MATH 147 Assignment #1

Due: Monday, September 26

1) a) The statement $p \vee \neg p$ can be translated as "either p is true, or not p is true."

Prove the statement

$$p \vee \neg p$$

by completing the following truth table. (The statement is proved if the column with the heading $p \vee \neg p$ contains only T's.)

p	$\neg p$	$p \vee \neg p$
T		
F		

b) Complete the following truth table:

p	$\neg p$	$p \wedge \neg p$
T		
F		

c) Complete the following truth table. (Recall that $s \Rightarrow t$ is true unless s is true and t is false.)

p	q	$p \vee q$	$p \wedge q$	$p \Rightarrow q$	$\neg p \Rightarrow q$
T	T				
T	F				
F	T				
F	F				

d) Prove that the statement $[(p \Rightarrow q) \wedge (\neg p \Rightarrow q)] \Rightarrow q$ by completing the following truth table:

p	q	p ⇒ q	¬ p ⇒ q	(p ⇒ q)∧ (¬p ⇒ q)	[(p ⇒ q)∧ (¬p ⇒ q)]⇒q
T	T				
T	F				
F	T				
F	F				

Note: This result says that whenever we have that both "p implies q", and "not p implies q", we must also have that q is true because, by part a), one of "p" or "not p" always holds.

2) Let

$$\begin{aligned} A &= \{n \in \mathbb{N} \mid n \text{ is even } \} \\ B &= \{n \in \mathbb{N} \mid n \text{ is divisible by 3 } \} \\ C &= \{n \in \mathbb{N} \mid n \text{ is divisible by 5 } \} \end{aligned}$$

Then we could write $S = \{n \in \mathbb{N} \mid n \text{ is odd and divisible by 5 } \}$ as

$$S = A^c \cap C$$

Using the above example as a guide, for each of the following sets express S as the union, intersection or complement of a combination of the sets A, B, C and A^c, B^c, C^c.

a) $S = \{n \in \mathbb{N} \mid n \text{ is not divisible by 3 } \}$.

b) $S = \{n \in \mathbb{N} \mid n \text{ is even and divisible by 3 but not by 5 } \}$.

c) $S = \{n \in \mathbb{N} \mid n \text{ is divisible by 3 or 5 but not by 15 } \}$.

3) Given any set $A \subseteq X$, we have two possible cases:

1) that $x \in A$ and
2) $x \notin A$.

We can show that for any set $A \subseteq X$, we have that

$$(A^c)^c = A$$

This is represented in the table below by the columns containing A and $(A^c)^c$ having the same values in both of the above cases.

Case	$x \in A$	$x \in A^c$	$x \in (A^c)^c$
1	Y	N	Y
2	N	Y	N

Problem: By using the table above as a guide, prove the second of DeMorgan's Laws:

$$(A \cap B)^c = (A^c \cup B^c)$$

(Complete the following table by entering Y if x belongs to the given set and N if x does not belong to the given set):

Case	$x \in A$	$x \in B$	$x \in (A^c)$	$x \in (B^c)$	$x \in (A \cap B)$	$x \in (A \cap B)^c$	$x \in (A^c \cup B^c)$
1	Y	Y					
2	Y	N					
3	N	Y					
4	N	N					

4) Prove that:

a) Let $a_1 = 1$ and for each $n \geq 1$ let

$$a_{n+1} = \sqrt{3 + 2a_n}$$

(This is called a recursively defined sequence. You may use the fact that if $0 \leq x \leq y$, then $\sqrt{x} \leq \sqrt{y}$.)
Prove that for every $n \in \mathbb{N}$, we have

$$0 \leq a_n \leq a_{n+1} \leq 3$$

b) Prove that $2^n + 3^n$ is divisible by 5 for each odd $n \in \mathbb{N}$. (Hint: Odd n's are of the form $2k - 1$ for $k \in \mathbb{N}$.)

5) Let $P(n)$ be the statement that in any collection of n real numbers $\{x_1, x_2, \dots, x_n\}$ all of the values are equal. We will prove by induction that $P(n)$ is always true. The base case $P(1)$ is obviously true.

Next assume that $P(k)$ is true and that we have a collection of $k+1$ numbers $\{x_1, x_2, \dots, x_k, x_{k+1}\}$. If we remove, x_k from the collection, then we are left with a collection $\{x_1, x_2, \dots, x_k\}$ of k numbers. Hence by $P(k)$ we have that

$$x_1 = x_2 = \cdots = x_k. \quad (*)$$

Next we remove x_1 from our original collection to get a new collection $\{x_2, \dots, x_k, x_{k+1}\}$ of k numbers so again we have

$$x_2 = \cdots = x_k = x_{k+1}.$$

But then we have that

$$x_1 = x_2 = x_{k+1}$$

so in fact by $(*)$ we get that

$$x_1 = x_2 = \cdots = x_k = x_{k+1}$$

and $P(k + 1)$ holds. As such, by induction $P(n)$ holds for all n.

Question: Is there something wrong with this proof or are all real numbers in fact equal? Choose the best answer:

1) Nothing is wrong, the proof is correct.
2) We fail to properly establish the base case $P(1)$ which in this case is false.
3) The argument used to show $P(k+1)$ assuming $P(k)$ is flawed.
4) None of the above.

Briefly justify your selection.

6) The Principle of Strong Induction states that if S is a subset of $\mathbb{N}$ that satisfies:

i) $1 \in S$;
ii) if $1, 2, \ldots, k \in S$, then $k+1 \in S$;

then $S = \mathbb{N}$.

a) Use the Principle of Mathematical Induction to prove Principle of Strong Induction.
(Hint: Let $P(n)$ be the statement that $\{1, 2, \ldots, n\} \subset S$)

b) The Well Ordering Principle for $\mathbb{N}$ states that every nonempty subset S of $\mathbb{N}$ has a least element. Prove this using the Principle of Strong Induction.
(Hint: Let S be a subset of $\mathbb{N}$ that does not have a least element. Let

$$T = \mathbb{N} \setminus S = \{n \in \mathbb{N} \mid n \notin S\} = S^c.$$

and show that $T = \mathbb{N}$ and hence that $s = \emptyset$.)

7) Let n be a natural number greater than or equal to 2. We say that n is prime if whenever n can be written as a product of the form $n = m_1 m_2$ where m_1 and m_2 are natural numbers, then either $m_1 = 1$ or $m_2 = 1$. Use the Well Ordering Principle to show that if $n \in \mathbb{N}$ and $n \geq 2$, then n is a product of primes.

Note: The factorization above is unique in the sense that if

$$n = p_1^{n_1} p_2^{n_2} \cdots p_j^{n_j}$$

and

$$n = q_1^{k_1} q_2^{k_2} \cdots q_l^{k_l}$$

where $p_1 < p_2 < \cdots < p_j$ and $q_1 < q_2 < \cdots < q_l$ are distinct primes, then $j = l$, $p_i = q_i$ and $n_i = k_i$ for each $1 \leq i \leq j$.

8) Recall that a function $f : X \to Y$ is said to be $1-1$ if whenever $x_1, x_2 \in X$ with $x_1 \neq x_2$, then we have $f(x_1) \neq f(x_2)$.

A function $f : X \to Y$ is said to be onto if

$$range(f) = \{f(x) \mid x \in X\} = Y.$$

Moreover, if $f : X \to Y$ is both $1-1$ and onto , then f is *invertible* with its inverse $f^{-1} : Y \to X$ being defined by

$$f^{-1}(y) = x \text{ if and only if } f(x) = y.$$

a) Find a $1-1$ and onto function from the integers $\mathbb{Z} = \{\cdots, -3, -2, -1, 0, 1, 2, 3, \cdots\}$ onto the natural numbers $\mathbb{N} = \{1, 2, 3, \cdots\}$ of the form

$$f(k) = \begin{cases} ? & \text{if } k > 0, \\ ? & \text{if } k = 0, \\ ? & \text{if } k < 0. \end{cases}$$

b) Use the uniqueness of the factorization in Question 7 to show that there exists a $1-1$ function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ with the range of f being a subset of the set of all natural numbers whose prime factors consist only of 2 and 3.

c) Let $\mathbb{Q} = \{\frac{k}{m} \mid k \in \mathbb{Z}, m \in \mathbb{N}, gcd(|k|, m) = 1 \text{ if } k \neq 0\}$. Show that there exists a $1-1$ function $f : \mathbb{Q} \to \mathbb{N}$.

d) Let $f : X \to Y$, $g : Y \to Z$. For each of the following statements indicate if it is true or false. If you believe it is true you do not have to justify your choice. If however, you think it is false provide a counter example.

i) If f and g are both 1-1, then so is $g \circ f$.

ii) If f and g are both onto, then so is $g \circ f$.

iii) If f and g are both invertible, then so is $g \circ f$ with

$$(g \circ f)^{-1} = f^{-1} \circ g^{-1}.$$

iv) If $g \circ f$ is 1-1, then so are f and g.

9) a) Prove that $\mathbb{R}$ has the *Archimedean Property II:*

Let $\epsilon > 0$. Then there exists an $n \in \mathbb{N}$ such that $0 < \frac{1}{n} < \epsilon$.

b) Let $0 \leq a < b$. Show that there exists a rational number r with $a < r < b$ and an irrational number γ with $a < \gamma < b$.

(Hint: Let $0 < a < b$. Let $\epsilon = b - a$. Choose $N_0 \in \mathbb{N}$ with $0 < \frac{1}{N_0} < \epsilon$. Let

$$S = \{k \in \mathbb{N} \mid a < \frac{k}{N_0}\}.$$

Show that $S \neq \emptyset$. You may also assume that $\sqrt{2}$ is irrational.)