DELIGNE–LUSZTIG THEORY

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ABSTRACT. We show how Deligne–Lusztig theory is used to produce irreducible representations of $\mathrm{SL}_2(\mathbb{F}_q)$. We begin by discussing sheaves and schemes, eventually moving on to the theory of group schemes and reductive linear algebraic groups. We look at the representation theory of $\mathrm{SL}_2(\mathbb{F}_q)$ and show how it runs into difficulties that are resolved by a clever technique of Drinfeld. We then explain how Deligne and Lusztig generalized this technique to other finite groups of Lie type and show that their method really does coincide with Drinfeld's in the particular case of $\mathrm{SL}_2(\mathbb{F}_q)$.

1. Sheaves and Schemes

Deligne–Lusztig theory, first developed in [3], is a method for constructing representations $\rho : G \to \mathrm{GL}(V)$ where V is a vector space and G is a finite Lie group. These ideas, which the namesake authors have written about from 1976 onwards, heavily relies on results about *ℓ-adic cohomology*, a tool developed by Alexander Grothendieck and many others throughout the 1960s. To discuss Deligne–Lusztig theory therefore requires an introduction to algebraic geometry. We will have to assume knowledge of category theory (e.g., functors, pullbacks, equalizers), commutative algebra (e.g., radical and homogeneous ideals), and representation theory (e.g., induced representations), and Lie theory (the content of this course) but will define the new terminology from algebraic geometry as it comes up. Because of the amount of algebraic geometry required to begin defining the basic objects of Deligne–Lusztig theory, we will omit proofs with impunity.

The material of the current section can be found in any textbook on modern algebraic geometry. The canonical reference is [4], and a good modern presentation can be found in [11].

The notion of *sheaf* is central to modern algebraic geometry. Sheaves were first invented by Jean Leray while he was in a prisoner of war camp during World War Two. (Amazingly, Leray invented the concepts of *spectral sequence* and *sheaf cohomology* at the same time! We will not discuss these, however.) Although Leray was concerned with what seemed like a rather specific problem in algebraic topology, Henri Cartan quickly saw that his work could be applied to algebraic geometry, and Cartan's doctoral student, Jean-Pierre Serre, wrote the foundational text [5] on the matter.

Very often in topology, one tries to study a topological space by looking at functions of a given type on open sets of that space. One might look at all (real- or complex-valued) functions on an open set, or one might restrict to continuous or smooth functions on the open set. We will see that these are examples of a *presheaf*.

Definition 1.1. Let X be a topological space, and let $\mathbf{C}$ be the category of sets, the category of commutative rings, or the category of abelian groups. (We consider only these categories for convenience in defining sheaves later on, although their definition can be given in greater generality than ours.) Let Open(X) be the category whose objects are open subsets of X and such that there is a morphism $U \to V$ if and only if $U \subseteq V$. A *presheaf* F on X is a contravariant functor from Open(X) to $\mathbf{C}$. If $U \subseteq X$ is open, elements of $F(U)$ are known as *sections* of F, and elements of $F(X)$ are known as *global sections*. If $V \subseteq U$ and $s \in F(U)$ is a section, we often use the notation $s|_V$ for $\text{res}_{V,U}(s)$.

We expand on this definition. It means that for every open subset U of X, there is an object $F(U)$ in $\mathbf{C}$, and for every inclusion of open subsets $V \subseteq U$, there is a *restriction morphism* $\text{res}_{V,U} : F(U) \to F(V)$ in $\mathbf{C}$. Moreover, the morphism $\text{res}_{U,U}$ coincides with the identity map for every U, and given open subsets $W \subseteq V \subseteq U$, we have $\text{res}_{W,V} \circ \text{res}_{V,U} = \text{res}_{W,U}$.

Example 1.2. (i) Suppose $U \subseteq \mathbb{R}^n$ is open. Let

$$F(U) := \{\text{functions } U \to \mathbb{R}\}.$$

Given $V \subseteq U$, let the restriction morphism $\text{res}_{U,V}$ be given by restricting a function on U to a function on V. Then F is a presheaf.

(ii) If we replace "functions" by "continuous functions" or "smooth functions" in (i), F is still a presheaf.

(iii) If instead $U \subseteq \mathbb{C}^n$ is open, and we define

$$G(U) := \{\text{holomorphic functions } U \to \mathbb{C}\}$$

and define the restriction morphisms as restriction of functions again, then G is a presheaf.

We are now ready to define a sheaf. The definition may look intimidating at first, but we will soon motivate it.

Definition 1.3. Let X be a topological space, let I be an indexing set, and suppose $\{U_i\}_{i \in I}$ is an open cover of X. Let F be a presheaf from X to some category $\mathbf{C}$ that is the category of sets, of commutative rings, or of abelian groups. We say F is a *sheaf* if it satisfies two additional conditions.

(i) If $s, t \in F(U)$ are sections and $s|_{U_i} = t|_{U_i}$ for every $i \in I$, then $s = t$.

(ii) Let $s_i \in F(U_i)$ for every $i \in I$. Suppose

$$s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$$

for all $i, j \in I$. Then there exists $s \in F(U)$ such that $s|_{U_i} = s_i$ for every $i \in I$.

Given two sheaves F, G on X with values in $\mathbf{C}$, a *morphism* $\phi : F \to G$ consists of a morphism $\phi_U : F(U) \to G(U)$ for every open subset $U \subseteq X$ such that each morphism ϕ_U commutes with the restriction maps.

The first of these conditions says that if two sections are equal on every element of an open cover, then they are equal. The second says that a family of sections equal on intersections of an open cover can be "glued" to produce a single section.

Example 1.4. (i) Presheaves of arbitrary, continuous, smooth, or holomorphic functions are still sheaves. For example, given an open cover $\{U_i\}$ of an open set $X \subseteq \mathbb{R}^n$, if $f|_{U_i} = g|_{U_i}$, where $f, g : X \to \mathbb{R}$ are continuous functions, then $f = g$. If $f_i \in U_i$ is a family of continuous functions and $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ for all i and j, then there is a continuous $f : X \to \mathbb{R}$ such that $f|_{U_i} = f_i$ for every i. These are basic results in real analysis, and similar results hold for smooth and holomorphic functions.

(ii) If $U \subseteq \mathbb{R}^n$ is open, let $F(U)$ consist of constant functions $U \to \mathbb{R}$ with the usual restriction maps. Then $F(U)$ is a presheaf but *not necessarily* a sheaf. Indeed, if U_1, U_2 are disjoint open subsets of $\mathbb{R}^n$, then if we let $s_1 \equiv 0$ on U_1 and $s_2 \equiv 1$ on U_2, both s_1 and s_2 are constant, but they cannot be glued to a constant function on $U_1 \cup U_2$.

(iii) If $U \subseteq \mathbb{R}^n$ is open and we let $F(U)$ consist of *locally* constant functions $U \to \mathbb{R}$, then F is a sheaf.

Part (iii) of the last example illustrates the intuition that a family of functions forms a sheaf if it is somehow determined by local data. This is the motivating idea of sheaf theory.

The following definition will come in handy later.

Definition 1.5. Suppose $f : X \to Y$ is a continuous map of topological spaces. Let $\mathrm{Sh}(X)$ denote the category of sheaves on X. The *direct image functor* $f_* : \mathrm{Sh}(X) \to \mathrm{Sh}(Y)$ is defined by

$$f_*F(U) = F(f^{-1}(U)).$$

This turns out to be a sheaf on Y.

Sheaf theory provides a powerful language for discussing algebraic geometry because the data of a sheaf gives us a family of sections for every open set of a particular space at once. Historically, *schemes* were defined using sheaves. To explain what a scheme is and why they are important, we first discuss classical algebraic geometry.

Classical algebraic geometers, who, roughly speaking, worked during the first half of the 20^{th} century, were concerned, as all algebraic geometers are, with solutions to families of equations of the form $f_j(x_1, ..., x_n) = 0$ where the f_j's are polynomials. Generally speaking, they studied solution sets of such equations over a field. The following example shows how a new point of view came about among them.

Example 1.6. Suppose we are interested in the line $x = 0$ in the *affine plane*, which for our purposes is $\mathbb{R}^2$. Notice that the locus of points cut out by this equation is the same as the locus cut out by $x^2 = 0$.

Because of examples like this one, algebraic geometers began to study particular ideals in polynomial rings, rather than simply studying families of polynomial equations. Rather than studying the line $x = 0$ in $\mathbb{R}^2$, it makes sense to study the ideal $(x) \subset \mathbb{R}[x, y]$. The fact that $x = 0$ corresponds to the same line as $x^2 = 0$ corresponds to the fact that the ideal (x) is the *radical* of the ideal (x^2).

Given an ideal I generated by n-variable polynomials, we let $V(I)$ denote the set of points P in k^n, for some field k, for which $f(P) = 0$ for every $f \in I$. Conversely, given a set of points V in k^n, we let $I(V)$ denote the ideal of n-variable polynomials vanishing at those points.

Definition 1.7. A subset of k^n that is of the form $V(I)$ for some ideal I in $k[x_1, ..., x_n]$ is called an *affine algebraic set*. A non-empty affine algebraic set is called *irreducible* if it cannot be written as the union of two proper algebraic subsets. An irreducible affine algebraic set is called an *affine variety*. (Note that if V is an affine variety, the ideal $I(V)$ is prime, as we will discuss shortly.) Given an affine variety V in n-dimensional affine space k^n,

$$k[V] := k[x_1, ..., x_n]/I(V)$$

is called the *affine coordinate ring* of V. A *projective variety* is a subset of some projective n-space $\mathbb{P}^n$ over a field k that is the vanishing locus of a finite family of homogeneous polynomials in $n+1$ variables $\{x_0, ..., x_n\}$ with coefficients in k that generate a prime ideal I. The quotient ring $k[x_0, ..., x_n]/I$ is then called the *homogeneous coordinate ring* of the projective variety.

The following is one of the central results of classical algebraic geometry.

Theorem 1.8. *Suppose $\{f_j\}$ is a family of polynomials on k^n that generate an ideal J in $k[x_1, ..., x_n]$. The functions V and I just described induce mutually inverse, inclusion-reversing bijections between maximal ideals in $k[x_1, ..., x_n]/J$ and points of $V(J)$, between prime ideals of $k[x_1, ..., x_n]/J$ and irreducible algebraic subsets of $V(J)$, and between radical ideals in $k[x_1, ..., x_n]$ and algebraic subsets of $k[x_1, ..., x_n]/J$.*

The notion of a scheme was created by Cartier, Chevalley, Grothendieck, and others because of their dissatisfaction with this correspondence. They noticed that in many situations, it was useful to use the full correspondence between prime ideals and irreducible algebraic subsets, rather than the correspondence between maximal ideals and points. Therefore, they began to regard prime ideals as "generic points" of a variety, which are a more fundamental object than the usual points corresponding to maximal ideals of the affine coordinate ring. The idea is that we can identify affine varieties with their affine coordinate rings, and then we can regard prime ideals as "generic points" of a ring! Thinking along these lines, they made three key generalizations.

(i) They allowed for the "affine coordinate ring" of a scheme to contain nilpotent elements, something that could never happen for affine varieties. Thus, in scheme theory, an ideal and its radical carry different geometric information, something that does not happen in the classical theory where they correspond to the same variety.

(ii) Rather than working over a base field, they worked over an arbitrary base ring. This approach is very fruitful because many situations in number theory require us to study objects like $\mathbb{Z}$, which do not form a field. In this way, scheme theory unifies classical algebraic geometry and number theory.

(iii) They observed that projective varieties can be viewed as multiple affine varieties "glued together" along certain maps. Therefore, taking their cue from the definition of a manifold, they allowed arbitrary schemes to be made up of simpler "affine schemes" that are identified along certain maps.

Because we are generalizing algebraic varieties, we cannot use them in the definition of a scheme. Instead, a scheme is defined as a topological space with some extra structure. Motivated by our discussion of sheaves, we wish to associate a commutative ring to each open subset of the topological space, which in many cases can be regarded as the ring of all/continuous/smooth/holomorphic functions on U.

Definition 1.9. A *ringed space* is a pair (X, O_X) where X is a topological space and O_X is a sheaf of commutative rings on X. The sheaf O_X is called a *structure sheaf.* A *morphism* from a ringed space (X, O_X) to a ringed space (Y, O_Y) is a pair (f, ϕ) where $f : X \to Y$ is a continuous map and $\phi : O_Y \to f_* O_X$ is a morphism of sheaves.

Because of the way in which sheaves depend on local data, it is often useful to consider the following construction.

Definition 1.10. Given a topological space X and a sheaf F on the open subsets of X, the stalk of F at $x \in X$, denoted F_x, is defined by

$$F_x = \varinjlim_{U \ni x} F(U).$$

The direct limit is indexed over all open sets $U \subseteq X$ containing x where $U < V$ in the ordering if $U \supseteq V$. By definition of the direct limit, an element of the stalk is an equivalence class of families of elements $s_U \in F(U)$, U an open subset of X, where two elements s_U and s_V are equivalent if their restrictions are equal on some neighbourhood of X contained in U and V.

Definition 1.11. A *locally ringed space* is a ringed space (X, O_X) such that every stalk of O_X is a local ring (i.e., has a unique maximal ideal). (Note that we do not require $O_X(U)$ to be a local ring for any open set U, and in fact this usually is not the case.) A *morphism* of locally ringed space is a morphism of ringed spaces (f, ϕ) such that the homomorphisms induced by ϕ map the maximal ideal of the local ring of $f(x) \in Y$ to the maximal ideal of the local ring of $x \in X$; ϕ is said to be a *local homomorphism.*

The following definitions help set up the next example.

Definition 1.12. If X and Y are affine varieties in k^n and k^m respectively, where k is the base field, then a *regular map* $f : X \to Y$ is the restriction of a polynomial map $k^n \to k^m$. If X and Y are arbitrary varities, a map $f : X \to Y$ is *regular at a point* x if there is an open neighbourhood U of x and V of $f(x)$ such that $f(U) \subseteq V$ and the restricted function $f : U \to V$ is regular as map on some affine charts of U and V. We then say that f is a *regular map* if it is regular at all points of X.

Definition 1.13. Suppose X and Y are varieties. A *rational map* $f : X \to Y$ is an equivalence class of pairs (f_U, U) where U is an open subset of X and $f_U : U \to Y$ is a regular map. Two pairs (f_U, U) and (f_V, V) are equivalent if they are equal on the intersection $U \cap V$.

Example 1.14. (i) Ringed spaces were already in use before the introduction of schemes. Algebraic varieties are usually equipped with the *Zariski topology*, under which the algebraic subsets of a variety are defined to be its closed sets. Suppose X is a variety with the Zariski topology, and for an open subset $U \subseteq X$, let $O_X(U)$ be the ring of rational maps defined on U that do not become infinite at any point of U. Then (X, O_X) is a locally ringed space.

(ii) Let R be a commutative ring. Recall that its *(prime) spectrum* $\mathrm{Spec}(R)$ consists, as a set, of all prime ideals of R. We equip $\mathrm{Spec}(R)$ with a topology, known as the *Zariski topology (for spectra)*, as follows. For $f \in R$, let D_f be the set of prime ideals not containing f, and let $\{D_f : f \in R\}$ be a basis of open sets for the topology. For $X := \mathrm{Spec}(R)$, a structure sheaf O_X is defined on this basis by $O_X(D_f) = R_f$, the localization of R by powers of f. One can prove that O_X is indeed a sheaf, that the localization of O_X at a prime ideal P is equal to the localization of R at P, and that (X, O_X) is then a locally ringed space.

We are finally ready to define schemes.

Definition 1.15. An *affine scheme* is a locally ringed space isomorphic to the spectrum $\mathrm{Spec}(R)$ of a commutative ring R. A *scheme* is a locally ringed space admitting a covering by open sets $\{U_i\}$ such that each U_i, as a locally ringed space, is an affine scheme. (Note that we think of Spec as a contravariant functor from rings to affine schemes. In fact, it turns out to define an equivalence of categories between the two, which shows that affine schemes really are the correct generalization of varieties.)

Example 1.16. For any commutative ring R, $\mathrm{Spec}(R)$, viewed as a locally ringed space, is a scheme.

The following notion will be useful later.

Definition 1.17. Given a scheme (X, O_X), an *open subscheme* if a scheme isomorphic to $(U, O_X|_U)$ for an open subset $U \subseteq X$. A morphism $f : X \to Y$ of schemes is a *closed immersion* if it is a homeomorphism onto a closed subset of Y and the induced map $O_Y \to f_*O_X$ is surjective. A *closed subscheme* of X is an equivalence class of closed immersions with codomain X where $f : Z \to X$ and $f' : Z' \to X$ are equivalent if there exists an isomorphism $g : Z \to Z'$ such that $f = f' \circ g$.

2. Group Schemes

Very often in geometry we encounter mathematical objects that simultaneously have an algebraic and a geometric structure. The theory of Lie groups provides an example, as do *abelian varieties*, whose simplest example are elliptic curves equipped with their group structure. It is not immediately clear how to make sense of such constructions in scheme theory. However, there is such a notion, that of a *group scheme*, which we will soon define.

The development of the theory of group schemes occurred in [4] as well as the seminar notes [7]. A good modern reference for this section is [2].

The category of schemes admits pullbacks, which are often called *fibered products* in this context. For our purposes, their most important property is that if $X \times_S Y$ is a fibered product of affine schemes, then

$$\mathrm{Spec}(X \times_S Y) = \mathrm{Spec}(X) \otimes_{\mathrm{Spec}(S)} \mathrm{Spec}(Y).$$

It is generally the case that a problem about schemes can be quickly reduced to a problem about affine schemes, so it is not very restrictive to consider only this case when discussing any categorical construction in scheme theory. It is often fruitful to consider schemes over an arbitrary base scheme, usually denoted S. Categorically, this means passing to the *comma category* of the category of schemes, whose elements are morphisms $X \to S$, where X is a scheme and S is our base scheme, and whose morphisms are morphisms $X \to Y$ of schemes that commute with the natural morphisms $X \to S$ and $Y \to S$. When viewing a scheme X as an element of this category, we refer to it as an *S-scheme*. This perspective is known as

Grothendieck's relative point of view.

We also remark that given two morphisms $f, g : X \to Y$ of schemes, the equalizer of f and g forms a closed subscheme of X. This fact will be useful later.

Definition 2.1. Let S be a scheme. Let X and S' be S-schemes. By definition of a pullback, the fibered product $X \times_S S'$ is equipped with a map to S', which means it can be viewed as an S'-scheme. We call it the *base change* of X to S'.

Example 2.2. Suppose k is a field and $\bar{k}$ is its algebraic closure. We have an inclusion map $k \to \overline{k}$, and since Spec is a contravariant functor that gives an equivalence of categories between rings and affine schemes, we have a corresponding map $\mathrm{Spec}(\bar{k}) \to \mathrm{Spec}(k)$ realizing $\mathrm{Spec}(\bar{k})$ as a $\mathrm{Spec}(k)$-scheme. Suppose X is a k-scheme. Then the base change $X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(\overline{k})$ is a $\bar{k}$-scheme. We denote this scheme by $X_{\bar{k}}$.

Definition 2.3. Suppose R is a commutative ring and X is an R-scheme (i.e., a scheme over the base scheme $\mathrm{Spec}(R)$). Suppose S is a commutative R-algebra. Let $X(S)$ denote the set of R-morphisms $\mathrm{Spec}(S) \to X$. The set $X(S)$ is called the set of S-points of X. If k is a field, $\bar{k}$ is its algebraic closure, and X is a scheme over k, then a morphism $\mathrm{Spec}(\bar{k}) \to X$ is called a *geometric point.*

Example 2.4. Suppose X is an affine variety over a field k defined by a system of polynomial equations $f_j(x_1, ..., x_n) = 0$. if R_X denotes its affine coordinate ring, then X can be identified with the affine scheme $\mathrm{Spec}(R_X)$. If $\bar{k}$ is a field extension of S, $X(\bar{k})$ can be identified with the set of solutions to the system $f_j(x_1, ..., x_n)$ for $(x_1, ..., x_n) \in \bar{k}^n$.

Definition 2.5. Let S be a scheme. A *group scheme* is an S-scheme G equipped with three morphisms $\mu : G \times_S G \to G$, $e : S \to G$, and $\iota : G \to G$ corresponding to the multiplication, identity, and inverse maps of a group, respectively. These maps satisfy the following properties.

(i) Associativity of multiplication is given by commutativity of the following diagram, where 1_G indicates the identity map:

$$\begin{array}{ccc} G \times_S G \times_S G & \xrightarrow{1_G \times \mu} & G \times_S G \\ \downarrow{\scriptstyle \mu \times 1_G} & & \downarrow{\scriptstyle \mu} \\ G \times_S G & \xrightarrow{\quad\mu\quad} & G \end{array}$$

(ii) Let $\Delta : G \to G \times_S G$ denote the diagonal map. Let $\pi : G \to S$ be the canonical morphism realizing G as an S-scheme. Then the defining property of the inverse map ι amounts to saying that the composition defined by the following two diagrams is equal to $e \circ \pi$:

$$G \xrightarrow{\Delta} G \times_S G \xrightarrow{1_G \times \iota} G \times_S G \xrightarrow{\mu} G$$

$$G \xrightarrow{\Delta} G \times_S G \xrightarrow{\iota \times 1_G} G \times_S G \xrightarrow{\mu} G$$

(iii) The defining property of the identity map $e : S \to G$ amounts to commutativity of the following diagram:

$$\begin{array}{ccccc} & & S \times_S G & & \\ & \nearrow & & \searrow^{e \times 1_G} & \\ G & & & & G \times_S G \xrightarrow{\mu} G \\ & \searrow & & \nearrow_{1_G \times e} & \\ & & G \times_S S & & \end{array}$$

Here the maps $G \to G \times_S S$ and $G \to S \times_S G$ are actually isomorphisms.

Example 2.6. We can view the *general linear group* as a group scheme in the following way. First, let

$$\mathrm{GL}_n := \mathrm{Spec}(\mathbb{Z}[\{x_{ij}\}_{1 \le i,j \le n}][1/d])$$

where $d := \det((x_{ij}))$. That is, GL_n is an affine scheme whose coordinate ring is the ring of polynomials in $n^2 + 1$ variables with coefficients lying in $\mathbb{Z}$, where the last variable is the inverse of the determinant of the matrix (x_{ij}) containing the first n^2 variables. (We include the inverse determinant because it appears in the formula for the inverse of a matrix.) The multiplication morphism giving the group structure is defined as follows. Motivated by the formula for an entry of the product of two matrices, we define a map

$$x_{ij} \mapsto \sum_k x_{ik} \otimes x_{kj}.$$

This map goes from $\mathbb{Z}[x_{ij}, 1/d] \to \mathbb{Z}[x_{ij}, 1/d] \otimes_{\mathbb{Z}} \mathbb{Z}[x_{ij}, 1/d]$. This induces a map $\mathrm{GL}_n \times \mathrm{GL}_n \to \mathrm{GL}_n$, as desired. In a similar way, all of the standard matrix groups may be regarded as group schemes.

We remark that it is possible to define a quotient of group schemes in several different ways. There are notions of *geometric quotient*, *categorical quotient*, and *geometric invariant theory (GIT) quotient*. The most general notion is that of a *quotient stack*. Because this area is quite technical, we will not go into the details here. We simply ask that the reader take it for granted that a quotient of group schemes can be regarded as a group scheme in the situations that arise in this article.

We can now define reductive group schemes, which are the central objects of study in Deligne–Lusztig theory.

Definition 2.7. Let k be a field with algebraic closure $\bar{k}$. If G is a smooth affine group scheme over $\bar{k}$, we say it is a *linear algebraic group* over k. One can prove that every linear algebraic group, viewed as a group, has a faithful representation $j : G \to \mathrm{GL}(V)$ for some vector space V. We say $g \in G(\bar{k})$ is *semisimple* (resp. *unipotent*) if $j(g)$, viewed as a linear endomorphism on V, is diagonalizable (resp. $j(g) - I_V$ is nilpotent). One can prove that for any $g \in G(\bar{k})$, there exist unique commuting elements $g_u, g_{ss} \in G(\bar{k})$ such that $g = g_{ss} g_u = g_u g_{ss}$, g_{ss} is semisimple, and g_u is unipotent. (This is the *Jordan decomposition* of g.) We say the linear algebraic group G is *unipotent* if $g = g_u$ for every $g \in G(\bar{k})$. We say G is *solvable* if $G(\bar{k})$ is solvable.

A *closed (resp. open) subgroup* of a linear algebraic group G is an algebraic group H that is a closed (resp. open) subscheme of G such that the inclusion map $H \to G$ is a morphism of algebraic groups. We say G is *reductive* if no non-trivial unipotent normal connected linear algebraic subgroups. If G is instead defined over k, we say it is *reductive* if the base change $G_{\bar{k}}$ is reductive. Similarly, G is *solvable* if the base change $G_{\bar{k}}$ is solvable.

The identity component of the maximal normal solvable subgroup of a linear algebraic group is called its *radical*. The set of unipotent elements in the radical is called the *unipotent radical* of the group. Reductive groups can equivalently be defined as groups with trivial unipotent radical.

Example 2.8. (i) Although we do not give a proof or construction, the general linear group $\mathrm{GL}(V)$ on a vector space V can be thought of as a group scheme. (For finite-dimensional V, this follows from the description we gave earlier of GL_n.) Moreover, $\mathrm{GL}(V)$ turns out to be a connected reductive linear algebraic group.

(ii) The additive group scheme $\mathbb{G}_a$ has the affine line $\mathbb{A}^1 = \mathrm{Spec} k[x]$ as its underlying scheme. Given a scheme T, $\mathbb{G}_a(T)$ is defined to be the additive group of global sections of the structure sheaf of T. (Since schemes are ringed spaces, the global sections of a scheme form a commutative ring, and thus in particular they have an additive group structure.) This group scheme is not reductive.

In the theory of Lie groups and algebras, a configuration of vectors known as a *root system* is associated to and classifies the group. Similarly, in the theory of reductive linear algebraic groups, a generalized version of a root system known as the *root datum* can be used to classify the group. We will not define root data, but the fact that reductive groups (or more precisely, connected *split* reductive groups) can be neatly characterized in terms of their root data is one reason for the central importance of reductive groups in the theory.

We will now describe a clever trick for generating finite group schemes.

Definition 2.9. Let p be a prime integer, let $n \in \mathbb{N}$, and let $q := p^n$. Any $\mathbb{F}_q$-algebra admits an endomorphism, known as the *Frobenius endomorphism* Frob_q given by the map $x \mapsto x^q$. If X is a scheme over $\mathbb{F}_q$, we can define the *absolute Frobenius morphism* of X as follows. Suppose $U = \mathrm{Spec}(R)$ is an open affine subset of X. Then R admits a Frobenius endomorphism, and given an affine open subset $V \subseteq U$, the Frobenius morphism on V is the restriction of the Frobenius morphism on R. Thus, we may glue the Frobenius morphisms on each open affine subset to produce the absolute Frobenius morphism on X.

Definition 2.10. Retaining the notation $q = p^n$ for p prime, let $k := \mathbb{F}_q$. Suppose $\bar{k}$ is the algebraic closure of k, and G is a connected reductive linear algebraic group over $\bar{k}$. We then have an absolute Frobenius morphism $F = \mathrm{Frob}_q : G \to G$. The fixed points G^F of the morphism can be defined as the equalizer of F and the identity morphism, and therefore, by an earlier comment, G^F may be viewed as a closed subscheme of G. We say a scheme G^F of this form is a *finite group scheme of Lie type* or a *finite Lie group scheme.*

Example 2.11. For $G = \mathrm{SL}_n$, which can be defined as a group scheme by analogy with the definition of GL_n given earlier, if $F = \mathrm{Frob}_q$, then $G^F = \mathrm{SL}_n(\mathbb{F}_q)$.

Now that we have some familiarity with group schemes, we will be a bit more flexible in treating them as groups or as schemes whenever one point of view is more convenient than the other.

In this article, we use irreducible representations of $\mathrm{SL}_2(\mathbb{F}_q)$ as a toy example. These were classified before the invention of Deligne–Lusztig theory. Such representations correspond to the conjugacy classes of $\mathrm{SL}_2(\mathbb{F}_q)$, which come in the following four families.

(i) There are 2 conjugacy classes whose elements are of the form

$$\begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}.$$

(ii) There are $\frac{q-3}{2}$ conjugacy classes whose elements are of the form

$$\begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix}$$

for $x \in \mathbb{F}_q^\times \setminus \{\pm 1\}$.

(iii) There are $\frac{q-1}{2}$ conjugacy classes whose elements are of the form

$$\begin{pmatrix} \omega & 0 \\ 0 & \omega^q \end{pmatrix}$$

where $\omega \in \mu_{q+1} \setminus \{\pm 1\}$, and μ_{q+1} denotes the group of $(q+1)^{\text{st}}$ roots of unity.

(iv) There are 4 conjugacy classes of the form

$$\begin{pmatrix} x & y \\ 0 & x^{-1} \end{pmatrix}$$

where $x \in \{\pm 1\}$ and $y \in \mathbb{F}_q^\times/(\mathbb{F}_q^\times)^2$, i.e., b is some representative in $\mathbb{F}_q^\times$ of a coset in $\mathbb{F}_q^\times/(\mathbb{F}_q^\times)^2$.

There are therefore $q+4$ distinct irreducible representations of $\mathrm{SL}_2(\mathbb{F}_q)$. The way these representations were historically constructed is by a technique that we now describe. There is a torus

$$T = \{\begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} | x \in \mathbb{F}_q^\times\} \subseteq \mathrm{SL}_2(\mathbb{F}_q).$$

This torus is isomorphic to $\mathbb{F}_q^\times$, and its irreducible representations therefore correspond to characters $\theta_i : T \to \mathbb{C}^\times$ for $i \in \{1, ..., q-1\}$. However, the representation $\mathrm{Ind}_T^G \theta_i$ for such a character turns out to be too big. Instead, we note that θ_i can be extended to a character on the Borel subgroup

$$B = \{\begin{pmatrix} x & y \\ 0 & x^{-1} \end{pmatrix} | x, y \in \mathbb{F}_q^\times\},$$

and we then consider $\mathrm{Ind}_B^G \theta_i$. This technique is known as *parabolic induction.* We summarize its result.

(i) If θ_i^2 is not the identity, then $\mathrm{Ind}_B^G \theta_i$ is irreducible of dimension $q+1$. These representations are known as *principal series.*

(ii) If θ_i^2 is the identity but θ is not, then $\mathrm{Ind}_B^G\theta_i$ is a direct sum of two irreducible representations each of dimension $\frac{q+1}{2}$. These two representations are known as *half principal series.*

(iii) If θ_i is the identity, then $\mathrm{Ind}_B^G\theta_i$ is a direct sum of the trivial representation and an irreducible representation of dimension q, which is known as the *Steinberg representation.*

Note that this leaves $\frac{q-1}{2}$ undiscovered irreducible representations. We should expect from our earlier list of conjugacy classes that the remaining representations can be obtained by somehow inducing characters from the conjugacy classes of the form

$$\begin{pmatrix} \omega & 0 \\ 0 & \omega^q \end{pmatrix}$$

to all of G. However, these conjugacy classes do not lie in our Borel subgroup. We will soon see how this leads us down the path to Deligne–Lusztig theory.

3. The Work of Drinfeld and Deligne–Lusztig

Our main references for this section and the last part of the previous section were [6] and [1]. The canonical references for the material on ℓ-adic cohomology are [8], [9], and [10].

Drinfeld made the following observation. Consider the affine variety X over $\mathbb{F}_q$ (where $q = p^n$ is a prime power) given by $xy^q - x^q y = 1$. The group μ_{q+1} of $(q+1)^{\text{th}}$ roots of unity acts on it by $\omega \cdot (x, y) := (\omega x, \omega y)$. Also, $\mathrm{SL}_2(\mathbb{F}_q)$ acts on it by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot (x, y) := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Indeed, this preserves the determinant because

$$\begin{aligned} &\det \begin{pmatrix} ax+by & (ax+by)^q \\ cx+dy & (cx+dy)^q \end{pmatrix} \\ &= (\det \begin{pmatrix} a & b \\ c & d \end{pmatrix})(\det \begin{pmatrix} x & x^q \\ y & y^q \end{pmatrix}) \\ &= \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} (xy^q - x^q y) \\ &= \det \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \end{aligned}$$

To proceed further, we need to consider the *ℓ-adic cohomology groups with compact support* $H^i_c(X) := H^i_c(X, \mathbb{Q}_\ell)$. Just defining these groups and proving their basic properties was a monumental work of Grothendieck and others. We therefore summarize the properties we will be using in the following proposition.

Proposition 3.1. *Suppose X is an algebraic variety over $\mathbb{F}_q$ for $q = p^n$, p prime. There exist ℓ-adic cohomology groups with compact support, denoted $H^i_c(X, \mathbb{Q}_\ell)$ or $H^i_c(X)$ for short. They may be regarded as finite-dimensional $\mathbb{Q}_\ell$–vector spaces, and they vanish for $i > 2\dim(X)$. (This last property would not hold in such generality for general ℓ-adic cohomology groups (not with compact support). This is why Deligne and Lusztig use the groups $H^i_c(X)$ rather than $H^i(X)$.) A group action on X induces a corresponding group action on $H^i_c(X)$. An automorphism σ on X induces a linear automorphism σ^* on $H^i_c(X)$.*

By the proposition, $\mu_{q+1} \times \mathrm{SL}_2(\mathbb{F}_q)$ acts on $H^i_c(X)$.

Definition 3.2. Let G be a finite group, let $\chi : G \to \mathbb{C}^\times$ be a character of G, and let $\rho : G \to \mathrm{GL}(V)$ be a representation of G for some vector space V. The *χ-isotypic subspace* of V is defined to be

$$V_\chi = \{v \in V | \rho(g)v = \chi(g)v \text{ for all } g \in G\}.$$

Definition 3.3. Let $\theta : \mu_{q+1} \to \mathbb{C}^\times$ be a character. Letting $V = H^i_c(X)$, we can regard elements of G and elements of $\mathrm{im}(\theta)$ as acting on V. The *Deligne–Lusztig (virtual) character* of θ is

$$R(\theta) := \sum_{i=0}^{2\dim(X)} (-1)^i H^i_c(X)_\theta,$$

where $H^i_c(X)_\theta$ denotes the θ-isotypical component of $H^i_c(X)$ and the sum is formal. (Technically, it can be formalized by defining *Grothendieck groups*, but we will not do that here.) By the last proposition, the sum only has finitely many non-zero terms.

We state the following result of Deligne and Lusztig in full generality, although we will only apply it to one specific case.

Theorem 3.4. *Suppose X is an algebraic variety over a field k and $\sigma : X \to X$ is an automorphism of finite order. If $\sigma = su$ where s, u are powers of σ of orders prime to p and a power of p, respectively, then*

$$Tr(\sigma^*, H^*_c(X)) = Tr(u^*, H^*_c(X^s)).$$

Here we let $Tr(\sigma^*, H^*_c(X)) = \sum_{i=0}^{2\dim(X)} (-1)^i Tr(\sigma^*, H^i_c(X))$, where $Tr(\sigma^*, H^i_c(X))$ is the trace of the linear endomorphism σ^* induced by σ on $H^i_c(X)$ induced by σ. Also, X^s denotes the set of fixed points of X under s.

Example 3.5. Suppose $1 \neq \omega \in \mu_{q+1}$. By the previous fixed point formula,

$$\mathrm{Tr}(\omega, H^*_c(X)) = \mathrm{Tr}(1, H^*_c(X^\omega)).$$

However, $X^\sigma = \emptyset$, as can be seen from the definition of X as the curve $xy^q - x^q y = 1$. Therefore, the trace of any non-identity element $\omega \in \mu_{q+1}$ on $H^*_c(X)$ is zero. This virtual character therefore corresponds to a multiple of the regular representation, which in this context turns out to be $\overline{\mathbb{Q}_\ell}[\mu_{q+1}]$. This implies that every θ-isotypic component in the sum for $R(\theta)$ has the same degree, and for $\theta = 1$ this degree turns out to be $1 - q$. It follows that $-R(\theta)$ is a degree $q - 1$ representation of $\mathrm{SL}_2(\mathbb{F}_q)$. It turns out that if $\theta^2 \neq 1$, then the representation is irreducible, and if $\theta^2 = 1$, it is a direct sum of two irreducible representations of degree $\frac{q-1}{2}$. These are exactly the representations of $\mathrm{SL}_2(\mathbb{F}_q)$ that we hoped to recover using Deligne–Lusztig theory.

Lang proved the following theorem, although in greater generality than we state it.

Theorem 3.6. *Suppose G^F is a finite group of Lie type. Let T be a maximal torus of G stable under F. Then there exists a Borel subgroup of G containing T and stable under F.*

Deligne and Lusztig invented the following construction to generalize Drinfeld's technique. Let T be a maximal torus of G stable under F. By Lang's theorem, there exists a Borel subgroup B of G containing T and stable under F. The group

$$W := N_G(T)/T$$

is known as the *Weyl group* of G corresponding to the maximal torus T. Because the action of G by conjugation on its Borel subgroups is transitive, there is a bijection between G/B and the Borel subgroups of G given by mapping the image of an element $g \in G$ in G/B to the conjugate gBg^{-1}. The following theorem is now useful.

Theorem 3.7. *(Bruhat decomposition.) Suppose G is a connected reductive linear algebraic group over an algebraically closed field $\bar{k}$. Let B be a Borel subgroup of G, and let W be a Weyl group of G corresponding to a maximal torus in B. Then*

$$G = BWB = \bigsqcup_{w \in W} BwB.$$

We can therefore identify W with the double cosets $B \setminus G/B$.

Definition 3.8. Let G be a connected reductive linear algebraic group, $g \in G$, and B a Borel subgroup of G. We say that the two Borel subgroups gBg^{-1} and B are *in relative position* w where w is the image of g in W along the isomorphism $W \simeq B \setminus G/B$ obtained from the Bruhat decomposition.

Note that G/B can itself be regarded as a projective variety over $\bar{k}$.

Definition 3.9. Let G be a connected reductive linear algebraic group over $\overline{\mathbb{F}_q}$. Let B be a Borel subgroup of G containing a maximal torus T. Let $w \in W \simeq B \setminus G/B$. Let $F : G \to G$ be the absolute Frobenius morphism. Let U be the unipotent radical of B. We define the variety

$$Y(w) := \{gU \in G/U | g^{-1}F(g) \in UwU\} \subseteq G/U.$$

The set

$$T^{wF} := \{t \in T | wF(t)w^{-1} = t\}$$

is a torus which acts on $Y(w)$ on the right. Let θ be a character of the torus T^{wF}, and let

$$R_w(\theta) := \sum_{i=0}^{2\dim(Y(w))} (-1)^i H^i_c(Y(w))_\theta.$$

The varieties $Y(w)$ are known as *Deligne–Lusztig varieties.*

Example 3.10. Take $G = \mathrm{SL}_2$ over $\mathbb{F}_q$. The torus

$$T = \{\begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} | x \in \mathbb{F}_q^\times\}$$

and the Borel subgroup

$$B = \{\begin{pmatrix} x & y \\ 0 & x^{-1} \end{pmatrix} | x, y \in \mathbb{F}_q^\times\}$$

are stable under the Frobenius morphism F. One finds that the Weyl group with respect to T is isomorphic to $S_2 = \{e, w\}$. If U is the unipotent radical of B, then elements of G/U can be identified with one-dimensional subspaces L of $\mathbb{F}_q^2$ along with the additional data of vectors $u \in \mathbb{F}_q^2/L$, $v \in L$ for which $\det(v, u) = 1$. Therefore, the Deligne–Lusztig variety $Y(w)$ consists of vectors v for which $\det(v, F(v)) = 1$. Writing $v = \begin{pmatrix} x \\ y \end{pmatrix}$, we obtain

$$\det \begin{pmatrix} x & x^q \\ y & y^q \end{pmatrix} = 1$$

and calculate the torus

$$T^{wF} = \{ \begin{pmatrix} x & 0 \\ 0 & x^q \end{pmatrix} | x \in \mathbb{F}_q^\times \} \simeq \mu_{q+1}.$$

This recovers Drinfeld's curve!

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