OUTER AUTOMORPHISMS OF FREE GROUPS

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Abstract. We discuss mapping class groups and Teichmüller spaces. We then use these constructions to motivate the definition of the Culler–Vogtmann Outer space, and we briefly discuss applications to the study of outer automorphisms of free groups.

1. Introduction.

Very often in mathematics, we like to study the symmetries of a particular object we care about. Since we care about the free group on n generators, F_n, it makes sense to study its automorphism group, $\text{Aut}(F_n)$. A particularly nice class of automorphisms is the set of *inner automorphisms*, $\text{Inn}(F_n)$. Elements of $\text{Inn}(F_n)$ are those automorphisms $\varphi_g : F_n \to F_n$ for $g \in G$ such that $\varphi_g(x) = gxg^{-1}$ for every $x \in G$. The following lemma shows us that the set of inner automorphisms is actually a group that is isomorphic to a certain familiar quotient of G. Inner automorphisms are therefore relatively easy objects to understand.

Lemma 1.1. *For any group G, $Inn(G)$ is a normal subgroup of $Aut(G)$, and $Inn(G) \simeq G/Z[G]$.*

Proof. Recall the notation $\varphi_g(x) = gxg^{-1}$. The identity automorphism is given by conjugation by the group identity. If $\varphi_g \in \text{Inn}(G)$ and $x \in G$, then it is easy to check that $(\varphi_g)^{-1} = \varphi_{g^{-1}} \in \text{Inn}(G)$. If $\varphi_g, \varphi_h \in \text{Inn}(G)$, then

$$\varphi_g \circ \varphi_h = \varphi_{gh},$$

so $\text{Inn}(G)$ is closed under composition. If $\sigma \in \text{Aut}(G)$, $g \in G$, and $x \in G$, then

$$\begin{aligned}
\sigma \circ \varphi_g \circ \sigma^{-1}(x) &= \sigma \circ \varphi_g(\sigma^{-1}(x)) \\
&= \sigma(g\sigma^{-1}(x)g^{-1}) \\
&= \sigma(g)x\sigma(g)^{-1} \\
&= \varphi_{\sigma(g)}(x).
\end{aligned}$$

This proves that $\text{Inn}(G)$ is a normal subgroup of $\text{Aut}(G)$.

Consider the map $\phi : G \to \text{Inn}(G)$ given by

$$\phi(g) = \varphi_g.$$

Because $\varphi_g \circ \varphi_h = \varphi_{gh}$, ϕ is a group homomorphism. Clearly $\text{im}(\phi) = \text{Inn}(G)$. Also,

$$\begin{aligned}
\ker(\phi) &= \{g \in G | \phi(g) = \text{id}\} \\
&= \{g \in G | gxg^{-1} = x \text{ for all } x \in G\} \\
&= Z[G].
\end{aligned}$$

By the first isomorphism theorem, $\mathrm{Inn}(G) \simeq G/Z[G]$. □

By the lemma, we can pass to the quotient group $\mathrm{Out}(G) = \mathrm{Aut}(G)/\mathrm{Inn}(G)$, known as the group of *outer automorphisms* of G. Note that $F_1 \simeq \mathbb{Z}$, so

$$\mathrm{Out}(F_1) \simeq \mathrm{Out}(\mathbb{Z}) \simeq \mathrm{Aut}(\mathbb{Z}) = \mathrm{GL}_1(\mathbb{Z}) \simeq \mathbb{Z}/2\mathbb{Z}.$$

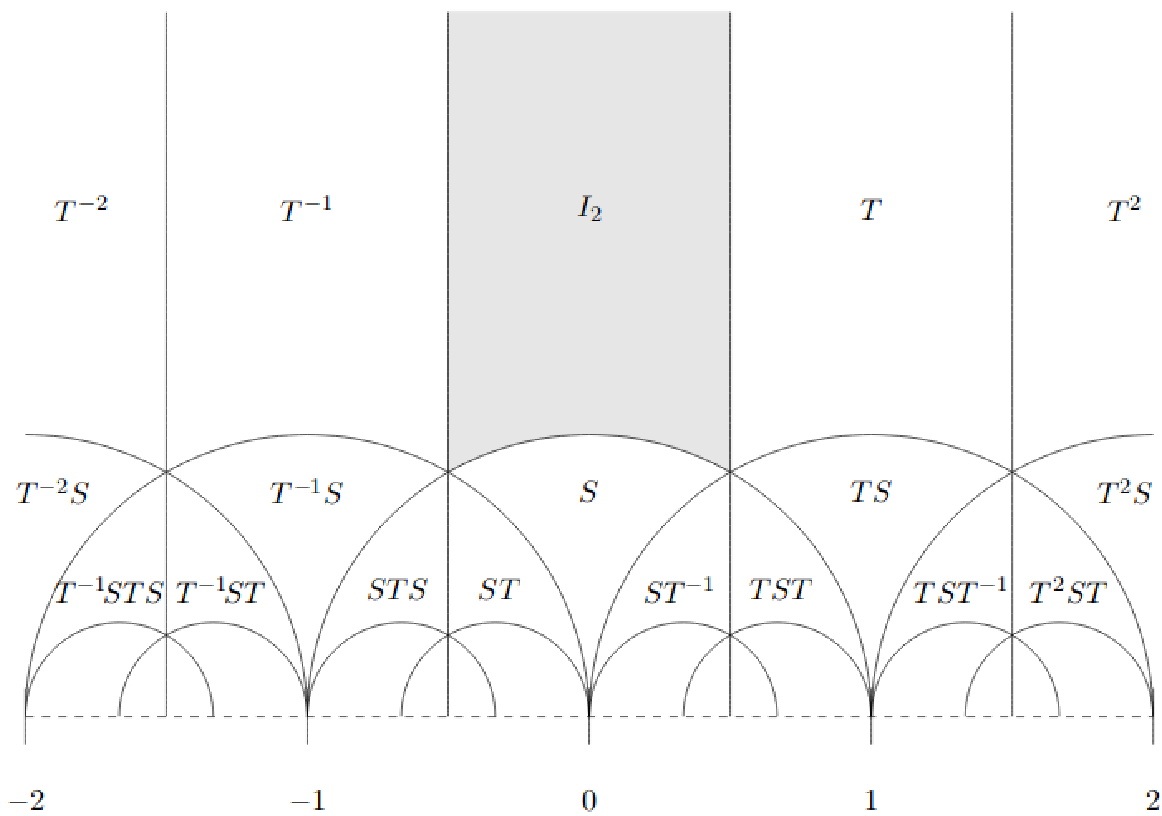
In fact, Nielsen proved in [5] that $\mathrm{Out}(F_2) \simeq \mathrm{GL}_2(\mathbb{Z})$ as well. For $n > 2$, however, we can only guarantee the existence of a surjection $\mathrm{Out}(F_n) \twoheadrightarrow \mathrm{GL}_n(\mathbb{Z})$ induced by the abelianization map $F_n \twoheadrightarrow \mathbb{Z}$.

Historically, the groups $\mathrm{GL}_n(\mathbb{Z})$ were studied by passing to the subgroup $\mathrm{SL}_n(\mathbb{Z})$ and then examining the action of $\mathrm{SL}_n(\mathbb{Z})$ on certain "homogeneous spaces". It is worth recalling some details of this setup.

There is an action of $\mathrm{SL}_2(\mathbb{Z})$ on the complex upper half plane $\mathbb{H}$ given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az+b}{cz+d} \in \mathbb{H}$$

for $z \in \mathbb{H}$ and $a, b, c, d \in \mathbb{Z}$ satisfying $ad - bc = 1$. The group is said to act via *fractional linear transformations*. Any discussion of this action would be incomplete without the iconic picture of its "fundamental domain". Ours is borrowed from [1].



Here S and T are two generators of $\mathrm{SL}_2(\mathbb{Z})$, and the labels indicate where products of the generators send the shaded region. The shaded region is the *fundamental domain*.

The next proposition lets us characterize $\mathbb{H}$ in a way that admits an easy generalization to higher dimensions. Before stating it, we prove a useful lemma.

Lemma 1.2. *Suppose a group G acts on a set X. If $x_1, x_2 \in X$ are in the same G-orbit, then their stabilizers G_{x_1} and G_{x_2} are conjugate. In particular, if the action of G on X is transitive, then every two stabilizers are isomorphic.*

Proof. Since x_1 and x_2 are in the same G-orbit, there exists $g \in G$ such that $gx_2 = x_1$. If $h \in G_{x_2}$, then $hx_2 = x_2$, so

$$ghg^{-1}x_1 = ghx_2 = gx_2 = x_1.$$

Thus, we obtain an injective homomorphism $\phi : G_{x_2} \to G_{x_1}$. A short calculation shows that its inverse is given by $h \mapsto g^{-1}hg$. Thus, G_{x_1} and G_{x_2} are conjugate.

If the G-action is transitive, then there is only one G-orbit, so for any $x_1, x_2 \in X$, G_{x_1} and G_{x_2} are conjugate. Since conjugate subgroups are isomorphic, any two stabilizers are isomorphic in this case. □

Now we are ready for a well-known proposition, which we do not prove.

Proposition 1.3. *If a group G acts transitively on a topological space X, then if $x \in X$ is arbitrary and G_x denotes the stabilizer of x, we have that*

$$G/G_x \simeq X,$$

where $\simeq$ denotes homeomorphism. (Notice that the choice of x is arbitrary because of the previous lemma.)

Although the action of $\mathrm{SL}_2(\mathbb{Z})$ does not have this property, the action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathbb{H}$ is in fact *transitive*, meaning it has only a single orbit. Letting $G = \mathrm{SL}_2(\mathbb{R})$, we calculate the stabilizer

$$\begin{aligned} G_i &= \{g \in G | gi = i\} \\ &= \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G | \frac{ai+b}{ci+d} = i\} \\ &= \{a, b, c, d \in \mathbb{R} | ad - bc = 1, a = d, b = -c\} \\ &= \mathrm{SO}_2(\mathbb{R}). \end{aligned}$$

Thus, by the proposition,

$$\mathbb{H} \simeq \mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2(\mathbb{R}).$$

More generally, it is possible to define actions

$$\mathrm{SL}_n(\mathbb{Z}) \curvearrowright \mathrm{SL}_n(\mathbb{R})/\mathrm{SO}_2(\mathbb{R}),$$

and early attempts at the study of $\mathrm{Out}(F_n)$ used its action on $\mathrm{SL}_n(\mathbb{R})/\mathrm{SO}_n(\mathbb{R})$ induced by the surjection $\mathrm{Out}(F_n) \twoheadrightarrow \mathrm{GL}_n(\mathbb{Z})$. However, as is mentioned in [9], this action turns out not to be *proper*, i.e., inverses of compact sets under it are not necessarily compact. It is thus rather badly behaved, so research on $\mathrm{Out}(F_n)$ had to move in a different direction.

2. Mapping Class Groups and Teichmüller Spaces.

The development of knot theory in the mid–20th century popularized the notion of "isotopy". To understand what this means, we first recall the definition of a homotopy.

Definition 2.1. Let X and Y be topological spaces, and let $f, g : X \to Y$ be continuous functions. A *homotopy* from f to g is a continuous map $H : X \times [0,1] \to Y$ with the following properties:

(i) $H(-,0) = f$
(ii) $H(-,1) = g$
(iii) $H(-,t)$ is a continuous function from X to Y for every $t \in [0,1]$.

By requiring $H(-,t)$ to have stronger properties than simply being continuous, we obtain the definition of an isotopy. First, we recall what an embedding is. Intuitively, if we have an embedding from a topological space X to a topological space Y, then we can realize X as a subspace of Y. The following definition makes this precise.

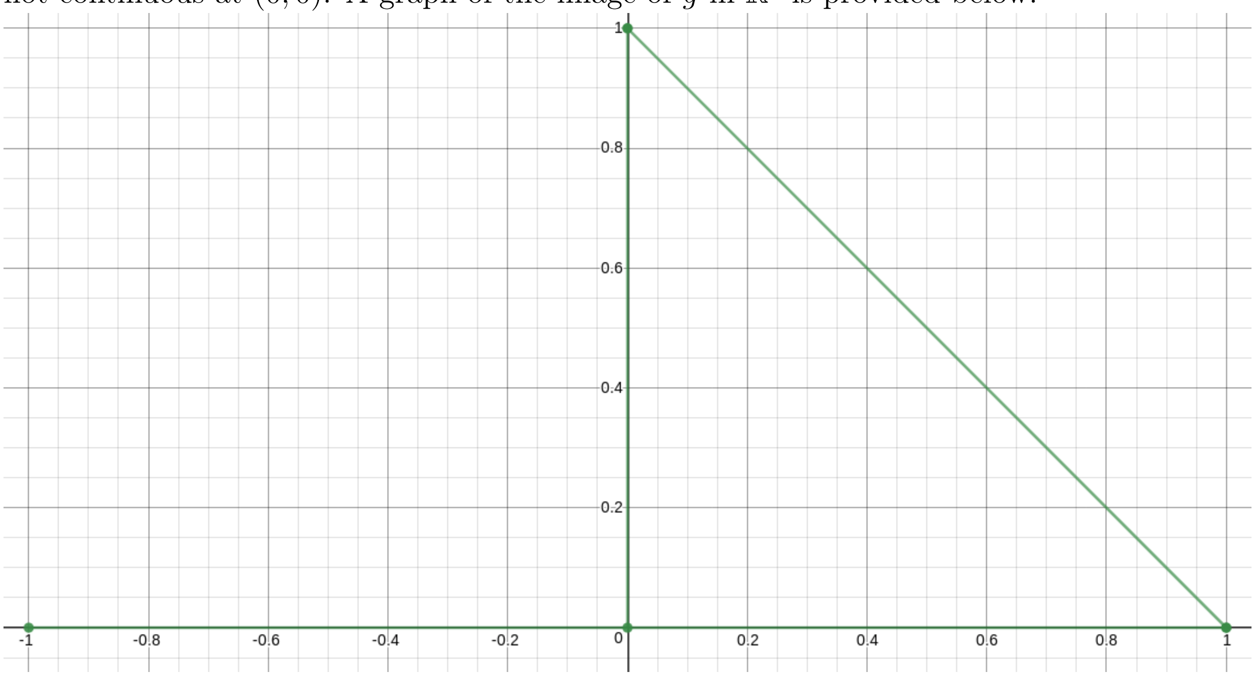
Definition 2.2. Let X and Y be topological spaces. An embedding $f : X \to Y$ is an injective continuous function that is a homeomorphism onto its image.

The last condition does not follow from the first two, as the following classic example shows.

Example 2.3. Consider $g : (0,1) \to \mathbb{R}^2$ defined by

$$g(n) = \begin{cases} (6n-1, 0) & \text{if } 0 < t \leq \frac{1}{3} \\ (2-3n, 3n-1) & \text{if } \frac{1}{3} \leq n \leq \frac{2}{3} \\ (0, 3-3n) & \text{if } \frac{2}{3} \leq n < 1. \end{cases}$$

Then g is injective and continuous but not a homeomorphism onto its image. Indeed, g^{-1} is not continuous at $(0,0)$. A graph of the image of g in $\mathbb{R}^2$ is provided below.



Definition 2.4. Let X and Y be topological spaces, and let $f, g : X \to Y$ be embeddings. An *isotopy* from f to g is a continuous map $H : X \times [0,1] \to Y$ with the following properties:

(i) $H(-,0) = f$
(ii) $H(-,1) = g$
(iii) $H(-,t)$ is an embedding from X to Y for every $t \in [0,1]$.

In particular, every homeomorphism is an embedding. This allows us to define the mapping class group of a surface.

Definition 2.5. Let S be a connected, closed, orientable surface equipped with a metric d. Let $\mathrm{Homeo}^+(S)$ be the group of orientation-preserving homeomorphisms of S. This is indeed a group under composition of homeomorphisms. We define a metric δ on it by

$$\delta(f,g) = \sup_{x \in S} d(f(x), g(x)).$$

In the topology induced by this metric, we can consider the connected component of the identity, which we denote by $\mathrm{Homeo}_0(S)$. This turns out to coincide with the homeomorphisms of S which are isotopic to the identity. It is also a normal subgroup of $\mathrm{Homeo}^+(S)$, so that we can define the *mapping class group*

$$\mathrm{MCG}(S) = \mathrm{Homeo}^+(S)/\mathrm{Homeo}_0(S).$$

Remark 2.6. Notice that the previous definition contains several claims we did not check. It is not even clear *a priori* that the connected component of the identity in $\mathrm{Homeo}^+(S)$, which is made up of homeomorphisms that can be deformed *through a path of homeomorphisms* to the identity, coincides with homeomorphisms isotopic to the identity, as our notion of isotopy involved a path of embeddings, not homeomorphisms. The two notions do turn out to coincide in this case, but to discuss this would take us too far afield. Instead, we defer to a good reference on mapping class groups, e.g., [3], for the details.

Calculating examples of mapping class groups is non-trivial since it requires a significant amount of algebraic topology. We therefore only state some mapping class groups without proof.

Example 2.7. The mapping class group of the sphere S^2 is isomorphic to $\mathbb{Z}/2\mathbb{Z}$. The mapping class group of the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ is isomorphic to $\mathrm{SL}(2,\mathbb{Z})$. Note that we discussed the group $\mathrm{SL}(2,\mathbb{Z})$ earlier, in the context of its action on the upper half plane $\mathbb{H}$. As we will see shortly, this is not a coincidence.

The notion of *Teichmüller space* was introduced by Oswald Teichmüller in his study of quasi-conformal mappings. The study of "moduli spaces", spaces which parameterize some family of surfaces or of differential structures on a surface, began with Riemann's 19^{th} century work on his namesake surfaces. Teichmüller's insight was that by looking at complex structures on a surface up to the action of homeomorphisms that are isotopic to the identity, we obtain a moduli space that is in many respects simpler to work with than previous moduli spaces. This is the Teichmüller space of the surface. In particular, Teichmüller placed a topology on the Teichmüller space of surfaces of genus $g \geq 2$ and proved these spaces are homeomorphic to balls of dimension $6g-6$.

The study of Teichmüller spaces was further developed by many geometers throughout the 1960s, culminating in Thurston's work on the relationship between Teichmüller spaces and mapping class groups in the late 1970s. We follow Thurston's definition of a Teichmüller space.

Definition 2.8. Let S be a connected, closed, orientable surface. The *Teichmüller space* $T(S)$ is the set of pairs (X,g) where X is a surface and $g : S \to X$ is a homeomorphism, defined up to isotopy. (We are implicitly using the fact that isotopy defines an equivalence relation, which we do not prove.) We refer to such a pair (X, g) as a *marked Riemann surface*, and we refer to the homeomorphism g as a *marking*.

As is the case for mapping class groups, Teichmüller spaces are non-trivial to calculate. We are only concerned with one example, however.

Example 2.9. The Teichmüller space of the torus $\mathbb{T}^2$ is $\mathbb{H}$, the complex upper half plane. Recall that the mapping class group of the torus was $\mathrm{MCG}(\mathbb{T}^2) = \mathrm{SL}_2(\mathbb{Z})$. We know that $\mathrm{SL}_2(\mathbb{Z})$ acts on $\mathbb{H}$. This suggests that we might generally have an action of mapping class groups on their corresponding Teichmüller spaces, and in fact we do.

Definition 2.10. Let S be a connected, closed, orientable surface equipped with a metric. Let $(X, f) \in T(S)$ and $h \in \mathrm{MCG}(S)$. We define an action $\mathrm{MCG}(S) \curvearrowright T(S)$ by

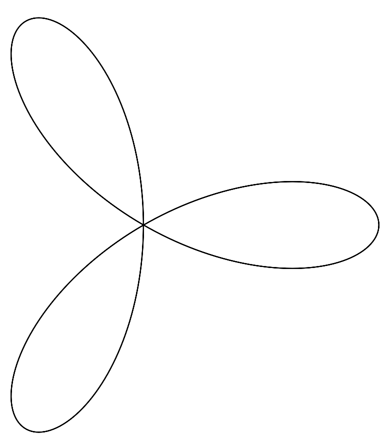
$$(X, f) \mapsto (X, f \circ h^{-1}).$$

We do not check that this gives a well-defined action. (Recall that (X, f) is only defined up to isotopy of f.) Instead, we defer the details to a standard reference like [3]. We also do not check that, in the particular case of $\mathbb{T}^2$, this action reproduces the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{H}$ by fractional linear transformations. An illustrated discussion of this fact is given at [7].

The construction of Outer space cleverly transports the relationship between mapping class groups and Teichmüller spaces to the setting of geometric group theory. The analogy is that Outer space is to $\mathrm{Out}(F_n)$ as Teichmüller spaces are to their mapping class groups. We will now dive into this analogy.

3. Outer Space and Applications.

This section concerns the main construction of the groundbreaking 1986 paper [2] by Culler and Vogtmann. We assume the reader can calculate the fundamental group of some basic shapes. In particular, let R_n be the topological space obtained by taking n copies of S^1, each with a distinguished point, and identifying them along their distinguished points. Below is a picture of the space R_3.



We also choose an orientation for each loop in R_n. Then $\pi_1(R_n) \simeq F_n$, as is proved, for example, in [4]. Basically, the n generators of F_n correspond to the n oriented loops, and an element of F_n corresponds to the concatenation of those loops (up to homotopy) in the order they appear in the word.

We wish to modify the definition of Teichmüller space in a way that is appropriate to our new setting. To accomplish this, we replace the role of the surface S in the definition of Teichmüller space with R_n and take the target X to be a finite graph. Since we essentially only care about R_n because of its fundamental group, we replace the notion of homeomorphism in the definition of Teichmüller space with the weaker notion of homotopy equivalence. We recall what this means.

Definition 3.1. Let X and Y be topological spaces. We say X and Y are *homotopy equivalent* if there exists continuous maps $f : X \to Y$ and $g : Y \to X$ such that $f \circ g$ is homotopic to the identity on Y and $g \circ f$ is homotopic to the identity on X.

Homotopy equivalence is an equivalence relation, and two homotopy equivalent spaces have the same sequence of homotopy groups, as is shown in [4]. In particular, they have the same fundamental group.

Remark 3.2. (This remark is an aside for readers knowledgeable about homotopy theory.) It would be interesting to know what happens if we replace the notion of homotopy equivalence in the definition of Outer space with the even weaker notion of *weak homotopy equivalence*. A weak homotopy equivalence $f : X \to Y$ is a continuous map that induces isomorphisms on all homotopy groups. One can then say that two spaces X and Y have the same *weak homotopy type* if they become isomorphic in the homotopy category obtained by turning the weak equivalences into isomorphism, or more formally, passing to the localization Ho(Top)/Ho(sSet) where Ho(Top) is the homotopy category of topological spaces, which we localize at Ho(sSet), the homotopy category of simplicial sets.

We are now ready to state the definition of Outer space.

Definition 3.3. Let R_n be the bouquet of n circles. As a set, *Outer space* $\mathcal{O}_n$ consists of pairs (g, Γ), modulo a certain equivalence relation and satisfying the following conditions:

(i) Γ is a finite graph with each vertex of degree at least 3.

(ii) The map $g : R_n \to \Gamma$ induces a homotopy equivalence between R_n and Γ, called the *marking*. (This means that there exists another map $h : \Gamma \to R_n$ such that g and h together induce a homotopy equivalence between R_n and Γ. This second map will be used in the remark below to give another characterization of points in Outer space.)

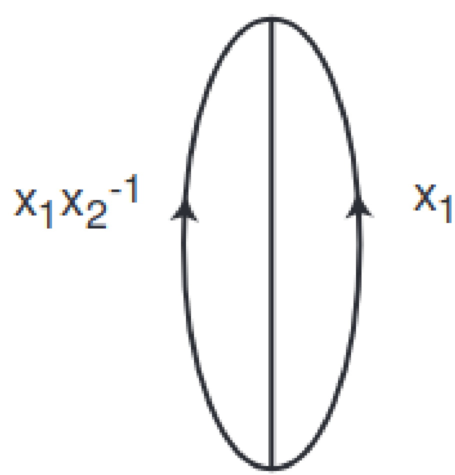
(iii) Each edge of Γ is assigned some positive real length, with the normalization condition that the sum of the lengths is 1, making Γ into a metric space via the path metric. That is, the distance between two points is the length of the shortest path connecting them, where the length is obtained by summing the lengths of the edges traversed or partially traversed.

We then impose an equivalence relation on this set by saying that points (g, Γ) and (g', Γ') are equivalent if there is an isometry $h : \Gamma \to \Gamma'$ such that $g \circ h$ is homotopic to g'.

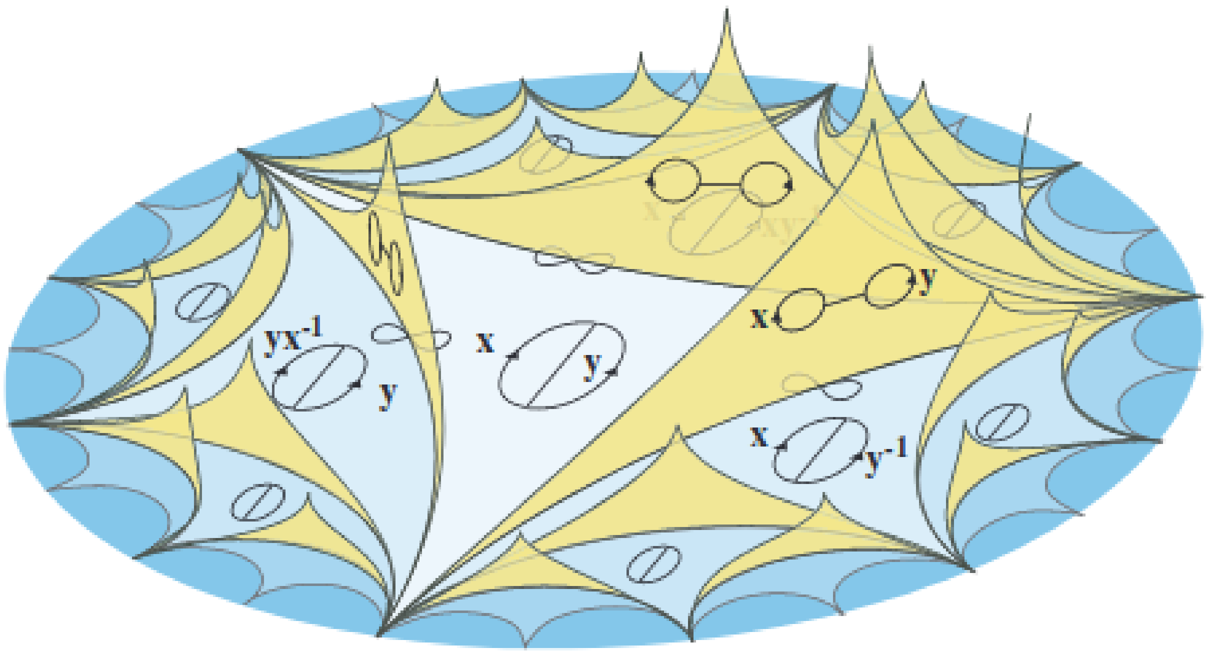
A topology is given on Outer space as follows. Let $\mathcal{C}$ be the set of conjugacy classes in F_n, or equivalently the set of cyclically reduced words in F_n. We define a map from $\mathcal{O}_n$ to $\mathbb{RP}^n$ in the following way. Suppose $(g, \Gamma) \in \mathcal{O}_n$. For each cyclically reduced word w, there is a unique cyclically reduced edge path loop in Γ homotopic to $g(w)$. The length of this loop can be viewed as an element of $\mathbb{RP}$. We thus obtain an injection $\mathcal{O}_n \to \mathbb{RP}^{\mathcal{C}}$, and we assign $\mathcal{O}_n$ the subspace topology obtained from $\mathbb{RP}^{\mathcal{C}}$. Outer space is therefore a topological space.

Remark 3.4. We can represent an element (g, Γ) of Outer space in a different way. First, we choose a maximal tree T of Γ. We orient each edge in $\Gamma \setminus T$ and label it by some word of F_n. The labels then uniquely determine a continuous map $h : \Gamma \to R_n$ sending T to the basepoint of R_n and sending each edge of $\Gamma \setminus T$ to the loop in R_n indicated by its label. The labels are chosen so that this map h is a homotopy inverse for g, i.e., the maps h and g induce the homotopy equivalence between R_n and Γ.

Note that a representation of (g, Γ) of this form, although useful and easy to visualize, is not unique because it depends on the choice of maximal tree and labels. We include a picture, borrowed from [8], of such a point.



Intuitively, we think of two points in Outer space as being close if they are obtained by "small deformations" of one another. These deformations do not necessarily have to be homeomorphisms. Identifying a short portion of two adjacent edges is also allowed. Because of the condition that the sum of the edge lengths has to be 1, Outer space is topologically the union of simplices. The following illustration of $\mathcal{O}_2$ is borrowed from [9].



It is reasonable to conjecture from such an illustration that $\mathcal{O}_n$ is contractible for all n. In fact, this is the main result of Culler and Vogtmann's paper [2].

We now describe the action of $\text{Out}(F_n)$ on $\mathcal{O}_n$.

Definition 3.5. The group of outer automorphisms of F_n, $\text{Out}(F_n)$, acts on $\mathcal{O}_n$ as follows. Let R_n be the bouquet of n circles. Given $\alpha \in \text{Out}(F_n)$, pick a representative $f : R_n \to R_n$ for α. Then let $(g, \Gamma)\alpha = (g \circ f, \Gamma)$.

Remark 3.6. Note that the value $(g \circ f, \Gamma)$ turns out not to depend on the choice of representative f. Also, the stabilizer of the point (g, Γ) can be proved to be isomorphic to the group of isometries of Γ, which is finite; this is proved in [6]. In this respect, the action of $\text{Out}(F_n)$ on $\mathcal{O}_n$ is much more nicely behaved than its action on $\text{SL}_n(\mathbb{R})/\text{SO}_n(\mathbb{R})$.

Aside from its inherent beauty and resolution of the problem of finding a nice action of $\text{Out}(F_n)$ on some space, the theory of Outer space has several applications. It can be shown that because the action of $\text{Out}(F_n)$ on Outer space has finite stabilizers, there is a finite-index, torsion-free, normal subgroup Γ of $\text{Out}(F_n)$ that acts freely on $\mathcal{O}_n$. In particular, its cohomology can be proved to be equal to that of the quotient $\mathcal{O}_n/\Gamma$ and to vanish in all dimensions except the dimension of $\mathcal{O}_n$. Similar observations, often using a certain deformation retract of $\mathcal{O}_n$, denoted K_n and known as the *spine* of Outer space, can be used to prove that $\text{Out}(F_n)$ has finitely-generated cohomology in all dimensions and has only finitely many conjugacy classes of finite subgroups. These results and many more are discussed in [8] and [9].

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