# DIFFERENTIALS 

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## 1. Differentials

In this short article, the notion of differentials on algebraic curves will be discussed. All of the following is based on the exposition in Chapter II Section 4 of Silverman.
Definition 1.1. Let $C$ be a smooth curve. The space of differential forms on $C$, $\Omega_{C}$, is the $\bar{K}(C)$ vector space of all symbols $d f$ for each $f$ in $\bar{K}(C)$ subject to the following relations,
(1) $d(f+g)=d f+d g$ for all $f, g$ in $\bar{K}(C)$
(2) $d(f g)=f d g+g d f$ for all $f, g$ in $\bar{K}(C)$
(3) $d \alpha=0$ for all $\alpha$ in $\bar{K}$

Formally $\Omega_{C}$ can be constructed by taking the free $\bar{K}(C)$ module of symbols $d f$ for all $f$ in $\bar{K}(C)$ and modding out by the submodule generated by the relations above. If $f=\frac{g}{h}$ then it is easy to see that $d f=\frac{1}{h} d g-\frac{f}{h} d h$.
Theorem 1.2. Let $C$ be a smooth plane curve.
(1) $\Omega_{C}$ is a one dimensional vector space over $\bar{K}(C)$
(2) Given $f$ in $\bar{K}(C)$, df generates $\Omega_{C}$ if and only if the field extension $\bar{K}(C) / \bar{K}(f)$ is finite and separable.
(3) Given a map between smooth curves $\phi: C_{1} \rightarrow C_{2}$, there is an induced map $\phi^{*}: \Omega_{C_{2}} \rightarrow \Omega_{C_{2}}$ such that $\phi^{*}$ is injective if and only if $\phi$ is separable.

Proof.
(1) Firstly let $C=V(F)$ for some $F \in K[x, y]$. Then

$$
0=d F=F_{x} d x+F_{y} d y
$$

where $F_{x}$ and $F_{y}$ denote the partial derivatives with respect to $x$ and $y$. Since $C$ is smooth, both of these partial derivatives cannot vanish thus w.l.o.g if $F_{y} \neq 0$

$$
d y=-F_{x}\left(F_{y}\right)^{-1} d x
$$

This shows that $\operatorname{dim} \Omega_{C} \leq 1$. To get the dimension to be one we need to show that $\Omega_{C}$ is not the zero vector space.

Construct a map $\psi: \bar{K}(C) \rightarrow \bar{K}(C), \psi(f)=f_{x}+f_{y}$. Now construct a map $\lambda: \Omega_{C} \rightarrow \bar{K}(C), \lambda(d f)=\psi(f)$ extended linearly. To see that $\lambda$ is well defined it suffices to show that $\lambda$ vanishes on the relations that the differential might satisfy. For example,

$$
\begin{aligned}
\lambda(d(x+y)-d x-d y) & =\lambda(d(x+y))-\lambda(d x)-\lambda(d y) \\
& =\psi(x+y)-\psi(x)-\psi(y) \\
& =1+1-1-1=0 . \\
& 1
\end{aligned}
$$

The other relations can be checked similarly. Note that $\lambda(d x)=\psi(x)=1$, thus $d x \neq 0$.
(2) First we show that separability implies that $d f$ generates the vector space $\Omega_{C}$. Choose $t \in \bar{K}(C)$ and $F(X, T)$ with minimal degree in $T$. Since the extension is separable $F_{T}(X, T) \neq 0$. Thus

$$
0=d F=F_{X}(f, t) d f+F_{T}(f, t) d t
$$

and so $d t=F_{X}(f, t)\left(F_{T}(f, t)\right)^{-1} d f$ and thus $d f$ is a basis.
Now for the other direction note that $\bar{K}(C) / \bar{K}(f)$ is finite since the transcendence degree of $\bar{K}(C) / \bar{K}$ and $\bar{K}(f) / \bar{K}$ are both one. If the characteristic of $K$ is zero the result is clear, thus assume $0 \neq p=\operatorname{char}(K)$. Given $t \in \bar{K}(C)$ we can find $F(X, T)$ with minimal degree in $T$ such that $F(x, t)=0$. Assume the extension is not separable thus $F_{T}(X, T)$ is zero and every power of $T$ in $F$ must be a multiple of $p$, so

$$
0=d F=F_{X}(f, t) d f+F_{T}(f, t) d t=F_{X}(f, t) d f
$$

Since $d f$ is a basis, $F_{X}(f, t)=0$ but this implies that every power of $X$ in $F$ must be a multiple of $p$ and thus we can find $G, H \in \bar{K}[X, Y]$ such that $F(X, T)=G\left(X^{p}, T^{p}\right)=H(X, T)^{p}$, which contradicts the irreducibility of $F$.
(3) Define $\phi^{*}(g d f)=\phi^{*}(g) d\left(\phi^{*}(f)\right)$ extended linearly. Now choose $f$ in $\bar{K}\left(C_{2}\right)$ so that $f$ generates $\Omega_{C_{2}}$. Note that

$$
\begin{aligned}
\phi^{*} \text { is injective } & \Leftrightarrow d\left(\phi^{*}(f)\right) \neq 0 \\
& \Leftrightarrow d\left(\phi^{*}(f)\right) \text { is a basis for } \Omega_{C} \\
& \Leftrightarrow \bar{K}\left(C_{1}\right) / \bar{K}\left(\phi^{*}(f)\right) \text { is separable } \\
& \Leftrightarrow \bar{K}\left(C_{1}\right) / \phi^{*} \bar{K}\left(C_{2}\right) \text { is separable }
\end{aligned}
$$

It is interesting to note that part 2 of the theorem above reasserts what it means for an extension to be separable, namely that the minimal polynomials of elements have non-zero derivative (and in our case form a basis).

Having studied some global properties of differentials, we now move to some more local properties which will allow us to consider the divisor of a differential.

Theorem 1.3. Let $C$ be a smooth curve and let $t$ be a uniformizer at the point $P$ of $C$.
(1) For every $\omega$ in $\Omega_{C}$ we can find a unique $g$ in $\bar{K}(C)$ so that $\omega=g d t$. We let $\omega / d t=g$.
(2) If $f$ in $\bar{K}(C)$ is regular at $P$ so is $d f / d t$.
(3) Note that $\operatorname{ord}_{P}(\omega)$ is independent of the choice of $t$. We let this common value be denoted as ord ${ }_{P}(\omega)$.
(4) Choose $f$ in $\bar{K}(C)$ such that $\bar{K}(C) / \bar{K}(f)$ is separable and $f$ vanishes at $P$.

Then if $p=\operatorname{char}(K)$ for all $g$ in $\bar{K}(C)$,

$$
\begin{gathered}
\operatorname{ord}_{P}(g d f)=\operatorname{ord}_{P}(f)+\operatorname{ord}_{P}(g)-1 \text { if } p=0, p \nmid n \\
\quad \operatorname{ord}_{P}(g d f) \geq \operatorname{ord}_{P}(f)+\operatorname{ord}_{P}(g) \text { otherwise }
\end{gathered}
$$

(5) For all but finitely many $P$ in $C$, $\operatorname{ord}_{P}(\omega)=0$.

Remark 1.4. Note that 4 asserts that the degree of vanishing of a differential of a function is one less then the degree of vanishing of the function. This is certainly a property we would want to capture.

Proof.
(1) From Lemma 2 in Silverman we know that if $t$ is a uniformizer at point $P$ then the extension $\bar{K}(C) / \bar{K}(t)$ is separable. Use this and 1.2 above to get the result.
(2) Clear.
(3) Let $s$ be another uniformizer at $P$. From the above, $d t / d s$ and $d s / d t$ are both regular at $P$. From this it follows that $\operatorname{ord}_{P}(d s / d t)=0$ and thus

$$
\omega=g d s=g\left(\frac{d s}{d t}\right) d t
$$

So the result follows.
(4) Let $f=u t^{n}$ where $n=\operatorname{ord}_{P}(f), n \geq 1$ and $u$ is a unit. It is clear that

$$
d f=\left(n u t^{n-1}+\frac{d u}{d t} t^{n}\right) d t
$$

Since $d u / d t$ is regular at $P$ then if $n \neq 0$ and $p \nmid n$, the power of $t$ in $n u t^{n-1}$ is the least and so

$$
\operatorname{ord}_{P}(g d f)=\operatorname{ord}_{P}(g)+n-1
$$

In the case where $p \mid n$, the result follows similarly.
(5) Choose $g$ so that $\bar{K}(C) / \bar{K}(g)$ is separable and let $\omega=f d g$. From a theorem in Hartshorne (see the Remark after this proof) the map $g: C \rightarrow \mathbb{P}^{1}$ ramifies at only finitely many points. Thus discarding these points and all points $Q \in C$ where $f(Q)=0, \infty, g(Q)=\infty$, we note that on all other points $P \in C, g-g(P)$ is a uniformizer at $Q$ and thus
$\operatorname{ord}_{P}(\omega)=\operatorname{ord}_{P}(f d g)=\operatorname{ord}_{P}(f d(g-g(P)))=\operatorname{ord}_{P}(f)+\operatorname{ord}_{P}(g-g(P))-1=0$.

Remark 1.5. This result is the geometric analogue of the result in function fields which states that only finitely many primes ramify in an integral extension of $\mathbb{Z}$. In general this result is a specific case of a result about integral extensions of Dedekind domains.

Example 1.6. We conclude this section with an example of computations of divisors and differentials on elliptic curves. In the following we will use the convention that lower cased letters refer to coordinate functions on a specific affine piece whereas upper case letters refer to projective coordinates. Let $e_{1}, e_{2}, e_{3} \in K$ where $\operatorname{char}(K) \neq 2$. Consider the elliptic curve

$$
y^{2}=\left(x-e_{1}\right)\left(x-e_{2}\right)\left(x-e_{3}\right)
$$

Let $P_{1}=\left(e_{1}, 0\right), P_{2}=\left(e_{2}, 0\right), P_{3}=\left(e_{3}, 0\right)$, and $P_{\infty}=[0: 1: 0]$. It is clear that $P_{\infty}$ is the only point at infinity on the curve. We now compute several divisors.

- $\operatorname{div}\left(x-e_{i}\right)=2 P_{i}-2 P_{\infty}$. We must consider the homogenized function $\left(X-e_{i} Z\right) / Z$. Note that $y$ is not the tangent line to $P_{i}$, so $y$ is a uniformizer at $P_{i}$. Since

$$
\frac{y^{2}}{\left(x-e_{j}\right)\left(x-e_{k}\right)}=\left(x-e_{i}\right), i \neq j \neq k
$$

$\operatorname{ord}_{P_{i}}\left(X-e_{i} Z\right)=2$ and so by Bezout's Theorem
$\operatorname{div}\left(\frac{X-e_{i} Z}{Z}\right)=\operatorname{div}\left(X-e_{i} Z\right)-\operatorname{div}(Z)=2 P_{i}+P_{\infty}-3 P_{\infty}=2 P_{i}-2 P_{\infty}$

- $\operatorname{div}(y)=P_{1}+P_{2}+P_{3}-3 P_{\infty}$. Again $y$ is the uniformizer at $P_{i}$ for each $i$ and these are the only points at which $y$ vanishes. Thus since $\operatorname{deg}(\operatorname{div}(y))=0$

$$
\operatorname{div}(y)=P_{1}+P_{2}+P_{3}-3 P_{\infty}
$$

- $\operatorname{div}(d x)=P_{1}+P_{2}+P_{3}-3 P_{\infty}$. Firstly consider $P \neq P_{\infty}, P_{i}$ for $i=1,2,3$. If $P=(a, b)$ it is clear that $x-a$ is not a tangent line at $P$ thus it is a uniformizer. So $\operatorname{ord}_{P}(d x)=\operatorname{ord}_{P}(d(x-a))=\operatorname{ord}_{P}(x-a)-1=0$. Now if $P=P_{i}$ for $i=1,2,3$ we know that $\operatorname{ord}_{P}(d x)=\operatorname{ord}_{P}\left(d\left(x-e_{i}\right)\right)=$ $\operatorname{ord}_{P}\left(x-e_{i}\right)-1=1$. This leaves the case when $P=P_{\infty}$. Note that $d x=-x^{2} d(1 / x)$ thus

$$
\operatorname{ord}_{P}(d x)=\operatorname{ord}_{P}\left(x^{2}\right)+\operatorname{ord}_{P}\left(\frac{1}{x}\right)-1
$$

and

$$
\operatorname{div}\left(\frac{1}{x}\right)=\operatorname{div}\left(\frac{Z}{X}\right)=\operatorname{div}(Z)-\operatorname{div}(X)=3 P_{\infty}-\operatorname{div}(X)
$$

When $x=0, y^{2}=-e_{1} e_{2} e_{3}$ thus we see that this accounts for a total intersection multiplicity of two. By Bezout's theorem $\operatorname{ord}_{P}(X)=1$ and so

$$
\operatorname{ord}_{P}\left(\frac{1}{x}\right)=3-1=2
$$

Thus $\operatorname{ord}_{P}(d x)=-4+2-1=-3$ and

$$
\operatorname{div}(d x)=P_{1}+P_{2}+P_{3}-3 P_{\infty}
$$

From this we can conclude that $\frac{d x}{y}$ is holomorphic and non-vanishing.

