THE GELFAND–RAIKOV THEOREM

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1. Introduction

In this article, we prove the *Gelfand–Raikov theorem*. Before we are able to state the theorem, we must give one definition.

Definition 1.1. We say that a subset S of the representations of a group G forms a *complete system* if for every non-identity element of $g \in G$, there exists a representation of G in S that does not send g to the identity map.

We are now ready to state the Gelfand–Raikov theorem.

Theorem 1.2. [1, Theorem 7] *For every locally compact group, there exists a complete system of irreducible unitary representations.*

This theorem was originally proved in [1], and we follow the proof given there closely, only modernizing some of the terminology. It is more common nowadays to phrase the theorem a slightly different way, which we prove is equivalent to Theorem 1.2.

Proposition 1.3. [2, Theorem 3.34] *The following statement is equivalent to Theorem 1.2: "The irreducible unitary representations of a locally compact group G separate the points of G. That is, if $x, y \in G$ are distinct, there exists an irreducible unitary representation φ such that $\varphi(x) \neq \varphi(y)$."*

Proof. Since x and y are distinct, $xy^{-1} \neq e$. By Theorem 1.2, there exists a complete system of irreducible unitary representations for G. In particular, there exists an irreducible unitary representation φ such that

$$\varphi(x)\varphi(y)^{-1} = \varphi(xy^{-1}) \neq \varphi(e) = I,$$

where I is the identity operator, which implies $\varphi(x) \neq \varphi(y)$. Therefore, Theorem 1.2 implies the statement.

Conversely, suppose the statement holds. Then if $x, y \in G$ are distinct, we can find an irreducible unitary representation φ such that $\varphi(x) \neq \varphi(y)$. But then

$$\varphi(xy^{-1}) = \varphi(x)\varphi(y)^{-1} \neq \varphi(e) = I,$$

so there exists a complete system of irreducible unitary representations of G. □

2. The Proof

Definition 2.1. Let G be a group. We say a function $\varphi : G \to \mathbb{C}$ is *positive-definite* if for all $g_1, ..., g_n \in G$ and $\lambda_1, ..., \lambda_n \in \mathbb{C}$, we have

$$\sum_{k=1}^{n} \sum_{\ell=1}^{n} \varphi(g_\ell^{-1} g_k)\lambda_k \overline{\lambda_\ell} \geq 0.$$

We now claim the following holds.

Proposition 2.2. [1, p. 2] *Given a topological group G, for each continuous positive-definite function $\varphi : G \to \mathbb{C}$, there corresponds a unitary representation of the group on some Hilbert space.*

Proof. We construct the required Hilbert space as follows. First, let S be the space of functions $G \to \mathbb{C}$ that have finite support. Given two functions $\lambda, \mu \in S$, we define an inner product

$$(\lambda, \mu) := \sum_h \sum_{h'} \varphi(h'^{-1}h)\lambda(h)\overline{\mu(h')}, \quad (1)$$

which converges because λ and μ have finite support. We then define an equivalence relation on S by the prescription that $\lambda \sim \mu$ if $(\lambda - \mu, \lambda - \mu) = 0$. Let

$$K := S/\sim .$$

We then set the usual norm

$$|\lambda| := \sqrt{\langle \lambda, \lambda \rangle},$$

and define

$$\mathcal{L}_2(\varphi) := \overline{K}^{|\cdot|}$$

to be the completion of K under the norm $|\cdot|$. This is then a complete inner product space, so it is a Hilbert space.

Given any $g \in G$, we have a corresponding translation operator $T_g : S \to S$ defined by

$$T_g(\lambda(h)) := \lambda(g^{-1}h).$$

We claim that for every $g \in G$, the translation operator T_g is unitary. We calculate

$$(T_g\lambda, T_g\mu) = \sum_h \sum_{h'} \varphi(h'^{-1}h)\lambda(g^{-1}h)\overline{\mu(g^{-1}h')},$$

and making the substitution $h \mapsto gh$, $h' \mapsto gh'$, we get

$$(T_g\lambda, T_g\mu) = \sum_h \sum_{h'} \varphi(h'^{-1}h)\lambda(h)\overline{\mu(h')} = (\lambda, \mu).$$

It follows that T_g is a unitary operator for every $g \in G$. Since functions $\lambda : G \to \mathbb{C}$ with finite support are dense in $\mathcal{L}_2(\varphi)$, T_g extends uniquely to an operator on $\mathcal{L}_2(\varphi)$. It remains to prove that the operator T_g, considered as acting on $\mathcal{L}_2(\varphi)$, is continuous in g.

Indeed, let $g, g' \in G$ and $\eta \in \mathcal{L}_2(\varphi)$. We have

$$|T_{g'}\eta - T_g\eta|^2 = |T_{g^{-1}g'}\eta - \eta|^2 = 2[(\eta, \eta) - \mathrm{Re}(T_{g^{-1}g'}\eta, \eta)].$$

We wish to prove that for every $\eta \in \mathcal{L}_2(\varphi)$, if $g' \to g$, then $|T_{g'}\eta - T_g\eta| \to 0$. From the equation above, it suffices to show that if $g \to e$, where e is the identity of G, then $(T_g\eta, \eta) \to (\eta, \eta)$. But by (1) and the continuity of φ, we have

$$(T_g\eta, \eta) = \sum_h \sum_{h'} \varphi(h'^{-1}gh)\eta(h)\overline{\eta(h')}$$

$$\to \sum_h \sum_{h'} \varphi(h'^{-1}h)\eta(h)\overline{\eta(h')} \text{ as } g \to e.$$

For any $\eta \in \mathcal{L}_2(\varphi)$ and $\epsilon > 0$, we can find some $\lambda \in \mathcal{L}_2(\varphi)$ such that $|\eta - \lambda| < \frac{\epsilon}{3}$. Choose some neighbourhood V of e such that $|T_g\lambda - \lambda| < \frac{\epsilon}{3}$ for every $g \in V$. Then, if $g \in V$, we have

$$|T_g\eta - \eta| \leq |T_g\eta - T_g\lambda| + |T_g\lambda - \lambda| + |\lambda - \eta| = 2|\lambda - \eta| + |T_g\lambda - \lambda| < \epsilon.$$

We also have $T_{gh} = T_gT_h$, so the operators T_g form a unitary representation of G on $\mathcal{L}_2(\varphi)$. This completes the proof. □

Definition 2.3. We define the *Kronecker delta function* of the identity, $\xi_0(h) \in \mathcal{L}_2(\varphi)$, by

$$\xi_0(h) := \begin{cases} 1 & \text{if } h = e, \\ 0 & \text{if } h \neq e. \end{cases}$$

The vectors $T_h\xi_0$ then generate $\mathcal{L}_2(\varphi)$. Indeed, if $\lambda \in \mathcal{L}_2(\varphi)$, then we can write

$$\lambda = \sum_h \lambda(h)T_h\xi_0.$$

It follows from this and Equation (1) that for every $g \in G$,

$$\varphi(g) = (T_g\xi_0, \xi_0). \quad (2)$$

We now provide a converse to Proposition 2.2.

Proposition 2.4. [1, p. 3] *For every unitary representation of a topological group G, there exists a collection of continuous positive-definite functions on G.*

Proof. Suppose U_g are unitary operators forming a unitary representation of G on a Hilbert space $\mathcal{H}$. Then for every $\xi \in \mathcal{H}$, the function

$$\varphi(g) := (U_g\xi, \xi)$$

will be continuous and positive-definite. □

Moreover, if $\xi \neq 0$, then $\varphi(g) \neq 0$. Suppose $\mathcal{H}$ has a vector ξ_0 such that $\{U_g\xi_0\}_{g\in G}$ generates $\mathcal{H}$. Then, letting

$$\varphi_0(g) := (U_g\xi_0, \xi_0),$$

the space $\mathcal{H}$ is isomorphic as a Hilbert space to the space $\mathcal{L}_2(\varphi_0)$ defined earlier. Indeed, an explicit isomorphism is given by

$$\mathcal{H} \ni \sum_h \lambda(h)U_h\xi_0 \mapsto \lambda(h) \in \mathcal{L}_2(\varphi_0).$$

Definition 2.5. Suppose that $\varphi, \psi : G \to \mathbb{C}$ are positive-definite functions. If $\varphi - \psi$ is positive-definite, we will write $\psi \ll \varphi$ or $\varphi \gg \psi$.

We will need the following result, which we do not prove but for which we provide a reference.

Proposition 2.6. [4] *If $\varphi : G \to \mathbb{C}$ is positive-definite, then for all $g, h \in G$, it satisfies*

$$|\varphi(g) - \varphi(h)|^2 \leq 2\varphi(e)[\varphi(e) - Re(\varphi(h^{-1}g))]. \quad (3)$$

Lemma 2.7. [1, p. 3–4] *Suppose that $\varphi, \psi : G \to \mathbb{C}$ are positive-definite, that φ is continuous, and that $\psi \ll \varphi$. Then ψ is also continuous.*

Proof. By Proposition 2.6, we have

$$|\varphi(g) - \varphi(h)|^2 \leq 2\varphi(e)[\varphi(e) - \mathrm{Re}(\varphi(h^{-1}g))].$$

But since $\psi \ll \varphi$, $\varphi(e) - \psi(e) \geq 0$, and $\varphi(e) - \psi(e) - \mathrm{Re}[\varphi(h^{-1}g) - \psi(h^{-1}g)] \geq 0$, so we have

$$|\psi(g) - \varphi(h)|^2 \leq 2\psi(e)[\psi(e) - \mathrm{Re}(\psi(h^{-1}g))] \leq 2\varphi(e)[\varphi(e) - \mathrm{Re}(\varphi(h^{-1}g))].$$

Therefore, the continuity of φ at e implies the continuity of ψ at every $g \in G$. $\square$

Definition 2.8. We say that a continuous positive-definite function φ is *elementary* if $\psi \ll \varphi$ implies that $\psi = \alpha\varphi$ for some $\alpha \in \mathbb{C}$. (In particular, constant functions are elementary.)

The usefulness of the above definition is that the elementary functions can be used to obtain irreducible representations of a group G, as we will now prove.

Theorem 2.9. [1, Theorem 1] *Suppose $\varphi : G \to \mathbb{C}$ is elementary. (In particular, this means it is continuous and positive-definite.) Then the unitary representation of G on the Hilbert space $\mathcal{L}_2(\varphi)$, as defined in Proposition 2.2, is irreducible.*

Proof. Let $P \in \mathcal{L}_2(\varphi)$ be a projection operator, and suppose that it commutes with the translation operator T_g for each $g \in G$. Set

$$\psi(g) := (T_g P\xi_0, \xi_0),$$

where ξ_0 is the Kronecker delta from Definition 2.3. Then ψ is a positive-definite function such that $\psi \ll \varphi$. Indeed, if $\lambda : G \to \mathbb{C}$ is a function that is non-zero precisely at $g_1, ..., g_n \in G$, then

$$\begin{aligned}\sum_{k=1}^{n}\sum_{\ell=1}^{n} \psi(g_\ell^{-1}g_k)\lambda(g_k)\overline{\lambda(g_\ell)} &= \sum_{k=1}^{n}\sum_{\ell=1}^{n}(\lambda(g_k)T_{g_k}P\xi_0, \lambda(g_\ell)T_{g_\ell}\xi_0) \\ &= (P\sum_{k=1}^{n}\lambda(g_k)T_{g_k}\xi_0, \sum_{\ell=1}^{n}\lambda(g_\ell)T_{g_\ell}\xi_0) = (P\lambda, \lambda)\end{aligned}$$

and

$$\sum_{k=1}^{n}\sum_{\ell=1}^{n} \varphi(g_\ell^{-1}g_k)\lambda(g_k)\overline{\lambda(g_\ell)} = (\lambda, \lambda).$$

However, we also have $0 \leq (P\lambda, \lambda) \leq (\lambda, \lambda)$. This implies that ψ and $\varphi - \psi$ are positive-definite. But φ is elementary, so $\psi = \alpha\varphi$ for some $\alpha \in \mathbb{C}$. It follows that $(P\lambda, \lambda) = \alpha(\lambda, \lambda)$ for every λ with finite support. But such λ are dense in $\mathcal{L}_2(\varphi)$, so $P = \alpha I$, where I is the identity operator. Since P is a projection, $P^2 = P$, so $\alpha = 0$ or 1, and we see that the only projection operators in $\mathcal{L}_2(\varphi)$ commuting with every T_g are $P = 0$ and $P = I$. But if there were a subspace of $\mathcal{L}_2(\varphi)$ closed under the action of the T_g, then the T_g would commute with the projection onto that subspace. It follows that our representation must be irreducible. $\square$

We now prove the converse of the previous theorem.

Theorem 2.10. [1, Theorem 2] *Positive-definite functions that give rise to irreducible unitary representations of a group G in the manner of Proposition 2.2 are elementary.*

Proof. Let U_g be the unitary operators on a Hilbert space $\mathcal{H}$ giving rise to an irreducible representation of G. Then if $\xi \in \mathcal{H}$ is non-zero, we have that $\{U_g\xi\}_{g\in G}$ generates $\mathcal{H}$. Suppose that $\varphi(g) := (U_g\xi, \xi)$, that ψ is positive-definite, and that $\psi \ll \varphi$. Suppose $\lambda, \mu : G \to \mathbb{C}$ are positive-definite functions with finite support, so that we can write $\lambda = \sum_h \lambda(h)U_h\xi$ and $\mu = \sum_h \mu(h)U_h\xi$. We then define the operator B by

$$(B\lambda, \mu) := \sum_h \sum_{h'} \psi(h'^{-1}h)\lambda(h)\overline{\mu(h')}.$$

It follows that B is self-adjoint and that

$$0 \leq (B\lambda, \lambda) \leq (\lambda, \lambda) = \sum_h \sum_{h'} \varphi(h'^{-1}h)\lambda(h)\overline{\lambda(h')}.$$

But the vectors λ are dense in $\mathcal{H}$, so B uniquely extends to a self-adjoint operator on $\mathcal{H}$. Furthermore, for every $g, h \in G$, we have

$$(U_g B U_h \xi, U_h \xi) = (BU_H \xi, U_{g^{-1}h}\xi) = \psi(h^{-1}gh) = (BU_g U_h \xi, U_h \xi),$$

and since $\{U_h \xi\}$ is a basis for $\mathcal{H}$, we have that for every $\eta \in \mathcal{H}$,

$$(U_g B\eta, \eta) = (BU_g \eta, \eta),$$

which implies that B commutes with every U_g. But the U_g correspond to an irreducible representation, so $B = \alpha I$ for some $\alpha \in \mathbb{C}$. Indeed, if the T_g commute with B, then they commute with its spectral projections (from the statement of the spectral theorem), so so these must be 0 or I. But this means that the spectrum of B must be concentrated at a point α, so $B = \alpha I$. It follows that

$$(BU_g \xi, \xi) = \alpha(U_g \xi, \xi),$$

which implies that $\psi = \alpha\varphi$. □

Given a topological group G, a sufficient condition that it have a complete system of irreducible unitary representations is that for each $g_0 \neq e$ in G, there exists an irreducible representation $\{U_g\}$ on some Hilbert space $\mathcal{H}_0$ such that $U_{g_0} \neq I$. This situation inspires the following definition.

Definition 2.11. We say there is a *complete system of elementary continuous positive-definite functions* on G if for every $g_0 \neq e$ in G, there exists an elementary continuous positive-definite function φ_0 such that $\varphi_0(g_0) \neq \varphi_0(e)$.

We now have the following result.

Theorem 2.12. [1, Theorem 3] *A topological group G admits a complete system of irreducible unitary representations if and only if it admits a complete system of elementary continuous positive-definite functions.*

Proof. If $g_0 \neq e$ in G and φ_0 is an elementary continuous positive-definite function such that $\varphi_0(g_0) \neq \varphi_0(e)$, then the representation $\{T_g\}$ that arises from φ_0 by Proposition 2.2 satisfies $T_{g_0}\xi_0 \neq \xi_0$, where ξ_0 is the Kronecker delta from Definition 2.3 by Equation (2) from that same Definition. Therefore, $T_{g_0} \neq I$.

On the other hand, if $\{U_g\}$ is an irreducible unitary representation such that $U_{g_0} \neq I$, then there exists a vector ξ_0 such that $(U_{g_0}\xi_0, \xi_0) \neq (\xi_0, \xi_0)$. Therefore, setting $\varphi_0(g) := (U_g \xi_0, \xi_0)$, we have $\varphi_0(g_0) \neq \varphi_0(e)$. □

In what follows, we will suppose that our group G is locally compact, as we can then equip it with a left-invariant Haar measure $m(E)$. For any $h \in G$, $m(Eh)$ is also a left-invariant Haar measure, so by uniqueness of Haar measures, $m(Eh) = \ell_h m(E)$ for some constant ℓ_h independent of E. It follows that $\ell_{gh} = \ell_g \ell_h$ and that ℓ_h is continuous in h.

Definition 2.13. We write $\mathcal{L}_1$ for the space of measurable absolutely integrable functions $x : G \to \mathbb{C}$ under the norm

$$||x|| := \int |x(h)|\, dh.$$

Then, for every $g \in G$, we have a left translation operator $T_g \in \mathcal{L}_1$ given by

$$T_g x(h) := x(g^{-1}h)$$

and a right translation operator $T_{g'} \in \mathcal{L}_1$ given by

$$T_{g'} x(h) = x(hg^{-1}).$$

Both of these operators are unitary. By the properties of Haar measures, for any $x \in \mathcal{L}_1$, as $g \to g_0$, we have

$$\int |x(g^{-1}h) - x(g_0^{-1}h)|\, dh \to 0 \text{ and } \int |x(hg^{-1}) - x(hg_0^{-1})|\, dh \to 0. \quad (4)$$

It follows that T_g and $T_{g'}$ are continuous in g and g', respectively.

Definition 2.14. Recall that if $x, y \in \mathcal{L}_1$, then the *convolution*

$$x * y := \int x(h^{-1}g)y(h)\, dh$$

exists for almost every $g \in G$ and is in $\mathcal{L}_1$, since $||x * y|| \leq ||x||||y||$. We write

$$x * (g) := \ell_g^{-1}\overline{x(g^{-1})}.$$

Then we have

$$\int x(g)\, dg = \int x(g^{-1})\ell_g^{-1}\, dg,$$

so whenever $x \in \mathcal{L}_1$, then $x^* \in \mathcal{L}_1$ as well, since $||x|| = ||x^*||$. We also see that $(x^*)^* = x$ and $(x * y)^* = y^* * x^*$.

Definition 2.15. A linear functional $L : \mathcal{L}_1 \to \mathbb{C}$ is *positive* if $L(x * x^*) \geq 0$ for every $x \in \mathcal{L}_1$. (Note that this implies $L(x * x^*)$ is always real.) In particular, if $\varphi : G \to \mathbb{C}$ is *essentially bounded*, i.e., equal to a bounded function except on a set of measure zero, and if φ is *integrally positive-definite*, i.e., if for every $x \in \mathcal{L}_1$, we have

$$\iint \varphi(h^{-1}g)x(g)\overline{x(h)}\, dg\, dh \geq 0,$$

then the functional

$$L_\varphi(x) := \int \varphi(g)x(g)\, dg$$

is positive.

We can now state the following theorem.

Theorem 2.16. [1, Theorem 4] *For every positive linear functional $L : \mathcal{L}_1 \to \mathbb{C}$, there exists some $\varphi : G \to \mathbb{C}$ such that*

$$L(x) = L_\varphi(x) = \int \varphi(g)x(g)\, dg \text{ for every } x \in \mathcal{L}_1.$$

Proof. First, we define a bilinear functional

$$(x, y) := L(x * y^*).$$

Because

$$L((x+\lambda y)*(x+\lambda y)^*) = L(x*x^*) + \lambda L(y*x^*) + \overline{\lambda}L(x*y^*) + \overline{\lambda}\lambda L(y*y^*),$$

we have

$$(x+\lambda y, x+\lambda y) = (x,x) + \lambda(y,x) + \overline{\lambda}(x,y) + \overline{\lambda}\lambda(y,y),$$

but $(x+\lambda y, x+\lambda y), (x,x), \overline{\lambda}\lambda(y,y) \in \mathbb{R}$ as L is positive, so $\lambda(y,x) + \overline{\lambda}(x,y) \in \mathbb{R}$. Setting $\lambda =: a+bi$, $(y,x) =: c+di$, and $(x,y) =: e+fi$, and taking imaginary parts, we get

$$ad + bc + af - be = 0.$$

But λ was arbitrary, so a and b are arbitrary. Taking $a=1$, $b=0$ gives $f=-d$, which then gives $c=e$, so $(y,x) = \overline{(x,y)}$.

Next, let $\mathcal{L}_2(L)$ denote the Hilbert space obtained from $\mathcal{L}_1$ by identifying x and y whenever $(x-y, x-y) = 0$ and then taking the completion with respect to the norm

$$|x| := \sqrt{(x,x)}.$$

If $x \in \mathcal{L}_1$, then we have

$$|x|^2 = L(x*x^*) \leq |L|||x||||x^*|| = |L|||x||^2,$$

so

$$|x| \leq \sqrt{|L|}||x||.$$

We claim that the space $\mathcal{L}_2(L)$ will be isomorphic as a Hilbert space to one of the previously-defined spaces $\mathcal{L}_2(\varphi)$ for a suitable continuous positive-definite function $\varphi : G \to \mathbb{C}$. We now show how to construct this φ. The closure of $\mathcal{L}_1$ in $\mathcal{L}_2(L)$ contains a distribution ξ_0, the Dirac delta function centred at the identity e, and the vectors $T_g\xi_0$ then generate $\mathcal{L}_2(L)$; the continuous positive-definite function φ such that $\mathcal{L}_2(L) \simeq \mathcal{L}_2(\varphi)$ is given by $\varphi(g) := (T_g\xi_0, \xi_0)$.

Let $\{V\}$ be the collection of neighbourhoods of the identity in G, ordered by inclusion. A *unit*, $e_V(g)$, is a function in $\mathcal{L}_1$ satisfying the conditions

$$e_V(g) \geq 0, e_V(g) = 0 \text{ outside } V, e_V(g^{-1}) = e_V(g), ||e_V|| = 1$$

for every $g \in G$. Then, for any $x \in \mathcal{L}_1$, we have

$$||x*e_V^* - x|| = ||x*e_V - x|| = \int |\int [T_hx(g) - x(g)]e_V(h)\,dh|\,dg$$

$$\leq \int (\int |T_hx(g) - x(g)|\,dg)e_V(h)\,dh \leq \sup_{h\in V} ||T_hx - x||.$$

Because of the first relation in Equation (4) from the paragraph before Definition 2.14, this implies that

$$||x*e_V^* - x|| \to 0 \text{ as } V \to e.$$

Therefore,

$$(x, e_V) = L(x*e_V^*) \to L(x) \text{ as } V \to e \text{ for all } x \in \mathcal{L}_1.$$

But e_V is uniformly bounded in $\mathcal{L}_2(L)$ ($|e_V| \leq \sqrt{|L|}$) and the $x \in \mathcal{L}_1$ are dense in $\mathcal{L}_2(L)$, so it follows from the previous relation that $\lim_{V\to e}(\eta, e_V)$ exists for every $\eta \in \mathcal{L}_2(L)$. Indeed, for every $\epsilon > 0$, we can choose $x \in \mathcal{L}_1$ satisfying

$$|\eta - x| < \frac{\epsilon}{3\sqrt{|L|}}.$$

We can then choose a neighbourhood V of e such that

$$|(x, e_{V_1}) - (x, e_{V_2})| < \frac{\epsilon}{3}$$

for every $V_1, V_2 \subseteq V$. But then we will have

$$\begin{aligned} |(\eta, e_{V_1}) - (\eta, e_{V_2})| &\leq |(\eta, e_{V_1}) - (x, e_{V_1})| + |(x, e_{V_1}) - (x, e_{V_2})| + |(x, e_{V_2}) - (\eta, e_{V_2})| \\ &\leq |\eta - x||e_{V_1}| + \frac{\epsilon}{3} + |\eta - x||e_{V_2}| < \epsilon. \end{aligned}$$

It follows that as $V \to e$, e_V converges weakly to some $\xi_0 \in \mathcal{L}_2(L)$ with $(x, \xi_0) = L(x)$ for every x. In particular, taking $x = e_V$ and letting $V \to e$, we get $(\xi_0, \xi_0) = \lim_{V \to e} L(e_V)$. But we have $e_V * e_V^* = e_V * e_V = e_{V^{-1}V}$, so $(e_V, e_V) = L(e_{V^{-1}V}) \to (\xi_0, \xi_0)$. We thus see that

$$|e_V - \xi_0|^2 = (e_V, e_V) - (\xi_0, e_V) - (e_V, \xi_0) + (\xi_0, \xi_0) \to 0,$$

and therefore e_V converges strongly to ξ_0 as $V \to e$.

We have that $T_g x * (T_g y)^* = x * y^*$, so $(T_g x, T_g x) = (x, x)$ for every $x \in \mathcal{L}_1$, and by denseness of $\mathcal{L}_1$, T_g uniquely extends to a unitary operator on $\mathcal{L}_2(L)$ for every $g \in G$. Moreover, for every $x in \mathcal{L}_1$,

$$|T_{g'} x - T_g x| \leq \sqrt{|L|}||T_{g'} x - T_g x|| \to 0 \text{ as } g' \to g,$$

so T_g is continuous in g on $\mathcal{L}_2(L)$. Indeed, if $\eta \in \mathcal{L}_2(L)$, then we can pick $x \in \mathcal{L}_1$ with $|x - \eta| < \frac{\epsilon}{3}$ and also pick a neighbourhood V of g with $|T_{g'} x - T_g x| < \frac{\epsilon}{3}$ for every $g' \in V$. We then have

$$\begin{aligned} |T_{g'} \eta - T_g \eta| &\leq |T_{g'} \eta - T_{g'} x| + |T_{g'} x - T_g x| + |T_g x - T_g \eta| \\ &= 2|\eta - x| + |T_{g'} x - T_g x| < \epsilon. \end{aligned}$$

Finally, $T_{gh} = T_g T_h$, so $\{T_g\}$ is a unitary representation of G. Hence $\varphi(g) = (T_g \xi_0, \xi_0)$ is continuous and positive-definite, by Proposition 2.4.

It remains to show that the functional L arises from $\varphi(g)$, i.e., that for every $x \in \mathcal{L}_1$, we have

$$L(x) = \int (T_g \xi_0, \xi_0) x(g)\, dg.$$

Indeed, e_V converges strongly to ξ_0, which implies that $L(T_g e_V) = (T_g e_V, \xi_0)$ converges uniformly to $(T_g \xi_0, \xi_0)$. Therefore,

$$\int (T_g \xi_0, \xi_0) x(g)\, dg = \lim_{V \to e} \int L(T_g e_V) x(g)\, dg, \quad (5)$$

but we also have

$$\int L(T_g e_V) x(g)\, dg = L(\int T_g e_V x(g)\, dg). \quad (6)$$

Indeed, suppose $E \subseteq G$ is measurable, and set $m_L(E) := L(f_e)$, where $f_E : G \to \{0, 1\}$ is the characteristic function of E. Then $m_L(E)$ is a countably-additive complex-valued measure and for every $y \in \mathcal{L}_1$, we have $L(y) = \int y(h)\, dm_L(h)$. By Fubini's theorem, we obtain

$$\begin{aligned} \int L(T_g e_V) x(g)\, dg &= \int (\int e_V(g^{-1}h)\, dm_L(h)) x(g)\, dg \\ &= \int (\int e_V(g^{-1}h) x(g)\, dg)\, dm_L(h) = L(\int T_g e_V x(g)\, dg), \end{aligned}$$

which is Equation (6). Using Equations (5) and (6), we calculate that

$$\int (T_g\xi_0, \xi_0)x(g)\,dg = \lim_{V\to e} L(\int T_g e_V x(g)\,dg) = \lim_{V\to e} L(\int e_V(g^{-1}h)x(g)\,dg)$$
$$= \lim_{V\to e} L(\int e_V(g^{-1})x(hg)\,dg) = \lim_{V\to e} L(\int e_V(g)x(hg)\,dg).$$

But by the second relation in Equation (4) from the paragraph before Definition 2.14, we have

$$||\int e_V(g)x(hg)\,dg - x(h)|| = \int |\int [x(hg) - x(h)]e_V(g)\,dg|\,dh$$
$$\leq \int (\int |x(hg) - x(h)|\,dh)e_V(g)\,dg \leq \sup_{g\in V} \int |x(hg) - x(h)|\,dh \to 0 \text{ as } V \to e.$$

This implies that

$$\lim_{V\to e} L(\int e_V(g)x(hg)\,dg) = L(x),$$

and the result follows. □

From Theorem 2.16 and the remarks preceding it, we obtain the following corollary.

Corollary 2.17. [1, p. 10] *Every essentially bounded, integrally positive-definite function differs from some continuous positive-definite function on a set of measure zero.*

Proof. Suppose ψ is essentially bounded and integrally positive-definite. Let φ be the continuous positive-definite function giving rise to the positive functional $L_\psi(x) = \int \psi(g)x(g)\,dg$ in Theorem 2.16. Then, for every $x \in \mathcal{L}_1$, we have

$$\int \psi(g)x(g)\,dg = \int \varphi(g)x(g)\,dg.$$

This implies that ψ and φ differ on a set of measure zero. □

Definition 2.18. We say that a positive-definite function φ is *normalized* if $\varphi(e) = 1$.

Observe that the collection $\mathcal{B}$ of continuous positive-definite functions φ with $\varphi(e) \leq 1$ is convex, i.e., if it contains φ and ψ, then for every $\lambda \in [0, 1]$, it contains $\lambda\varphi + (1 - \lambda)\psi$.

We define an *extreme point* of a convex set to be a point which is not an interior point of any line segment contained in the set. In particular, the function $\varphi \equiv 0$ is an extreme point of $\mathcal{B}$

We claim the following.

Lemma 2.19. [1, p. 10–11] *Any extreme point of $\mathcal{B}$ other than the zero function is a normalized elementary positive-definite function.*

Proof. Suppose φ is an extreme point of $\mathcal{B}$ that is not identically zero. Suppose also that $\psi \ll \varphi$, ψ is not identically zero, and ψ is not equal to φ. These conditions imply that $\varphi - \psi$ is positive-definite and is not equal to φ or the zero function. Since the absolute value of a positive-definite function is maximized at $g = e$, we have that $\varphi(e) > \psi(e) > 0$. We also have that $\varphi(e) = 1$ because otherwise φ would be an interior point of the line segment

$[0, \frac{\varphi(g)}{\varphi(e)}]$. We therefore have

$$\varphi(g) = \psi(e)\frac{\psi(g)}{\psi(e)} + [1 - \psi(e)]\frac{\varphi(g) - \psi(g)}{1 - \psi(e)},$$

and thus, by extremality of φ, we have that

$$\frac{\psi(g)}{\psi(e)} = \frac{\varphi(g) - \psi(g)}{1 - \psi(e)}.$$

This implies that $\psi(g) = \psi(e)\varphi(g)$, so $\varphi(g)$ is elementary. □

Conversely, we have the following.

Lemma 2.20. [1, p. 11] *Every normalized elementary continuous positive-definite function is an extreme point of $\mathcal{B}$.*

Proof. Suppose φ is elementary and positive-definite, satisfying $\varphi(e) = 1$, and suppose further that $\varphi = \lambda\psi + (1 - \lambda)\chi$ for some $\psi, \chi \in \mathcal{B}$. Then we have $\psi(e) = \chi(e) = 1$, and, since $\lambda\psi \ll \varphi$, we have $\lambda\psi = \alpha\varphi$ for some $\alpha \in \mathbb{C}$. By choosing $g = e$, we get $\alpha = \lambda$, so $\psi = \varphi$ and furthermore $\chi = \varphi$. It follows that φ is an extreme point of $\mathcal{B}$. □

From the two preceding lemmas, it follows that we can think of normalized elementary continuous positive-definite functions as non-zero extreme points of the set $\mathcal{B}$ of continuous positive-definite functions φ with $\varphi(e) \leq 1$. We will now show there are "enough" such extreme points.

We equip the set of continuous positive-definite functions with the weak topology, identifying them with their corresponding linear functionals on $\mathcal{L}_1$. More precisely, given a continuous positive-definite function φ_0, we define a neighbourhood of it to be the set, given any finite collection $x_1, ..., x_n \in \mathcal{L}_1$ and any $\epsilon > 0$, of all continuous positive-definite functions φ such that

$$|\int \varphi_0(g)x_k(g)\,dg - \int \varphi(g)x_k(g)\,dg| < \epsilon \quad (k = 1, ..., n). \qquad (7)$$

We claim the following theorem holds.

Theorem 2.21. [1, Theorem 5] *The set of continuous positive-definite functions φ with $\varphi(e) \leq 1$ is the smallest weakly closed convex set containing every normalized elementary continuous positive-definite function as well as the zero function $\varphi \equiv 0$.*

Proof. We write $\mathrm{Re}(\mathcal{L}_1)$ for the closed linear subspace spanned over the reals by functions $x \in \mathcal{L}_1$ such that $x^* = x$.Suppose $L : \mathcal{L}_1 \to \mathbb{C}$ is a positive linear functional. Then, as we saw earlier, $L(y * x^*) = \overline{L(x * y^*)}$, so taking $y = e_V$ and letting $V \to e$, we obtain $L(x^*) = \overline{L(x)}$ for every $x \in \mathcal{L}_1$. It follows that a positive linear functional on $\mathcal{L}_1$ can be thought of as a positive real linear functional on $\mathrm{Re}(\mathcal{L}_1)$. On the other hand, given a real linear functional on $\mathrm{Re}(\mathcal{L}_1)$, we can extend it uniquely to a linear functional on $\mathcal{L}_1$ by imposing

$$L(x) := L(\mathrm{Re}(x)) + iL(\mathrm{Im}(x)) \text{ with } \mathrm{Re}(x) := \frac{x + x^*}{2} \text{ and } \mathrm{Im}(x) := \frac{x - x^*}{2i}.$$

It follows that positive linear functionals differing on $\mathcal{L}_1$ also differ on $\mathrm{Re}(\mathcal{L}_1)$, and that the norm and weak topologies on the space of positive linear functionals do not depend on whether the functionals are defined on $\mathcal{L}_1$ or on $\mathrm{Re}(\mathcal{L}_1)$.

Given a positive linear functional $L = L_\varphi$, we have $|L_\varphi| = \operatorname{ess\,sup}|\varphi(g)|\varphi(e)$. (Recall that the *essential supremum* is defined by taking the infimum of all *essential upper bounds*, i.e., upper bounds that the function only exceeds on a set of measure zero.) It follows that $\mathcal{B}$ can be thought of as the convex set $\mathcal{B}'$ of positive linear functionals L on $\mathrm{Re}(\mathcal{L}_1)$ with $|L| \leq 1$. But $\mathcal{B}'$ is weakly closed in the space of real linear functionals defined on $\mathrm{Re}(\mathcal{L}_1)$, so by teh Krein–Milman theorem (see [3] for a reference), $\mathcal{B}'$ is the weakly closed convex hull of its extreme points, and therefore $\mathcal{B}$ is also. But then the correspondence between extreme points of $\mathcal{B}$ and normalized elementary continuous positive-definite functions gives the result. □

We now recall a construction of continuous positive-definite functions on any group that admits a Haar measure (and in particular, on a locally compact group G). We write $\mathcal{L}_2$ for the Hilbert space of square-integrable measurable functions under the inner product

$$(x, y) := \int x(h)\overline{y(h)}\, dh.$$

As usual, we have translation operators $U_g x(h) := x(g^{-1}h)$, which give a unitary representation of G in $\mathcal{L}_2$. It follows that for any $x \in \mathcal{L}_2$, the function $(U_g x, x) = \int x(g^{-1}h)\overline{x(h)}\, dh$ is continuous and positive-definite.

Therefore, on a locally compact group G, for each non-identity element g_0, there exists a continuous positive-definite function φ_0 such that $\varphi_0(g_0) \neq \varphi_0(e)$. What's more, if V is a neighbourhood of e, then there exists a normalized continuous positive-definite function vanishing outside V. Indeed, if W is a neighbourhood with $WW^{-1} \subseteq V$, then we can pick a function $x_W(h) \in \mathcal{L}_2$ with $(x_W, x_W) = 1$ and $x_W(h) = 0$ outside W. Then the function $\varphi(g) := (U_g x_W, x_W)$ satisfies all the required properties.

Next, we prove the following.

Theorem 2.22. [1, Theorem 6] *Let G be a locally compact group. Then there exists a complete system of elementary continuous positive-definite functions on G.*

Proof. Suppose for the sake of contradiction that there exists some non-identity element $g_0 \in G$ with $\zeta(g) = 1$ for every normalized elementary continuous positive-definite function ζ. Pick a real normalized continuous positive-definite function φ_0 with $\varphi_0(g_0) \neq 1$. (By the remarks preceding this theorem, we can do this because $g_0 \neq e$.) Choose a neighbourhood V of the identity and some $\epsilon > 0$ such that

$$1 - \varphi_0(g) < \epsilon \text{ and } |\varphi_0(g_0) - \varphi_0(g_0 g)| < \epsilon \text{ whenever } g \in V. \quad (9)$$

Next, consider a neighbourhood of φ_0 defined by the functions $x_1 := \frac{1}{m(V)} f_V$, $x_2 := \frac{1}{m(V)} f_V(g_0^{-1}g)$ and by the ϵ we have chosen. (Recall that the suitable notion of neighbourhood was defined in Equation (7) in the preamble to Theorem 2.21.) By Theorem 2.21, this neighbourhood will contain a function of the form $\varphi = \lambda_1\zeta_1 + ... + \lambda_k\zeta_k$ for some normalized elementary continuous positive-definite functions $\zeta_1, ..., \zeta_k$ and some $\lambda_1, ..., \lambda_k \in \mathbb{R}$ with $\lambda_i \geq 0$ for every $i \in \{1, ..., k\}$ and $\lambda_1 + ... + \lambda_k \leq 1$. This same neighbourhood will contain the real part $\psi := \mathrm{Re}(\varphi)$, which is positive-definite if φ is. By assumption, we have that $\psi(g_0) = \psi(e) \leq 1$. By our choices, we have

$$|\int [\varphi_0(g) - \psi(g)]x_1(g)\, dg| = |\frac{1}{m(V)} \int_V [\varphi_0(g) - \psi(g)]\, dg| < \epsilon.$$

By the first inequality in (9), this gives

$$1-\psi(e)=\frac{1}{m(V)}\int_V[1-\psi(e)]\,dg\leq\frac{1}{m(V)}\int_V[1-\psi(g)]\,dg$$
$$\leq\frac{1}{m(V)}\int_V[1-\varphi_0(g)]\,dg+|\frac{1}{m(V)}\int_V[\varphi_0(g)-\psi(g)]\,dg|<2\epsilon. \quad (10)$$

Moreover, we have

$$\varphi_0(g_0g)-\psi(g_0g)=[\varphi_0(g_0)-\psi(g_0)]-[\varphi_0(g_0)-\varphi_0(g_0g)]+[\psi(g_0)-\psi(g_0g)],$$

so

$$|\frac{1}{m(V)}\int_V[\varphi_0(g_0g)-\psi(g_0g)]\,dg|\geq|\varphi_0(g_0)-\psi(g_0)|-|\frac{1}{m(V)}\int_V[\varphi_0(g_0)-\varphi_0(g_0g)]\,dg|$$
$$-|\frac{1}{m(V)}\int_V[\psi(g_0)-\psi(g_0g)]\,dg|.$$

By successively applying the Cauchy–Schwarz inequality, Proposition 2.6, and (10) from this proof, we obtain

$$|\frac{1}{m(V)}\int_V[\psi(g_0)-\psi(g_0g)]\,dg|^2\leq\frac{1}{m(V)}\int_V|\psi(g_0)-\psi(g_0g)|^2\,dg$$
$$\leq\frac{2}{m(V)}\int_V[\psi(e)-\psi(g)]\,dg\leq\frac{2}{m(V)}\int[1-\psi(g)]\,dg<4\epsilon.$$

The second inequality in (9) gives

$$|\frac{1}{m(V)}\int_V[\varphi_0(g_0)-\varphi_0(g_0g)]\,dg|<\epsilon.$$

Therefore, by the inequality $1-\psi(e)<2\epsilon$ (from (10)), we obtain

$$|\frac{1}{m(V)}\int_V[\varphi_0(g_0g)-\psi(g_0g)]\,dg|>|\varphi_0(g_0)-\psi(g_0)|-\epsilon-2\sqrt{\epsilon}$$
$$=|\varphi_0(g_0)-\psi(e)|-\epsilon-2\sqrt{\epsilon}>[1-\varphi_0(g_0)]-[1-\psi(e)]-\epsilon-2\sqrt{\epsilon}$$
$$>1-\varphi_0(g_0)-3\epsilon-2\sqrt{\epsilon}. \quad (11)$$

But by our choices we have

$$|\int[\varphi_0(g)-\psi(g)]x_2(g)\,dg|=|\frac{1}{m(V)}\int_V[\varphi_0(g_0g)-\psi(g_0g)]\,dg|<\epsilon. \quad (12)$$

By (11) and (12),

$$1-\varphi_0(g_0)<4\epsilon+2\sqrt{\epsilon}.$$

However, by assumption, $1-\varphi_0(g_0)$ is a constant positive real number, so as $\epsilon\to 0$, we obtain a contradiction. This proves the result. □

Finally, we recall and prove the Gelfand–Raikov theorem.

Theorem 2.23. [1, Theorem 7] *For every locally compact group, there exists a complete system of irreducible unitary representations.*

Proof. By Theorem 2.22, there exists a complete system of elementary continuous positive-definite functions on our group. But by Theorem 2.12, there exists a complete system of irreducible unitary representations on our group. □

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