi_1(q_1))$ where $p =: h(q)$. This gives us the h we want. Define

$$\chi_1(p_1, q_1) := i_0 \circ \varphi_1(p_1) + \tilde{h}(q_1).$$

By construction, one can check that $\chi_0 \circ \chi_1 = 0$ and $\operatorname{im}(\chi_1) = \ker(\chi_0)$.

Lemma 21.2. *If P, Q, U are projective and $F : \mathcal{C} \to \mathcal{D}$ is an additive functor between abelian categories, and if $0 \to P \xrightarrow{f} U \xrightarrow{g} Q \to 0$ is exact, then*

$$0 \to FP \to FU \to FQ \to 0$$

is also exact.

Proof. Since Q is projective, there exists a section $s : Q \to U$ such that $g \circ s = \mathrm{id} : Q \to Q$. We next construct $t : U \to P$. We have an epimorphism $f : P \to \ker(g) = \operatorname{im}(f)$, and we want to construct a map from U to $\ker(g)$. You can do this by taking $\mathrm{id} - s \circ g$. We claim this maps into $\ker(g)$. Indeed, we need to check that

$$g(\mathrm{id} - s \circ g)(u) = 0$$

for $u \in U$. But we have

$$g \circ (\mathrm{id} - s \circ g) = g - g \circ s \circ g = g - \mathrm{id}_Q \circ g = 0,$$

so it works. Since U is projective, we get the desired map $t : U \to P$. Then $f \circ t = \mathrm{id}_U - s \circ g$, or

$$f \circ t + s \circ g = \mathrm{id}_U.$$

Now, $t \circ s = 0$ implies $Ft \circ Fs = 0$; we want $t \circ s$ to be a section. We know g is an epimorphism, so we calculate

$$t \circ s \circ g = t \circ (\mathrm{id}_U - f \circ t)$$

$$= t - t \circ f \circ t = 0,$$

so we get it. We now apply F to get

$$0 \to FP \xrightarrow{Ff} FU \xrightarrow{Fg} FQ \to 0,$$

and we need to check it's exact. We also have maps $Ft : FU \to FP$ and $Fs : FQ \to FU$. Since F is additive, we still have

$$Ff \circ Ft + Fs \circ Fg = \mathrm{id}_{FU}.$$

Without loss of generality, we have an inclusion of $\mathcal{D}$ into R-Mod of the sort described in the Mitchell embedding theorem, so

$$FU \simeq FP \oplus FQ.$$

We thus get a map $\phi : FU \to FP \oplus FQ$ by $x \mapsto (Ft(x), Fg(x))$. We also get a map $\psi : FP \oplus FQ \to FU$ by $(\alpha, \beta) \mapsto Ff(\alpha) + Fs(\beta)$. We calculate

$$\Psi \circ \Phi(x) = \Psi(Ftx, Fgx)$$

$$\begin{aligned} &= Ff \circ Ft(x) + Fs \circ Fg(x) \\ &= (Ff \circ Ft + Fs \circ Fg)(x) \\ &= \mathrm{id}(x) \\ &= x. \end{aligned}$$

This completes the proof. (In fact, Fs is a section.) $\square$

Corollary 21.3. *Suppose $F : \mathcal{C} \to \mathcal{D}$ is a right-exact additive functor between abelian categories with enough projectives. If $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is exact in $\mathcal{C}$, then we get a long exact sequence*

$$\begin{array}{ccccccc} \ldots \longrightarrow L_1FA \longrightarrow L_1FB \longrightarrow L_1FC \\ FA \longrightarrow FB \longrightarrow FC \longrightarrow 0 \end{array}$$

Proof. Step 1. We extend $0 \to A \to B \to C$ to get the following short exact sequence of projective resolutions:

$$\begin{array}{ccccc} 0 & & 0 & & \\ \downarrow & & \downarrow & & \\ P. & \longrightarrow & A & \longrightarrow & 0 \\ \downarrow{\scriptstyle i.} & & \downarrow{\scriptstyle f} & & \\ U. & \longrightarrow & B & \longrightarrow & 0 \\ \downarrow{\scriptstyle \pi.} & & \downarrow{\scriptstyle g} & & \\ Q. & \longrightarrow & C & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \\ 0 & & 0 & & \end{array}$$

Step 2. Let $P., U., Q.$ be the truncations, so for example

$$P. = \cdots \to P_2 \to P_1 \to P_0 \to 0.$$

Then if $0 \to P. \xrightarrow{i.} U. \xrightarrow{\pi.} Q. \to 0$ is exact, this implies that

$$0 \to FP. \xrightarrow{Fi.} FU. \xrightarrow{F\pi.} FQ. \to 0$$

is exact.

Step 3. This gives a long exact sequence

$$\cdots \longrightarrow H_2(FP_\cdot) \longrightarrow H_2(FU_\cdot) \longrightarrow H_2(FQ_\cdot) \longrightarrow$$

$$H_1(FP_\cdot) \longrightarrow H_1(FU_\cdot) \longrightarrow H_1(FQ_\cdot) \longrightarrow$$

$$H_0(FP_\cdot) \longrightarrow H_0(FU_\cdot) \longrightarrow H_0(FQ_\cdot) \longrightarrow 0$$

□

Dually, if $F : \mathcal{C} \to \mathcal{D}$ is left-exact and additive and if $\mathcal{C}$ has enough injectives, then given $C \in \mathrm{Ob}(\mathcal{C})$, there exists an injective resolution

$$0 \to C \to I^0 \to I^1 \to I^2 \to \cdots .$$

Truncate to get

$$0 \to I^0 \to I^1 \to I^2 \to \cdots .$$

We calculate $H^0(I^\cdot) = C$ and $H^i(I^\cdot) = 0$ for all $i \geq 1$. We have a right-derived functor $R^iG(C) = H^i(GI^\cdot)$ with the cochain complex

$$0 \to GI^0 \to GI^1 \to \cdots .$$

We have $R^0G = G$, and if C is injective, $R^iGC = 0$ for all $i \geq 1$. Indeed, consider the exact sequence

$$0 \to C \to I_0 = C \to 0$$

where C is injective. The result is independent of the injective resolution.

Example 21.4. I'm going to write $\{1, i, -1, -i\}$ for $\mathbb{Z}/4\mathbb{Z}$ and $\{\pm 1\}$ for $\mathbb{Z}/2\mathbb{Z}$ in this example. Take $M := \{1, i, -1, -i\}$ and $N := \{\pm 1\} \times \{\pm 1\}$.

If M is an R-module with R commutative, then $\mathrm{Hom}(M, -) : R\text{-Mod} \to R\text{-Mod}$ is an additive left-exact functor. We define

$$\mathrm{Ext}^i_R(M, -) := R^i\mathrm{Hom}(M, -).$$

We will compute $\mathrm{Ext}^i_{\mathbb{Z}}(M, N)$ for this example.

We want to include N in some injective object I. We characterized injective abelian groups in an assignment. We have a map $N \to H_2 := \{\omega \mid \omega^{2^n} = 1\}$, where the latter is the Prüfer 2-group and is injective. We get a short exact sequence

$$0 \to N \to H_2 \times H_2 \to H_2 \times H_2/N \simeq H_2 \times H_2 \to 0.$$

Here the map $H_2 \times H_2 \to H_2 \times H_2/N \simeq H_2 \times H_2$ is given by $(x, y) \mapsto (x^2, y^2)$. Applying $\mathrm{Hom}(M, -)$, we get an exact sequence

$$0 \to \mathrm{Hom}(M, H_2 \times H_2) \to \mathrm{Hom}(M, H_2 \times H_2) \to 0.$$

We calculate

$$\mathrm{Ext}^0_{\mathbb{Z}}(M, N) = \mathrm{Hom}(M, N) \simeq N$$

and

$$\mathrm{Ext}^i_{\mathbb{Z}}(M, N) = 0$$

for all $i \geq 2$. What is $\mathrm{Ext}^1_{\mathbb{Z}}(M, N)$? It suffices to determine $\mathrm{Hom}(M, H_2 \times H_2)$. This turns out to be isomorphic to $M \times M$. So we get an exact sequence

$$0 \to M \times M \xrightarrow{\phi} M \times M \xrightarrow{\psi} 0.$$

We have

$$H^1 = \ker(\psi)/\mathrm{im}(\phi) = M \times M/\mathrm{im}(\phi).$$

So we have to find $\mathrm{im}(\phi)$. Now given a map $f \in \mathrm{Hom}(M, H_2 \times H_2)$, the map $\mathrm{Hom}(M, H_2 \times H_2) \to \mathrm{Hom}(M, H_2 \times H_2)$ is given by composing with the map $H_2 \times H_2 \to H_2 \times H_2$ given by $(x, y) \mapsto (x^2, y^2)$. So $\mathrm{Hom}(M, H_2 \times H_2) \simeq M \times M/(\{\pm 1\} \times \{\pm 1\})$. So

$$\begin{aligned} H^1 &= M \times M/\mathrm{im}(\phi) \\ &\simeq M \times M/(\{\pm 1\} \times \{\pm 1\}) \\ &\simeq (M/\{\pm 1\}) \times (M/\{\pm 1\}) \\ &\simeq \{\pm 1\} \times \{\pm 1\}. \end{aligned}$$

Suppose R and S are rings, not necessarily commutative. Let M be a right R-module and N a left S-module. We have a functor $F : R\text{-Mod} \to S\text{-Mod}$ given by $N \mapsto M \otimes_R N$. Tor is the left-derived functor of this functor.

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Let R and S be unital but not necessarily commutative rings. Let M be a right R-module. Let N be a left S-module. Consider the functor $M \otimes - : R\text{-Mod} \to S\text{-Mod}$. Recall that $M \otimes N$ is a free S-module on symbols $e_{m,n}$ where $m \in M$ and $n \in N$ modulo the equivalence relation $\sim$ given by

$$e_{mr,n} \sim e_{m,rn},$$

$$e_{sm_1+m_2,n} \sim se_{m_1,n} + e_{m_2,n},$$

$$e_{m,n_1+n_2} \sim e_{m,n_1} + e_{m,n_2}.$$

Remark 22.1. Note that if R is commutative, then $M \otimes_R N \simeq N \otimes_R M$. In general,

$$\mathrm{Tor}^R_i(M, N) \simeq \mathrm{Tor}^R_i(N, M).$$

The functor $M \otimes - : R\text{-Mod} \to S\text{-Mod}$ is right exact and additive. Also,

$$\mathrm{Tor}^R_i(M, -) = L_i(M \otimes -).$$

The functor $\operatorname{Tor}_i(M, -)$ measures how far M is from being flat as an R-module.

Theorem 22.2. *The following are equivalent.*

(i) M is flat (i.e., if $N' \xrightarrow{f} N$ is one-to-one, then so is the induced map $M \otimes N' \to M \otimes N$).

(ii) $Tor_i^R(M, N) = 0$ for all $i \geq 1$ and all $N \in Ob(R\text{-}Mod)$.

(iii) $Tor_1^R(M, N) = 0$ for all $N \in Ob(R\text{-}Mod)$.

Proof. (ii) $\implies$ (iii): Immediate.

(i) $\implies$ (ii): Assume M is flat. Choose a projective resolution

$$\cdots \to P_2 \to P_1 \to P_0 \to N \to 0.$$

Then by flatness of M, the following sequence is exact:

$$\cdots \to M \otimes P_2 \to M \otimes P_1 \to M \otimes P_0 \to M \otimes N \to 0.$$

Now $\operatorname{Tor}_i(M, N) = H_i(M \otimes P.)$ where $P.$ is the truncated complex. Thus, we consider the exact sequence

$$\cdots \to M \otimes P_2 \to M \otimes P_1 \to M \otimes P_0 \to 0.$$

Thus, $\operatorname{Tor}_i(M, N) = 0$ if $i \geq 1$ and $M \otimes N$ if $i = 0$.

(iii) $\implies$ (i): Let $0 \to N' \xrightarrow{f} N$ be exact, so f is a monomorphism. Without loss of generality, we can assume f is an inclusion. Then we have a short exact sequence

$$0 \to N' \hookrightarrow N \to N/N' \to 0.$$

Applying $M \otimes -$, we get a long exact sequence

$$\cdots \longrightarrow \operatorname{Tor}_1(M, N/N') \longrightarrow M \otimes N' \longrightarrow M \otimes N \longrightarrow M \otimes (N/N') \longrightarrow 0$$

But $\operatorname{Tor}_1(M, N/N') = 0$, so get an exact sequence

$$0 \to M \otimes N' \to M \otimes N \to M \otimes (N/N') \to 0,$$

which implies M is flat. ∎

In fact, we can refine this.

Theorem 22.3. *The following are equivalent.*
(i) M is flat. [Jason writes that there is an (S, R)-bimodule structure induced by the map $M \otimes - : R\text{-}Mod \to S\text{-}Mod$, but this isn't part of the theorem statement.]

(ii) $M \otimes_R I \to M \otimes_R R \simeq M$ *is one-to-one for all left ideals* I *of* R*.*

(iii) $Tor_1(M, R/I) = 0$ *for all left ideals* I *of* R*.*

Proof. (i) $\implies$ (ii): Immediate.

(ii) $\implies$ (iii): We have a short exact sequence

$$0 \to I \to R \to R/I \to 0.$$

Applying $M \otimes -$ gives a long exact sequence

$$\begin{array}{ccccccc} \cdots \longrightarrow \operatorname{Tor}_1^R(M, I) & \longrightarrow & \operatorname{Tor}_1^R(M, R) & \longrightarrow & \operatorname{Tor}_1^R(M, R/I) & & \\ & & & \swarrow & & & \\ M \otimes I & \longrightarrow & M \otimes R & \longrightarrow & M \otimes (R/I) & \longrightarrow & 0 \end{array}$$

The map $M \otimes I \to M \otimes R$ in this sequence is one-to-one by assumption. We get an exact sequence

$$\operatorname{Tor}_1^R(M, R) \to \operatorname{Tor}_1^R(M, R/I) \to 0.$$

Recall that if P is projective, then $\operatorname{Tor}_i^R(M, P) = 0$ for all $i \geq 1$. In particular, since R is projective, we get $\operatorname{Tor}_1^R(M, R/I) = 0$.

(iii) $\implies$ (i): Assume (i) does not hold. Then there exists an exact sequence $0 \to N' \xrightarrow{f} N$ such that $0 \to M \otimes N' \xrightarrow{\text{id} \otimes f} M \otimes N$ is not exact. Pick $x = m_1 \otimes n_1 + \cdots + m_s \otimes n_s$ in $\ker(\text{id} \otimes f)$. Let $N_0 := Rn_1 + \cdots + Rn_s \subseteq N'$. There is a map $\text{id} \otimes f|_{N_0} : M \otimes N_0 \to M \otimes N$ given by $x \mapsto 0$. Without loss of generality, take $N' = N_0$ and N_0 finitely-generated.

By assumption,

$$x = m_1 \otimes n_1 + \cdots + m_s \otimes n_s (= m_1 \otimes f(n_1) + \cdots + m_s \otimes f(n_s))$$

is zero in $M \otimes N$. Thus, $x \sim 0$. We have

$$e_{m_1, n_1} + \cdots + e_{m_s, n_s}$$

$$= c_1(e_{a_1 r_1, b_1} - e_{a_1, r_1 b_1}) + \cdots + c_\ell(e_{a_s r_s, b_s} - e_{a_s, r_s b_s}) + b_1(e_{\theta_1 + \theta_1', \psi_1} - e_{\theta_1, \psi_1} - e_{\theta_1', \psi_1}) + \cdots + b_t(e_{\theta_t + \theta_{t'}, \psi_t} - \cdots) + \cdots .$$

Let $\tilde{N}$ be the submodule of N given by B_0 and containing the b_i's and ψ_i's. Then $x = 0$ in $M \otimes \tilde{N}$ since the relations giving $x = 0$ in $M \otimes N$ are implied by relations in $M \otimes \tilde{N}$. Without loss of generality, we can assume $N' = N_0$ and $N = \tilde{N}$ are finitely-generated.

Then we have a filtration

$$N' = N_0 \subseteq N_1 \subseteq \cdots \subseteq N_r = N$$

where N_i / N_{i-1} is cyclic, i.e., generated by one element. Notice that if $M \otimes N' \to M \otimes N$ is not one-to-one, then we can consider the composition

$$\mathrm{id} \otimes f : M \otimes N_0 \to M \otimes N_1 \to M \otimes N_2 \to \cdots \to M \otimes N_r,$$

and we see that there exists i such that $M \otimes N_i \to M \otimes N_{i+1}$ is not one-to-one. We consider the short exact sequence

$$0 \to N_i \to N_{i+1} \to N_{i+1}/N_i \to 0.$$

We get a short exact sequence

$$0 \to I \to R \xrightarrow{\psi} N_{i+1}/N_i \to 0$$

since $I = \ker(\psi)$, so

$$R/I \simeq N_{i+1}/N_i.$$

Now, tensoring the exact sequence $0 \to N_i \to N_{i+1} \to N_{i+1}/N_i \to 0$ on the left with M, we get a long exact sequence

$$\cdots \longrightarrow \mathrm{Tor}_1^R(M, R/I) \longrightarrow M \otimes N_i \longrightarrow M \otimes N_{i+1} \longrightarrow M \otimes (R/I) \simeq M \otimes (N_{i+1}/N_i) \longrightarrow 0$$

Then since $\mathrm{Tor}_1^R(M, R/I) = 0$ by assumption, we get that M is flat. □

Corollary 22.4. *Let $R := k[t]/(t^2)$. Let M be an R-module. Then M is flat if and only if $M/\bar{t}M \simeq \bar{t}M$ where $\bar{t} = t + (t^2) \in R$.*

Proof. Note that M is flat if and only if $\mathrm{Tor}_1^R(M, R/J) = 0$ for all ideals J of R. Ideals of $k[t]/(t^2)$ are in one-to-one correspondence with ideals of $k[t]$ that contain t^2, and $k[t]$ is a PID so "containing t^2" is the same as "dividing t^2". Thus we just get the ideals (t), (t^2), and (1). In $R = k[t]/(t^2)$, these correspond to $(\bar{t})$, (0), and R, respectively. Now, $\mathrm{Tor}_1(M, (0)) = 0$ and $\mathrm{Tor}_1(M, R) = 0$. (They both have free resolutions of length one.) So M is flat if and only if $\mathrm{Tor}_1(M, R/(\bar{t})) = 0$.

We have a projective resolution

$$\cdots \to R \to R \to R \to R/(\bar{t}) \to 0$$

where each map $R \to R$ is given by $1 \mapsto \bar{t}$. Applying $M \otimes -$ gives the exact sequence

$$\cdots \to M \otimes R \to M \otimes R \to M \otimes R/(\bar{t}) \to 0.$$

Now, M is flat if and only if $H_1(M \otimes R.) = 0$ since $H_1(M \otimes R.) = \mathrm{Tor}_1(M, R/(\bar{t}))$. Since $M \otimes_R R \simeq M$, the previous exact sequence gives an exact sequence $M \to M \to M$ where each map is $m \mapsto \bar{t}m$. Since the image is isomorphic to M modulo the kernel, $\bar{t}M \simeq M/\bar{t}M$. But the kernel modulo $\bar{t}M$ is isomorphic to $\mathrm{Tor}_1(M, R/(\bar{t}))$. Thus, the kernel is $\bar{t}M$ since $\mathrm{Tor}_1(M, R/(\bar{t})) = 0$. □

Corollary 22.5. *Let R be a commutative PID. Then M is flat if and only if M is torsion-free.*

Example 22.6. If $R = \mathbb{Z}$, then $\mathbb{Z}/5\mathbb{Z}$ is not flat because we can consider the exact sequence

$$0 \to \mathbb{Z} \xrightarrow{\times 5} \mathbb{Z},$$

and then

$$0 \to \mathbb{Z} \otimes \mathbb{Z}/5\mathbb{Z} \xrightarrow{0} \mathbb{Z} \otimes \mathbb{Z}/5\mathbb{Z}$$

is not exact.

Proof. If M is not torsion-free, then M is not flat. For the other direction, pick $0 \neq m \in M$ and $0 \neq r \in R$ such that $rm = 0$. Since we are in an integral domain, $x \mapsto rx$ is injective. We can consider the exact sequence

$$0 \to R \xrightarrow{x \mapsto rx} R$$

and tensor with M to get

$$0 \to M \otimes_R R \to M \otimes_R R$$

where the map $M \otimes R \to M \otimes R$ is given on non-zero elements by $0 \neq m \otimes 1 \mapsto m \otimes r = (mr) \otimes 1 = 0$. Thus, M is not flat. □

Corollary 22.7. *A $\mathbb{Z}$-module is injective if and only if it is divisible, projective if and only if it is free, and flat if and only if it is torsion-free.*

Proof. Show $\mathrm{Tor}_1^R(M, R/I) = 0$ for all ideals I of R. Since R is a PID, each ideal $I = Ra$ for some $a \in R$. Thus, $\mathrm{Tor}_1^R(M, R/Ra) = 0$ for all $0 \neq a \in R$, and when $a = 0$ it already vanishes. We have a short exact sequence

$$0 \to R \xrightarrow{r \mapsto ar} R \xrightarrow{\pi} R/Ra \to 0.$$

Tensoring with M gives the long exact sequence

$$\cdots \longrightarrow \mathrm{Tor}_1^R(M, R) \longrightarrow \mathrm{Tor}_1^R(M, R) \longrightarrow \mathrm{Tor}_1^R(M, R/Ra) \longrightarrow M \otimes R \longrightarrow M \otimes R \longrightarrow \otimes R/(Ra) \longrightarrow 0$$

We have the exact sequence

$$0 \to \mathrm{Tor}_1(M, R/Ra) \to M \otimes R \xrightarrow{m \otimes x \mapsto m \otimes ax} M \otimes R.$$

To show

$$\mathrm{Tor}_1(M, R/Ra) = 0,$$

it suffices to show that the map $m \otimes x \mapsto m \otimes ax$ is one-to-one. But $M \otimes R = M \otimes_R R \simeq M$, and our map $M \to M$ is given by $m \mapsto am$. This map is injective since $a \neq 0$ and M is torsion-free. Thus,

$$\mathrm{Tor}_1^R(M, R/Ra) = 0$$

for all $a \in R$, so M is flat. $\square$

23 Nov. 28, 2019

Jason starts by talking about how Tor is involved in calculating intersections of varieties. He gives the example of two lines in $\mathbb{C}^3$ that do not intersect. This is the generic situation. We have $L_1 \cap L_2 = \emptyset$ if and only $I(L_1) + I(L_2) = \mathbb{C}[x, y, z]$.

Now, R is commutative, then $R/I \otimes R/J \simeq R/(I + J)$ since we have a bilinear onto map $R/I \times R/J \to R/(I + J)$ given by $(x + I, y + J) \mapsto xy + I + J$, and this induces the desired isomorphism.

Notice that I and J are comaximal iff $I + J = R$ iff $R/I \otimes R/J \simeq R/(I + J) = (0)$ iff $\mathrm{Tor}_0^R(R/I, R/J) = (0)$.

Similarly, less generic intersections can be described by the vanishing of $\mathrm{Tor}_i^R(R/I, R/J) = 0$ for $i \geq 1$. See Exercise 4 of Assignment 4. Jason mentions Serre's intersection formula. The idea is that the length of each module in the sum should give you the multiplicity of the part of the intersection that's "bad to dimension $\geq i$", i.e., where the overlap of two components is dimension i in each component. Or something like that. Then it makes sense you have an alternating sum.

Suppose M is an R-module. Let $G := \mathrm{Hom}(M, -)$. Then $R^iG = \mathrm{Ext}_R^i(M, -)$. Equivalently, we can consider the functor

$$\bar{G} := \mathrm{Hom}(-, N) : R\text{-Mod}^{\mathrm{op}} \to \mathrm{Ab}.$$

This is left exact and additive. We want to calculate $R^i\bar{G}$. We choose an injective resolution in R-Mod$^{\mathrm{op}}$:

$$0 \to M \to I^0 \to I^1 \to I^2 \to \cdots .$$

This is just a projective resolution in R-Mod:

$$\cdots \to P^2 \to P^1 \to P^0 \to M \to 0.$$

We apply $\mathrm{Hom}(-, N)$ to get:

$$0 \to \mathrm{Hom}(M, N) \to \mathrm{Hom}(P^0, N) \to \mathrm{Hom}(P^1, N) \to \cdots .$$

We truncate our projective resolution and apply $\bar{G}$ to get

$$\bar{G}P^{\cdot} = \cdots \to \mathrm{Hom}(P^1, N) \to \mathrm{Hom}(P^0, N) \to 0.$$

We have that $H^i(\bar{G}, P^{\cdot}) = \mathrm{Ext}_R^i(M, N)$.

Example 23.1. Let $R := \mathbb{C}[x]$, $M := \mathbb{C}[x]/(x^3 - 1)$, $N := \mathbb{C}[x]/(x^2 - 1)$. Let's compute $\operatorname{Ext}^i_R(M, N)$ for all $i \geq 0$.

We use the classification of indecomposable injective $\mathbb{C}[x]$-modules. Recall the classification of indecomposable injective $\mathbb{Z}$-modules: we got $\mathbb{Q}$ and the Prüfer p-group $\mathbb{Z}(p^\infty)$ for any prime p, and that's it. Jason explains that this is because $\mathbb{Z}(p^\infty) = \varinjlim \mathbb{Z}/p^n\mathbb{Z}$ So we got something for each non-zero prime ideal of $\mathbb{Z}$. And for $\mathfrak{p} = (0)$, we instead take the direct limit of $\operatorname{Frac}(R/\mathfrak{p}) \to \operatorname{Frac}(R/\mathfrak{p}^2) \to \cdots$, and since $\mathfrak{p}^2 = \mathfrak{p}$, this is just $\operatorname{Frac}(\mathbb{Z}) = \mathbb{Q}$. So we get every indecomposable injective this way!

You can do the same thing for $\mathbb{C}[x]$. The prime ideals are of the form (0) or $(x - \lambda)$ for some $\lambda \in \mathbb{C}$. In the first case, we just get $\operatorname{Frac}(\mathbb{C}[x]/(0)) = \mathbb{C}(x)$. In the second case, we get the direct limit of

$$\mathbb{C}[x]/(x-\lambda) \to \mathbb{C}[x]/(x-\lambda)^2 \to \cdots .$$

Let's call this I_λ. Now we can continue with the example.

First, we take a projective resolution of M:

$$0 \to 0 \to \mathbb{C}[x] \to \mathbb{C}[x] \to \mathbb{C}[x]/(x^3 - 1) \to 0.$$

Here the map $\mathbb{C}[x] \to \mathbb{C}[x]$ is given by $1 \mapsto (x^3 - 1)$ and the map $\mathbb{C}[x] \to \mathbb{C}[x]/(x^3 - 1)$ is given by $1 \mapsto 1 + (x^3 - 1)$. Applying $\operatorname{Hom}(-, N)$, we get

$$0 \to \operatorname{Hom}(\mathbb{C}[x], \mathbb{C}[x]/(x^2 - 1)) \to \operatorname{Hom}(\mathbb{C}[x], \mathbb{C}[x]/(x^2 - 1)) \to 0 \to 0.$$

Thus,

$$\begin{aligned} \operatorname{Ext}^0_{\mathbb{C}[x]}(M, N) &= \operatorname{Hom}(M, N) \\ &= \operatorname{Hom}(\mathbb{C}[x]/(x^3 - 1), \mathbb{C}[x]/(x^2 - 1)) \\ &\simeq \mathbb{C}[x]/(x - 1). \end{aligned}$$

Jason explains this last isomorphism on the board, but he says maybe it's easier to check on our own, and I don't write it down. Also, $\operatorname{Ext}^i_{\mathbb{C}[x]}(M, N) = 0$ for all $i \geq 2$.

What is Ext^1? We have the sequence:

$$\operatorname{Hom}_{\mathbb{C}[x]}(\mathbb{C}[x], \mathbb{C}[x]/(x^2 - 1)) \to \operatorname{Hom}_{\mathbb{C}[x]}(\mathbb{C}[x], \mathbb{C}[x]/(x^2 - 1)) \to 0,$$

which can be rewritten as

$$P_0 := \mathbb{C}[x]/(x^2 - 1) \xrightarrow{f} \mathbb{C}[x]/(x^2 - 1) =: P_1 \to 0.$$

What is f? It's not the identity. We go back to our projective resolution. The map $\mathbb{C}[x] \to \mathbb{C}[x]$ is given there by $1 \mapsto (x^3 - 1)$. So the corresponding map

$$\operatorname{Hom}(\mathbb{C}[x], \mathbb{C}[x]/(x^2 - 1)) \to \operatorname{Hom}(\mathbb{C}[x], \mathbb{C}[x]/(x^2 - 1))$$

is given by sending ψ where $\psi : P_0 \to \mathbb{C}[x]/(x^2-1)$ to the map that precomposes ψ with the map $P_1 \to P_0$. So f sends $\bar{1}$ to (x^3-1). It follows that

$$\begin{aligned}&\mathrm{Ext}^1(M,N)\\ &= \ker(\mathbb{C}[x]/(x^2-1) \to 0)/\mathrm{im}(f)\\ &= (\mathbb{C}[x]/(x^2-1))/((x^3-1)\mathbb{C}[x]/(x^2-1))\\ &\simeq \mathbb{C}[x]/(p(x))\end{aligned}$$

since the above module is some cyclic $\mathbb{C}[x]$-module,

$$= \mathbb{C}[x]/(x-1)$$

since $p(x)$ has to divide $(x-1)$ but can't be 1. This completes the example.

Theorem 23.2. *The following statements are equivalent.*

(i) M is projective.
(ii) $\mathrm{Ext}^i_R(M,N) = 0$ for all N and all $i \geq 1$.
(iii) $\mathrm{Ext}^1_R(M,N) = 0$ for all N.

Proof. (i) $\implies$ (ii): Since M is projective, we have a projective resolution

$$0 \to 0 \to P_0 = M \to M \to 0.$$

In the usual way, we get $\mathrm{Ext}^i(M,N) = H^i(\mathrm{Hom}(P^\cdot, N))$. Since the first two terms of our projective resolution are 0, we get $H^i(\mathrm{Hom}(P^\cdot, N)) = 0$ for all $i \geq 1$.

(ii) $\implies$ (iii): Immediate.

(iii) $\implies$ (i): Given a short exact sequence

$$0 \to A \to B \to C \to 0,$$

we apply $\mathrm{Hom}(M,-)$ to get a long exact sequence

$$0 \longrightarrow \mathrm{Hom}(M,A) \longrightarrow \mathrm{Hom}(M,B) \longrightarrow \mathrm{Hom}(M,C) \longrightarrow \mathrm{Ext}^1(M,A) = 0 \text{ by (iii)} \longrightarrow \cdots$$

So $\mathrm{Hom}(M,-)$ is exact, which implies M is projective. □

We now discuss Yoneda's Ext. Suppose A, B are R-modules. Consider short exact sequences

$$\alpha := 0 \to A \to X \to B \to 0$$

and

$$\beta := 0 \to A \to Y \to B \to 0.$$

We say $\alpha \sim \beta$ if there exists $f : X \to Y$ such that the following diagram commutes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & X & \longrightarrow & B & \longrightarrow & 0 \\ & & \big\downarrow{\scriptstyle \mathrm{id}} & & \big\downarrow{\scriptstyle f} & & \big\downarrow{\scriptstyle \mathrm{id}} & & \\ 0 & \longrightarrow & A & \longrightarrow & Y & \longrightarrow & B & \longrightarrow & 0 \end{array}$$

This gives an equivalence relation, although we do not prove it. *Yoneda's* Ext^1, denoted $E^1_{\mathrm{Yon}}(A,B)$, is the collection of short exact sequences of the above form in A and B modulo equivalence. Then one can show that $E^1_{\mathrm{Yon}}(A,B) \simeq \mathrm{Ext}^1_R(A,B)$. More generally, we can consider exact sequences

$$0 \to A \to X_1 \to \cdots \to X_i \to B \to 0$$

and

$$0 \to A \to Y_1 \to \cdots \to Y_i \to B \to 0,$$

and we can say they're equivalent, denoted $\sim$ again, if we have maps $f_i : X_i \to Y_i$ for each i making the obvious diagram commute. Then we define $E^i_{\mathrm{Yon}}(A,B)$ to be exact sequences in A and B of this form modulo $\sim$.

Someone asks what the abelian group structure of $E^i_{\mathrm{Yon}}(A,B)$ is, since apparently there is one if we're comparing it to Ext. Jason doesn't remember off the top of his head, but it's called the *Baer sum* and is on Wikipedia at `https://en.wikipedia.org/wiki/Ext_functor#The_Baer_sum_of_extensions`.

In particular, one can show that if we take $A = B = C$, we get that $\bigoplus \mathrm{Ext}^i_R(A,A)$ is a ring. If R is a local k-algebra with maximal ideal $\mathfrak{m}$ and with $R/\mathfrak{m} \simeq k$, then

$$\bigoplus_{i \geq 0} \mathrm{Ext}^i_R(k,k)$$

is called the *Yoneda Ext algebra of* R.

We now discuss group cohomology. Let G be a group. Let

$$R := \mathbb{Z}[G] = \left\{ \sum_{g \in G} a_g g \mid a_g \in \mathbb{Z}, a_g = 0 \text{ for all but finitely many } g \right\}$$

be the group ring of G (with the usual operations). Let M be a left R-module. Then we have a functor $I : R\text{-Mod} \to \mathrm{Ab}$ given by

$$I(M) = M^G := \{ m \in M \mid g \cdot m = m \text{ for all } m \in M \}.$$

On Assignment 4, we show I is left exact and additive but not right exact. We can form the right derived functors

$$R^i I(M) =: H^i(G, M).$$

Now, $I \simeq \mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}, -)$. We have

$$H^i(G, M) = R^i I(M) \simeq R^i\mathrm{Hom}(\mathbb{Z}, M) = \mathrm{Ext}^i_{\mathbb{Z}[G]}(\mathbb{Z}, M).$$

24 Dec. 3, 2019

THE ASSIGNMENT AND FINAL PROJECT ARE NOW BOTH DUE DEC. 11 BY 5 PM.

Let G be a group, A a G-module. Recall that $\mathrm{Ext}^i(\mathbb{Z}, A) = H^i(G, A)$. We have the group of G-invariants, denoted A^G. We can also define the group of *coinvariants*,

$$A_G := A/\langle g \cdot a - a \mid a \in A\rangle.$$

Taking coinvariants is right exact. We have

$$A \otimes_{\mathbb{Z}[G]} \mathbb{Z}$$

$$\begin{aligned} &\simeq A \otimes_{\mathbb{Z}[G]} \mathbb{Z}[G]/\langle g - 1 \mid g \in G\rangle \\ &\simeq A/\langle (g-1)a \mid a \in A\rangle \\ &= A_G. \end{aligned}$$

The i^{th} homology group, denoted $H_i(G, A)$, can be defined as the left derived functor of the right exact and additive coinvariants functor $A \mapsto A_G$. Then

$$H_i(G, A) \simeq \mathrm{Tor}_i^{\mathbb{Z}[G]}(\mathbb{Z}, A).$$

Notice that to compute $\mathrm{Ext}^i_{\mathbb{Z}[G]}(\mathbb{Z}, A)$, we can take a projective resolution of $\mathbb{Z}$. We can even do it with the following standard free resolution of $\mathbb{Z}$. First, we have a map $g \mapsto 1$, which gives an exact sequence $\mathbb{Z}[G] \xrightarrow{\psi_0} \mathbb{Z} \to 0$. Now $\ker(\psi_0) = \langle g - 1 \mid g \in G\rangle$. So we extend to an exact sequence

$$\mathbb{Z}[G \times G] \xrightarrow{\psi_1} \mathbb{Z}[G] \xrightarrow{\psi_0} \mathbb{Z} \to 0.$$

We claim that $\mathbb{Z}[G \times G]$ is a free $\mathbb{Z}[G]$-module. Indeed, G acts on it by $g \cdot (g_1, g_2) = (gg_1, gg_2)$, and inside $\mathbb{Z}[G \times G]$, there is the diagonal $\Delta(G) = \{(g, g) \mid g \in G\}$. We can write

$$\mathbb{Z}[G \times G] = \bigoplus_{g \in G} (g, 1) \cdot \mathbb{Z}[\Delta(G)].$$

Indeed,

$$G \times G = \bigcup_{g \in G} (g, 1) \cdot \Delta(G).$$

By decomposing G as the union of cosets $H_1, \ldots, H_s$, we can write

$$\mathbb{Z}[G] = \{\sum_{g \in G} a_g g\} = \{\sum_{i=1}^{s} \sum_{h \in H_i} a_h h\}.$$

This gives the desired decomposition. Why do we have $\mathrm{im}(\psi_1) \supseteq \ker(\psi_0)$? Note that $x = \sum_{g \in G} c_g g$ is in $\ker(\psi_0)$ if and only if $\sum c_g = 0$. Then

$$\begin{aligned} x &= \sum_{g \in G} c_g (g - 1) \\ &= \sum_{g} c_g \psi_1(g, 1) \\ &= \sum c_g g - \sum c_g 1 \\ &= x. \end{aligned}$$

Thus, $x \in \mathrm{im}(\psi_1)$.

In general, we have a free resolution

$$\cdots \to \mathbb{Z}[G^3] \xrightarrow{\psi_2} \mathbb{Z}[G^2] \xrightarrow{\psi_1} \mathbb{Z}[G] \xrightarrow{\psi_0} \mathbb{Z} \to 0.$$

The map $\psi_i : \mathbb{Z}[G^{i+1}] \to \mathbb{Z}[G^i]$ is given by

$$\psi_i(g_0, g_1, \ldots, g_i) = \sum_{j=0}^{i} (-1)^j (g_0, g_1, \ldots, \hat{g}_j, \ldots, g_i)$$

where the hat indicates that $\hat{g}_j$ is excluded from the tuple. Now we can apply $\mathrm{Hom}_{\mathbb{Z}[G]}(-, A)$ to get

$$0 \to \mathrm{Hom}(\mathbb{Z}[G], A) \to \mathrm{Hom}(\mathbb{Z}[G^2], A) \to \cdots .$$

Then we can compute $H^i(G, A)$.

We now discuss the correspondence between vector bundles and projective modules. (This is due to Serre and Swan.)

Definition 24.1. Let S be a connected compact real manifold. A *real vector bundle of rank n over S* is a topological space V with a continuous map $\pi : V \to S$ such that:

(i) For every $x \in S$,

$$\pi^{-1}(x) = \{v \in V \mid \pi(v) = x\}$$

is a real vector space of dimension n.

(ii) For every $x \in S$, there exists an open neighbourhood $U \ni x$ in S and a homeomorphism $\varphi : U \times \mathbb{R}^n \to \pi^{-1}(U)$ such that $\pi \circ \varphi = p$ where $p : U \times \mathbb{R}^n \to U$ is projection onto the first coordinate. (In particular, this implies that $\pi^{-1}(U) \simeq U \times \mathbb{R}^n$.) Also, for every $y \in U$, $\varphi|_{\{y\}\times\mathbb{R}^n} : \{y\} \times \mathbb{R}^n \to \pi^{-1}(\{y\})$ is a lienar isomorphism of vector spaces.

The vector bundle V is *trivial* if $V \simeq S \times \mathbb{R}^n$, i.e., condition (ii) can be arranged with $U = S$.

Given a connected compact real manifold S, we let $C(S)$ be the continuous functions $f : S \to \mathbb{R}$ equipped with pointwise addition and multiplication. Then there is a correspondence between vector bundles V of rank n over S and finitely-generated projective $C(S)$-modules $P(V)$ of rank n.

How do we produce $P(V)$? Let $\pi : V \to S$ be the map coming from the definition of a vector bundle. A *section* s of (V, π) is a continuous map $s : S \to V$ such that $\pi(s(x)) = x$ for all $x \in S$. Let $P(V)$ be the set of sections $s : S \to V$. If $s \in P(V)$ and $f \in C(S)$, then define

$$(f \cdot s)(x) := f(x)s(x).$$

This equips $P(V)$ with a $C(S)$-module structure (with the usual pointwise addition: for $s_1, s_2 \in P(V)$, $(s_1 + s_2)(x) = s_1(x) + s_2(x)$). The following theorem is due to Swan.

Theorem 24.2. *If S is a connected compact real manifold and V is a vector bundle of rank n over S, then $P(V)$ is a rank n projective $C(S)$-module. Moreover, the functor $V \mapsto P(V)$ gives an equivalence of categories between vector bundles over S and finitely-generated projective $C(S)$-modules.*

Recall Swan's example. Let $A := \mathbb{R}[x, y, z]/(x^2 + y^2 + z^2 - 1)$ be the coordinate ring of S^2. Swan produced a projective but not free A-module P such that $P \oplus A \simeq A^3$ but $P \not\simeq A^2$. Recall the following theorem.

Theorem 24.3. *Suppose R is a unital commutative ring, Spec(R) is connected, and P is a stably free and rank 1 R-module. Then P is free.*

Proof. (i) Given an R-module M, define the i^{th} *exterior (wedge) product* by

$$\bigwedge^i M := M \otimes \cdots M/L$$

where there are i terms in the tensor product and where

$$L := \langle m_{\sigma(1)} \otimes \cdots \otimes m_{\sigma(i)} = \text{sgn}(\sigma) m_1 \otimes \cdots \otimes m_i \text{ for all } \sigma \in S_i \rangle$$

where S_i is the symmetric group on i elements.

(ii) If M is free and finitely-generated, then $M \simeq R^n$ for some $n \in \mathbb{N}$, and we have

$$\bigwedge^i M \simeq \bigwedge^i R^n \simeq R^{\binom{n}{i}}.$$

Why? We sketch the argument. We can write $R^n = Re_1 \oplus \cdots Re_n$. Then $R^n \otimes R^n \otimes \cdots R^n$ has basis

$$\{e_{j_1} \otimes e_{j_2} \otimes \cdots \otimes e_{j_i} \mid 1 \leq j_1, \ldots, j_i \leq n\}.$$

Note that given a tensor of the form

$$e_{j_1} \otimes e_{j_2} \otimes \cdots \otimes e_{j_i},$$

we can apply the relations defining the wedge product to rearrange it so that the indices are in "alphabetical order" (i.e., $1, 2, 3, \ldots$) and the coefficient in front is the sign of the corresponding permutation. Thus , we can show that $\bigwedge^i R^n$ has a basis

$$\{e_{j_1} \otimes \cdots \otimes e_{j_i} \mid j_1 < j_2 < \cdots < j_i\}.$$

This has size $\binom{n}{i}$.

(iii) We claim that

$$\bigwedge^i (A \oplus B) \simeq \bigoplus_{j=0}^{i} \bigwedge^j (A) \otimes \bigwedge^{i-j} (B).$$

Indeed, $(A \oplus B)^{\otimes i}$ is spanned by $z_1 \otimes \cdots \otimes z_i$ where each z_k is in A or B and, since we work mod L, we can move the z_k's that are in A to the left and the z_k's that are in B to the right.

(iv) If P is projective of rank d, then

$$\bigwedge^s P = 0$$

for all $s > d$. Indeed, locally P is free as an R_m-module where m is the maximal ideal we localize at. Then

$$(\bigwedge^s P)_m \simeq \bigwedge^s P_m \simeq \bigwedge^s R_m^d \simeq R_m^{\binom{d}{s}} = (0).$$

(v) If P is finitely-generated and stably free of rank 1, then there exists n such that $P \oplus R^n \simeq R^{n+1}$. Applying $\bigwedge^{n+1}$ to the right-hand side gives

$$\bigwedge^{n+1} R^{n+1} \simeq R^{\binom{n+1}{n+1}} = R.$$

Applying $\bigwedge^{n+1}$ to the left-hand side gives

$$\bigwedge^{n+1} (P \oplus R^n)$$

$$\simeq \bigoplus_{i=0}^{n+1} \bigwedge^i P \otimes_R \bigwedge^{n+1-i} R^n \text{ (by (iii))}$$

$$
\begin{aligned}
&\simeq \Big(\bigwedge^{0} P \otimes \bigwedge^{n+1} R^n\Big) \oplus \Big(\bigwedge^{1} P \otimes \bigwedge^{n} R^n\Big) \text{ (by (iv))} \\
&\simeq \bigwedge^{1} P \otimes \bigwedge^{n} R^n \\
&\simeq \bigwedge^{1} P \otimes R \\
&\simeq \bigwedge^{1} P \\
&\simeq P.
\end{aligned}
$$

So $R \simeq P$, which completes the proof. $\square$

We now discuss the Govorov–Lazard theorem.

Theorem 24.4. *Let R be a unital commutative ring. Then an R-module M is flat if and only if $M = \varinjlim F_i$ where the F_i's are free modules.*

Proof. The backward direction is quick. We now prove the forward direction (namely that M flat implies $M = \varinjlim F_i$).

We define a category $\mathcal{C}$ whose objects are finitely-generated free modules F equipped with R-module homomorphisms $\psi : F \to M$. The morphisms $(F, \psi) \xrightarrow{f} (G, \phi)$ are given by requiring the following diagram to commute:

$$
\begin{array}{ccc}
F & \xrightarrow{\psi} & M \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \mathrm{id}} \\
G & \xrightarrow{\phi} & M
\end{array}
$$

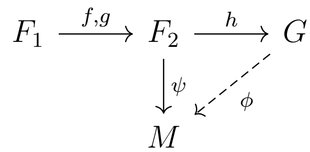
We work up to isomorphism to get a small category.

Given a finite sequence in M, we have a map I that sends that sequence to $\mathcal{C}$. It is defined by $I((m_1, \ldots, m_r)) := R^r$, which is equipped with a map ψ to M given by $e_i \mapsto m_i$. If M is flat, we will show this category is filtered. Then it follows that $\varprojlim I \simeq M$. (This follows from the same argument as in Assignment 2.)

To show it's filtered, we need flatness. Suppose $I((m_1, \ldots, m_r)) = F_1 \xrightarrow{\psi_1} M$ and $I((m'_1, \ldots, m'_s)) = F_2 \xrightarrow{\psi_2} M$. We verify the first part of the definition of being filtered using the following diagram:

$$
\begin{array}{ccccc}
F_1 & & & & F_2 \\
& \searrow & & \swarrow & \\
& & F_1 \oplus F_2 & & \\
& & \downarrow & & \\
& & M & &
\end{array}
\qquad (F_1 \to M,\ F_2 \to M)
$$

Jason writes $I((m_1, \ldots, m_r, m'_1, \ldots, m'_s))$ on the board to accompany this drawing. Then, to check the second part of the definition of being filtered, he draws this diagram:

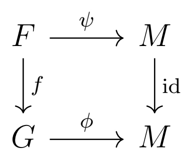


Jason writes "∃" next to F_1 in that diagram. The arrow labelled "f, g" means it's actually a double arrow indicating an equalizer of those two maps. Jason writes, "(G, ϕ) in our cat". We will continue this proof next class.

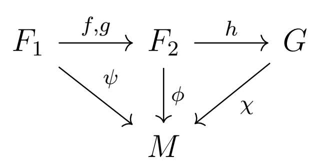
Jason then says he will give us 15 minutes to fill out course evaluations. He writes "`http://evaluate.uwaterloo.ca`" on the board. He says, "I will now leave the room for a while, maybe for the next 15 minutes, maybe for the rest of my life." Then he leaves. I have already filled out the course evaluation. The next lecture is on Thursday, in another room.

25 Dec. 5, 2019

Last time we were looking at the hard part of the Govorov–Lazard theorem. We made a category whose objects are free modules equipped with morphisms to a fixed module M and whose morphisms make the following square commute:



We used tricks to make this into a small category. Now, recall that the Govorov–Lazard theorem says we can realize M as a filtered colimit $\varinjlim F_i = M$ of free modules. The difficult part is showing this category is filtered; the rest follows using the same arguments we used to prove that a module is a filtered colimit of finitely presented modules. To complete the proof, we must show that this category is filtered. Consider the following diagram:



We want to show that if we have a coequalizer (like the triangle with F_1, F_2, M), there exists a finitely-generated free module G and morphisms $\chi : G \to M$, $h : F_2 \to G$ such that the diagram above commutes.

Let $F_1 = Re_1 \oplus \cdots \oplus Re_s$. For this to make sense, we need h to vanish on $f(e_1) - g(e_1), \ldots, f(e_s) - g(e_s)$. We now prove the following result.

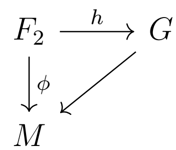
Theorem 25.1. *Let R be a unital commutative ring. Let M be an R-module. The following are equivalent.*

(i) M is flat.

(ii) If M is finitely-generated and free, $\beta : F \to M$ is an R-module homomorphism, and K is a cyclic submodule of $\ker(\beta)$ *(i.e., it has a single generator), then there exists a finitely-generated free module G and a homomorphism $\gamma : G \to M$ such that* $\ker(\gamma) \supseteq K$*.*

(iii) is the same as (ii) but with K finitely-generated instead of cyclic.

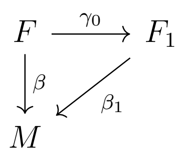
Notice that we can make the map h vanish on $f(e_i) - g(e_i)$ for $i \in \{1, \ldots, s\}$ by setting $K := \langle f(e_1) - g(e_1), \ldots, f(e_s) - g(e_s)\rangle \subseteq \ker(\phi)$ and applying the theorem to the diagram



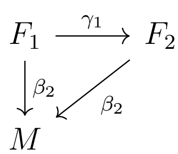
to get $\ker(h) \supseteq K$. So this theorem will follow once we prove the most recent theorem, which we do now.

Proof. (iii) $\implies$ (ii): Immediate.

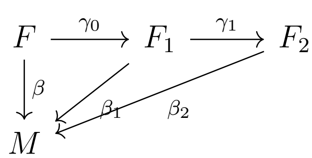
(ii) $\implies$ (iii): Suppose (ii) is true and we have an R-module homomorphism $\beta : F \to M$. Suppose $K = \langle c_1, \ldots, c_r\rangle$ with $c_i \in \ker(\beta)$ for each i. By (ii), there exists a finitely-generated free module F_1 and maps $\gamma : F \to F_1$, $\beta_1 : F_1 \to M$ such that the following diagram commutes:



Then $c_1 \in \ker(\gamma_0)$ and $\langle c_1\rangle$ is cyclic. Now we look at this commutative diagram:



Applying (ii) again, there exists a finitely-generated free module F_2 and maps $\gamma_1 : F_1 \to F_2$, $\beta_2 : F_2 \to M$ with $\gamma_1 \circ \gamma_0(c_2) = 0$ and $\gamma_0(c_2) \in \ker(\beta_1)$. Concatenating these diagrams, we get the following commutative diagram:



Then $\gamma_1 \circ \gamma_0$ vanishes on c_1 and c_2, so we have proved the results for modules with two generators. We can keep going in this way, repeatedly applying (ii), to complete the proof.

It remains to show that (i) holds if and only if (ii) and (iii) hold. For this we need the *equational criterion for flatness.* This states that if R is a unital commutative ring and M is an R-module, then M is flat if and only if for every relation

$$\sum_{i=1}^{s} n_i m_i = 0$$

with $m_i, n_i \in R$ for all i, there exists $t \geq 1$ and a matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1s} \\ \dots & \dots & \ddots & \dots \\ a_{t1} & a_{t2} & \cdots & a_{ts} \end{pmatrix} \in M_{t\times s}(R)$$

and $m'_1, m'_2, \dots, m'_t \in M$ such that

$$[m'_1, m'_2, \dots, m'_t]A = [m_1, m_2, \dots, m_s]$$

and

$$A \begin{bmatrix} n_1 \\ \vdots \\ n_s \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix},$$

i.e.,

$$\sum_j a_{ij} n_j = 0$$

for all i and

$$m_j = \sum_i a_{ij} m'_i$$

for all j.

Note that if we have this, then

$$\sum_j n_j m_j$$

$$= \sum_j n_j \left(\sum_i a_{ij} m'_i \right)$$

$$= \sum_j \sum_i a_{ij} n_j m'_i$$

$$= \sum_i \left(\sum_j a_{ij} n_j \right) m'_i$$

$$= 0.$$

One can show that (i) $\implies$ (ii) and (iii) $\implies$ (i) with this criterion; we leave this as an exercise. It remains to prove this criterion.

Proof. Suppose M is not flat. Then there exists a finitely-generated ideal I of R such that $I \otimes M \to M$ given by $i \otimes m \mapsto m$ is not injective. So there exist $n_1, \ldots, n_s \in I$ such that

$$n_1 \otimes m_1 + \cdots n_s \otimes m_s \neq 0$$

but

$$n_1 m_1 + \cdots + n_s m_s = 0.$$

But from the criterion, it follows that there exists $A = (a_{ij})$ and $[m'_1, \ldots, m'_t]$ such that

$$[m'_1, \ldots, m'_t]A = [m_1, \ldots, m_s]$$

and

$$A \begin{bmatrix} n_1 \\ \vdots \\ n_s \end{bmatrix} = 0.$$

That means that

$$\begin{aligned} 0 &\neq \sum_{j=1}^{s} n_j \otimes m_j \\ &= \sum_{j=1}^{s} \left(n_j \otimes_R \left(\sum_{i=1}^{t} a_{ij} m'_i \right) \right) \\ &= \sum_{j=1}^{s} \sum_{i=1}^{t} a_{ij} n_j \otimes m'_i \\ &= \sum_{i=1}^{t} \left(\sum_{j=1}^{s} a_{ij} h_j \right) \otimes m'_i \\ &= \sum_{i=1}^{t} 0 \otimes m'_i \\ &= 0. \end{aligned}$$

To prove the converse, suppose M is flat. Then for all finitely-generated ideals I of R, we have

$$\mathrm{Tor}_1^R(R/I, M) = (0),$$

and $I \otimes M \to M$ given by $i \otimes m \mapsto m$ is one-to-one for all finitely-generated ideals I. In particular,

$$\sum_{\ell=1}^{s} i_\ell \otimes m_\ell = 0 \iff \sum_{\ell=1}^{s} i_\ell m_\ell = 0.$$

Now suppose $\sum_{i=1}^{s} n_i m_i = 0$, where $n_i, m_i \in R$ for all i. Let $I := \langle n_1, \dots, n_s \rangle \subseteq R$. Then we have an exact sequence

$$F \xrightarrow{f} R^s \xrightarrow{g} I \to 0$$

for some free module with generating set $\{e_1, \dots, e_s\}$, where the map $F \to R^s$ is given by $e_i \mapsto n_i$. Applying $- \otimes_R M$ gives us the exact sequence

$$F \otimes_R M \xrightarrow{f \otimes \mathrm{id}_M} R^2 \otimes_R M \xrightarrow{g \otimes \mathrm{id}_M} I \otimes_R M \to 0.$$

Notice that under the map $g \otimes \mathrm{id}_M$, the element $e_1 \otimes m_1 + \cdots + e_s \otimes m_s$ maps to $n_1 \otimes m_1 + \cdots + n_s \otimes m_s = 0$. Thus,

$$e_1 \otimes m_1 + \cdots + e_s \otimes m_s \in \ker(g \otimes \mathrm{id}_M) = \mathrm{im}(f \otimes \mathrm{id}_M).$$

It follows that there exists $u := y_1 \otimes m_1' + \cdots + y_t \otimes m_t' \in F \otimes M$ such that

$$(f \otimes \mathrm{id}_M)(u) = e_1 \otimes m_1 + \cdots + e_s \otimes m_s.$$

Then

$$\begin{gathered} f(y_1) \otimes m_1' + \cdots + f(y_t) \otimes m_t' \\ = e_1 \otimes m_1 + \cdots + e_s \otimes m_s. \end{gathered}$$

Note that $f(y_i) \in R^s$, so we can write it as

$$f(y_i) = \sum_{j=1}^{s} a_{ij} e_j$$

for some $a_{ij} \in R$. But we also have $g(f(y_i)) = 0$ since $g \circ f = 0$ by exactness. Thus,

$$0 = g\left(\sum_{j=1}^{s} a_{ij}\right)$$

$$= \sum_{j=1}^{s} a_{ij} n_j.$$

We can therefore write

$$(a_{ij}) \begin{bmatrix} n_1 & \vdots & n_s \end{bmatrix} = 0.$$

On the other hand,

$$\begin{gathered} e_1 \otimes m_1 + \cdots + e_s \otimes m_s \\ = \sum_{i=1}^{t} f(y_i) \otimes m_i' \\ = \sum_{i=1}^{t} \left(\sum_{j=1}^{s} a_{ij} e_j\right) \otimes m_i'. \end{gathered}$$

Comparing coefficients, we get

$$m_i = \sum_{i=1}^{t} a_{ij} m'_i.$$

This completes the proof. □

Now that we have completes one nested proof, we have completed the second. □

Now that we have completed two nested proofs, we have completed the third and last. □

We now discuss some open problems. Suppose $\sigma : \mathbb{C}^3 \to \mathbb{C}^3$ is a polynomial automorphism given by

$$(x, y, z) \mapsto (p(x, y, z), q(x, y, z), r(x, y, z)).$$

A necessary condition for σ to be a polynomial automorphism is that its Jacobian is nonzero. It is not known whether this is sufficient, although Jason thinks it probably is; this is the *Jacobian conjecture*. Now, suppose L_1 and L_2 are two lines in $\mathbb{C}^3$ with corresponding ideals $I_1, I_2 \subseteq \mathbb{C}[x, y, z]$. Let's look at

$$\{n \mid \mathrm{Tor}_1(\mathbb{C}[x, y, z]/(\sigma^*)^n(I_1), \mathbb{C}[x, y, z]/I_2) \neq 0\}.$$

This is morally the same as looking at

$$S := \{n \mid \sigma^n(C_1) \cap C_2 \neq 0\}.$$

Question. Is it true that either S has zero density, i.e., that

$$\lim_{n\to\infty} \frac{|S \cap [-n, n]|}{2n + 1} \to 0$$

or that there exist $a > 0$ and b such that $an + b \in S$ for all but finitely many n?

More general question. Let X be an irreducible noetherian topological space. Let $\sigma : X \to X$ be a homeomorphism, and let Y, Z be closed subsets of X. We have a sheaf $\mathcal{O}_X$ of noetherian rings. For $i \geq 1$, do we have that

$$\{n \mid \mathrm{Tor}_i^{\mathcal{O}_X}(\mathcal{O}_{\sigma^n(Y)}, \mathcal{O}_Z) \neq 0\}$$

is a finite union of infinite arithmetic progressions and a set of zero density?

Jason believes this more general question is easy to answer once you've answered the simpler one. He does not know how to answer either.

Now for a different problem. We say that rings R and S are *Morita equivalent* if R-Mod $\simeq$ S-Mod. We write $R \simeq_M S$ if this is the case. For example, we showed that R and $M_n(R)$ are Morita equivalent for any $n \geq 1$ even though $R \not\simeq M_n(R)$.

Given a unital commutative ring R, we construct its *derived category* $D(R)$ as follows. First, start with the usual category of chain complexes of R-modules with the usual morphisms. Second, pass to the homotopy category of chain complexes by identifying morphisms that are chain homotopic. Third, pass to the derived category $D(R)$ by localizing at the set of quasi-isomorphisms. What is a quasi-isomorphism, and what is localization? A *quasi-isomorphism* is a morphism of chain complexes $A. \to B.$ such that the induced morphisms $H^n(A.) \to H^n(B.)$ of cohomology groups (or equivalently, the induced morphisms $H_n(A.) \to H_n(B.)$ of homology groups) are isomorphisms for all n. Localization of categories is a generalization of localization of rings, and you can look it up on Wikipedia (`https://en.wikipedia.org/wiki/Localization_of_a_category`).

We say R and S are *derived equivalent* if $D(R) \simeq D(S)$. We write $R \simeq_D S$ if this is the case. Note that isomorphism implies Morita equivalence implies derived equivalence.

The following conjecture of Zariski is still open. Let X be an affine variety such that $X \times \mathbb{C} \simeq \mathbb{C}^{n+1}$. Does it follow that $X \simeq \mathbb{C}^n$? This is known to be true for $n = 1$ and $n = 2$, but it is open even for $n = 3$.

Passing to coordinate rings, we can write $R[t] \simeq \mathbb{C}[t_1, \ldots, t_{n+1}]$ implies $R \simeq \mathbb{C}[t_1, \ldots, t_n]$ in the situation described in Zariski's conjecture. We say R is *cancellative* (resp. *Morita cancellative*, resp. *derived cancellative*) if $R[t] \simeq S[t \implies R \simeq S$ (resp. $R[t] \simeq_M S[t] \implies R \simeq_M S$, resp. $R[t] \simeq_D S[t] \implies R \simeq_D S$).

We have lots of questions about these properties. For example, let X be a complex smooth curve, and let D be a divisor on X. (We do not care about whether we use Weil or Cartier divisors because it's smooth.) If $0 \neq f \in \mathbb{C}(X)$, then

$$\operatorname{div}(f) = \sum_{P \text{ a zero of } f} n_P[P] - \sum_{P \text{ a pole of } f} n_P[P]$$

where n_P denotes the order of the zero or pole P. For example, if $X = \mathbb{P}^1$, then $C(\mathbb{P}^1) = \mathbb{C}(t)$, and if $f(t) := t^2/(t-1)$, then $\operatorname{div}(f) = 2[0] - [1] - [\infty]$.

Now, given two divisors, we write

$$\sum a_P[P] \geq \sum b_P[P]$$

if and only if $a_P \geq b_P$ for all P. If D is a divisor, we write

$$L(D) := \{0\} \cup \{f \in \mathbb{C}(X)^* \mid \operatorname{div}(f) \geq -D\}.$$

For example, if $X = \mathbb{P}^1$ and $D = 2[\infty]$, then $\mathbb{C}(\mathbb{P}^1) = \mathbb{C}(t)$, and

$$\begin{aligned} L(D) &= \{0\} \cup \{f \mid \operatorname{div}(f) \geq -2[\infty]\} \\ &= \operatorname{span}\{1, t, t^2\}. \end{aligned}$$

(In fact, $L(D)$ is always finite-dimensional, and you can compute it using the Riemann–Roch theorem.) Given a divisor $D := \sum a_P[P]$ and an automorphism $\sigma : X \to X$, we can define the *pullback of D along σ* by

$$\sigma^*(D) = \sum a_P[\sigma^{-1}(P)].$$

Given the data of X, σ, and D as above, we can define a ring

$$B(X, \sigma, D) := \bigoplus_{i=0}^{\infty} L(D + \sigma^*(D) + (\sigma^2)^*(D) + \cdots + (\sigma^{i-1})^*(D)).$$

Call the i^{th} term in this direct sum L_i. Given $f \in L_i$ and $g \in L_j$, define $f * g := f \cdot g(\sigma^i) \in L_{i+j}$. With this operation, we call this ring the *twisted homogeneous coordinate ring* of X.

There's a theorem that

$$\bigoplus_{i=0}^{\infty} \operatorname{Ext}^i_R(\mathbb{C}, \mathbb{C})$$

is finite-dimensional. We can then ask what its dimension is, in terms of X, σ, D. This is sort of open-ended, and not much is known about it.