THE ARVESON–DOUGLAS CONJECTURE

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ABSTRACT. The Arveson–Douglas conjecture relates a statement about Hilbert space operators to a concrete claim in algebraic geometry. In this article, we give an exposition of the conjecture and discuss recent progress on it. We begin with a discussion of the Drury–Arveson Hilbert space and relate it to the Hardy space $H^2(\mathbb{B})$. We discuss contractive tuples and essential normality, motivating the latter concept with a question of Halmos. We then examine the conjecture and recent progress on it, as well as its relation to K-homology.

1. Introduction.

The Arveson–Douglas conjecture, first formulated by Arveson in 1998, relates a property of certain Hilbert space operators to a statement in pure algebraic geometry. The modern statement of the conjecture, due to Douglas, is considerably more general than Arveson's original formulation. To introduce the conjecture, we first need some background knowledge.

In what follows, we let $z = (z_1, ..., z_d)$ be a d-tuple of complex numbers. Then $\mathbb{C}[z] = \mathbb{C}[z_1, ..., z_d]$. For $\alpha \in \mathbb{N}^d$, we let

$$z^\alpha := z^{\alpha_1} ... z^{\alpha_d}.$$

Definition 1.1. The *Drury–Arveson Hilbert space* H^2_d, also known as the *d-shift space*, is the completion of $\mathbb{C}[z]$ with respect to

$$\langle z^\alpha, z^\beta \rangle = \delta_{\alpha\beta} \frac{\alpha_1! ... \alpha_d!}{(\alpha_1 + ... + \alpha_d)!}, \alpha, \beta \in \mathbb{N}^d,$$

where δ is the Kronecker delta function.

The Drury–Arveson Hilbert space arises in the following natural way. Consider the space $\mathbb{C}\langle z_1, ..., z_d \rangle$ of non-commutative polynomials with complex coefficients in the variables $z_1, ..., z_d$. As a $\mathbb{C}$–vector space, $\mathbb{C}\langle z_1, ..., z_d \rangle$ has a basis of non-commutative monomials in $z_1, ..., z_d$. For monomials m and n in this basis, define

$$\langle m, n \rangle := \begin{cases} 1, & m = n \\ 0, & m \neq n. \end{cases}$$

This extends by bilinearity to an inner product on all of $\mathbb{C}\langle z_1, ..., z_d \rangle$. Consider the abelianization map $\mathbb{C}\langle z_1, ..., z_d \rangle \to \mathbb{C}[z_1, ..., z_d]$ obtained by quotienting out by the commutator subgroup. Given $\alpha \in \mathbb{N}^d$ and $z^\alpha \in \mathbb{C}[z]$, where $z = (z_1, ..., z_d)$ as usual, exactly $\binom{\alpha_1 + ... + \alpha_d}{\alpha_1, ..., \alpha_d} = \frac{(\alpha_1 + ... + \alpha_d)!}{\alpha_1! ... \alpha_d!}$ distinct non-commutative monomials will be sent to z^α by the abelianization map. The coefficient $\frac{\alpha_1! ... \alpha_d!}{(\alpha_1 + ... + \alpha_d)!}$ in the inner product of H^2_d is thus a weighting factor coming from the abelianization of the standard orthonormal basis for $\mathbb{C}\langle z_1, ..., z_d \rangle$.

The space H_d^2 can be viewed as a multivariable generalization of the classical Hardy space $H^2(\mathbb{B})$ on the complex open unit disk $\mathbb{B}$. We will denote this space H^2 for convenience. The space H^2 was historically defined as the Hilbert space of holomorphic functions on the complex open unit disk whose mean square value on the circle of radius r remains bounded as $r \to 1$ from below. We can write this latter condition for a function f as

$$\sup_{0 \le r < 1} \left(\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^2 \, d\theta\right)^{1/2} < \infty$$

for every $0 \le r < 1$. Equivalently, H^2 can be defined as follows. Let $\mathcal{A}$ be the algebra of single-variable holomorphic polynomials. Given $f \in \mathcal{A}$, there is a Taylor expansion

$$f(z) = a_0 + a_1 z + ... + a_n z^n.$$

Impose the norm

$$||f|| = |a_0|^2 + |a_1|^2 + ... + |a_n|^2.$$

By the polarization identity, we obtain an inner product on $\mathcal{A}$. Then H^2 is simply the completion of $\mathcal{A}$ under this inner product; indeed, this is the construction used in Arveson's 1998 paper [2].

Although there are simpler ways of seeing why H_d^2 is a generalization of this Hardy space, we choose to give a demonstration that gives a certain perspective on Drury–Arveson Hilbert space we often encountered in the research literature. First, we introduce a family of Hilbert spaces, each of which can be specified by giving a corresponding function, known as its "kernel". Both H^2 and H_d^2 belong to this family.

Definition 1.2. Suppose X is a set and H is a Hilbert space of real-valued (resp. complex valued) functions on X. Given $x \in X$, there is an *evaluation functional* $L_x : H \to \mathbb{R}$ (resp. $L_x : H \to \mathbb{C}$) given by

$$L_x(f) = f(x) \text{ for every } f \in H.$$

If for every $x \in X$, L_x is continuous on all of H, or equivalently if each L_x is a bounded operator, we say that H is a *reproducing kernel Hilbert space (RKHS)*. By the Riesz representation theorem, if H is an RKHS, then for every $x \in X$, there exists $K_x \in H$ such that

$$f(x) = L_x(f) = \langle f, K_x \rangle \text{ for every } f \in H.$$

For $y \in X$, we have

$$K_x(y) = L_y(K_x) = \langle K_x, K_y \rangle.$$

This motivates us to define a function $K : X \times X \to \mathbb{C}$ by

$$K(x, y) := \langle K_x, K_y \rangle.$$

The function K is known as the *reproducing kernel* of the RKHS H.

Definition 1.3. It follows from the definition that the reproducing kernel K is symmetric and positive-definite. In this context, positive-definiteness means that

$$\sum_{i,j=1}^{n} c_i c_j K(x_i, x_j) \ge 0$$

for any $n \in \mathbb{N}$, $x_1, ..., x_n \in X$, and $c_1, ..., c_n \in \mathbb{C}$. Any function $K : X \times X \to \mathbb{C}$ satisfying this property is said to be a *positive-definite kernel.*

The following theorem, which we do not prove, provides a converse.

Theorem 1.4. *(Moore, Aronszajn.) Given a symmetric, positive-definite kernel $K : X \times X \to \mathbb{C}$, there is a unique Hilbert space of functions on X for which K is a reproducing kernel.*

Proof. The theorem and its proof first appeared in [1], where Aronszajn attributes it to E. H. Moore. □

A third and classical definition of the Hardy space H^2 is as the RKHS of the *Szego kernel*

$$K(z_0, w_0) = \frac{1}{1 - \langle z_0, w_0 \rangle}$$

where $\langle z_0, w_0 \rangle$ is the standard inner product of $z_0, w_0 \in \mathbb{B}$. By analogy, given $z, w \in \mathbb{B}^d$, we could let $\langle z, w \rangle$ be their dot product and then consider the RKHS of the kernel

$$K'(z, w) := \frac{1}{1 - \langle z, w \rangle}.$$

The resulting Hilbert space turns out to be the Drury–Arveson Hilbert space. A good reference for more details is [15].

For each of $z_1, ..., z_d \in \mathbb{C}[z]$, there corresponds a multiplication operator M_{z_i} on $\mathbb{C}[z]$ defined by

$$M_{z_i} p(z) = z_i p(z) \text{ for every } p(z) \in \mathbb{C}[z].$$

The d-tuple $(M_{z_1}, ..., M_{z_d})$ extends to a *contractive* d-tuple on H^2_d. We recall what this means.

Definition 1.5. Let H be a Hilbert space. An n-tuple of bounded operators $(T_1, ..., T_n)$ on H is said to be *contractive* if

$$T_1 T_1^* + ... + T_n T_n^* \leq I,$$

i.e., if $I - (T_1 T_1^* + ... + T_n T_n^*)$ is a positive operator. This condition is equivalent to having the operator $(T_1, ..., T_n)$, which we view as an operator from $H^{\oplus n}$ to H, be a contraction.

We remark that one can always dilate a contractive tuple to a tuple of isometries with orthogonal ranges. Moreover, given a contractive tuple of *commuting* operators, while it is not in general possible to dilate it to a tuple of mutually commuting isometries, there is a canonical dilation due to Arveson ([2]) and Drury ([7]). It is no coincidence that H^2_d is named after these same two authors, as its construction arose from consideration of tuples of commuting contractions on the part of each author. A good survey article on commuting tuples is [4].

Definition 1.6. Given an ideal $I \trianglelefteq \mathbb{C}[z]$, since I is closed under multiplication, I is an invariant subspace for $M_{z_1}, ..., M_{z_d}$. We therefore obtain a decomposition

$$H^2_d = I^\perp \oplus \bar{I}$$

with respect to which

$$M_{z_i} = \begin{pmatrix} A_i & 0 \\ * & * \end{pmatrix}$$

for each $1 \leq i \leq d$. Let $\overline{p(z)}$ stand for the image of $p(z) \in \mathbb{C}[z]$ in $\mathbb{C}[z]/I$. We view $I^\perp$ as the completion $\overline{\mathbb{C}[z]/I}$, and we therefore view the d-tuple $(A_1, ..., A_d)$ as the extension of the

d-tuple $(L_1, ..., L_d)$ of operators

$$L_i\overline{p(z)} := \overline{z_i p(z)}$$

on $\mathbb{C}[z]/I$ to the completion $\overline{\mathbb{C}[z]/I}$. Note that the operators A_i, A_j for $1 \leq i, j \leq d$ commute.

We say that the d-tuple $(A_1, ..., A_d)$ is *induced* by the ideal $I \trianglelefteq \mathbb{C}[z]$.

A remarkable theorem shows that the construction of $(A_1, ..., A_d)$ is far more general than it seems at first glance.

Theorem 1.7. *(Arveson, Müller–Vasilescu.) Suppose $(T_1, ..., T_d)$ is a contractive d-tuple of commuting operators. Then there exists an ideal $I \trianglelefteq \mathbb{C}[z]$ whose induced d-tuple of operators $(A_1, ..., A_d)$ coincides with $(T_1, ..., T_d)$.*

Proof. The result is proved in the paper [2] by Arveson and the paper [14] of Müller and Vasilescu. □

Because of this theorem, it suffices to study tuples $(A_1, ..., A_d)$ induced by some ideal $I \triangleleft \mathbb{C}[z]$.

To state the Arveson–Douglas conjecture, we must first define a particularly interesting class of ideals. However, the definition of these ideals involves the *Schatten p-class.* We recall what that is.

Definition 1.8. Suppose H_1, H_2 are separable Hilbert spaces and $T : H_1 \to H_2$ is a bounded operator. Let $p \in [1, \infty)$, and define the p^{th} *Schatten norm* to be

$$||T||_p := (\text{tr}(|T|^p))^{1/p},$$

where $|T| = \sqrt{T^*T}$ is defined by functional calculus as usual. If the operator T has finite p^{th} Schatten norm, we say it lies in the p^{th} *Schatten class*, $\mathcal{S}^p$.

The space $\mathcal{S}^p$ forms a Banach space with respect to the p^{th} Schatten norm. We think of the space "$\mathcal{S}^\infty$" as coinciding with the compact operators, which have finite p^{th} Schatten norm for every $1 \leq p < \infty$. Moreover, $\mathcal{S}^1$ is the trace class operators, and $\mathcal{S}^2$ is the space of *Hilbert–Schmidt operators*, i.e., those with finite *Hilbert–Schmidt norm.* Indeed, this norm is precisely the 2^{nd} Schatten norm.

The following definition is motivated by work of Halmos, which we will discuss shortly.

Definition 1.9. We say an operator T is *essentially normal* if the commutator

$$[T^*, T] = T^*T - TT^*$$

is compact. For $p \in [1, \infty)$, we say T is *p–essentially normal* if $[T^*, T] \in \mathcal{S}^p$.

Definition 1.10. Let $I \trianglelefteq \mathbb{C}[z]$ be an ideal, and let $(A_1, ..., A_d)$ be the d-tuple of operators induced by I. We say I is *essentially normal* (resp. *p–essentially normal*) if $[A_i^*, A_j]$ is compact (resp. in $\mathcal{S}^p$) for all $1 \leq i, j \leq d$.

To motivate these notions, we discuss a theorem of Weyl and a problem posed by Halmos in response to it.

Theorem 1.11. *(Weyl.) Every self-adjoint bounded operator on a separable Hilbert space is the sum of a diagonal operator, with respect to some orthogonal basis, and a compact operator.*

Proof. See [17]. □

In *Ten problems in Hilbert space* ([11]), Halmos asked whether every normal operator on an *arbitrary* Hilbert space is the sum of a diagonal operator and a compact operator. The question was quickly settled in the affirmative by Berg ([5]) and Sikonia ([16]). Halmos wondered whether anything could be said about operators that are the sum of a normal operator and a compact operator. One observation is that if T is the sum of a normal operator and a compact operator, then $T^*T - TT^*$ is compact. This is what motivated the definition of essentially normal operators. Once those had been defined, the notion of p–essential normality was a natural one to investigate because the Schatten classes lie inside the class of compact operators.

Example 1.12. Not every essentially normal operator is normal. There are many examples, but one is the unilateral shift S, which acts by $Se_n = e_{n+1}$. Thus,

$$S^*S - SS^* = I - SS^* = e_0e_0^*,$$

which is a rank 1 projection, hence compact. However, its *Fredholm index* is

$$\operatorname{ind}(S) := \dim\ker(S) - \dim\ker(S^*) = -1,$$

and for any normal operator N we have $\ker(N) = \ker(N^*)$, hence $\operatorname{ind}(N) = 0$. Moreover, the Fredholm index is invariant under compact perturbations, so the sum of a normal and compact operator also has Fredholm index -1. Therefore, S is not even the sum of a normal operator and a compact operator.

An excellent article on Halmos's question and related problems is [6].

Recall that given an ideal I, we have an associated algebraic set

$$V(I) = \{x \in \mathbb{C}^d | f(x) = 0 \text{ for all } f \in I\}.$$

We are now ready to state the conjecture that is the main subject of this article.

2. The State of the Conjecture.

Conjecture 2.1. *(Arveson, Douglas.) Suppose $I \trianglelefteq \mathbb{C}[z]$ is a homogeneous ideal, i.e., an ideal generated by homogeneous polynomials. Then I is p–essentially normal for every $p > \dim(V(I))$.*

It is quite remarkable that the presence of an operator-theoretic property for a homogeneous ideal should depend solely on the dimension of its associated algebraic set, a purely geometric object. In fact, the conjecture is thought to hold even for non-homogeneous ideals, but the homogeneous case is difficult enough to currently be the main task at hand.

There are many partial results on the Arveson–Douglas conjecture.

Theorem 2.2. *(Arveson, 1998.) The Arveson–Douglas conjecture holds for $I = (0)$. In this case, $V(I) = \mathbb{C}^d$, so given a homogeneous ideal $I \trianglelefteq \mathbb{C}[z]$, it is p–essentially normal for every $p > d$.*

Proof. See [2]. □

A few years later, Arveson extended his result.

Theorem 2.3. *The conjecture is true for ideals generated by monomials z^α for $\alpha \in \mathbb{N}^d$.*

Proof. The theorem is announced [3], but the result seems to be unpublished. □

A few years later, there was a new breakthrough

Theorem 2.4. *(Guo–Wang, 2008.) The conjecture is true for $d \leq 3$, and for ideals generated by one homogeneous polynomial.*

Proof. See [10]. □

The method of proof is essentially a commutative algebra argument and the study of a certain exact sequence that relates the Arveson–Douglas conjecture to a statement about K-homology. Indeed, in [10], the conjecture is shown to imply exactness of the sequence

$$0 \to \mathcal{K} \to C^*(A_1, ..., A_d) + \mathcal{K} \to C(V(I) \cap \partial\mathbb{B}_d) \to 0.$$

Here $\mathcal{K}$ is the class of compact operators on the Hilbert space associated to the ideal $I \trianglelefteq \mathbb{C}[z]$. This exact sequence yields an odd K-homology element for $V(I) \cap \partial\mathbb{B}_d$. This is one of a series of analogies between the Arveson–Douglas conjecture and the Baum–Connes conjecture, which proposes a link between the K-theory of reduced group C^*-algebras and the K-homology of the classifying space of proper actions of the group.

In 2012, there was another breakthrough on the Arveson–Douglas conjecture with a series of results by Kennedy and Kennedy–Shalit.

Theorem 2.5. *(Kennedy, 2012). The Arveson–Douglas conjecture is true for ideals generated by homogeneous polynomials in mutually disjoint variables.*

Proof. See [12]. □

We recall some terminology from algebraic geometry.

Definition 2.6. Recall that a subset of $\mathbb{C}^d$ is an *affine variety* if it is of the form $V(I)$ for some ideal I and is *irreducible*, i.e., cannot be written as a union of two proper subsets of the form $V(I_1)$ and $V(I_2)$ for ideals I_1, I_2. Each affine variety (and indeed each algebraic set) V has an associated ideal

$$I(V) = \{f \in \mathbb{C}[z] | f(z) = 0 \text{ for every } z \in V\}.$$

The *coordinate ring* of an algebraic set V is the quotient of the polynomial ring by $I(V)$.

Theorem 2.7. *(Kennedy–Shalit, 2012.) Let V and W be homogeneous varieties (i.e., varieties of the form $V(I)$ for homogeneous ideals I) in $\mathbb{C}^d$ with isomorphic coordinate rings. Then the Arveson–Douglas conjecture holds for V if and only if it holds for W.*

Proof. See [13]. □

Two recent breakthroughs by Engliš–Eschmeier (2013) ([9]) and by Douglas–Tang–Yu (2014) ([8]) rely on an index theorem due to Boutet de Monvel to make further progress on the conjecture. It is not surprising that index theory should make an appearance in work on the Arveson–Douglas conjecture because of the conjecture's connection to Baum–Connes theory. Indeed, the Atiyah–Singer index theorem can be viewed as a special case of the Baum–Connes conjecture.

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