1 Smooth curves

1.1 Definition. Let C be an affine plane curve in $\mathbb{A}^2$ given by $f \in \Bbbk[x, y]$ and let $p = (a, b) \in C$. Then p is said to be a *smooth point* (or *simple point*) of C if $\nabla f(a, b) \neq (0, 0)$. A point that is not smooth is called a *singular point* (or *multiple point*). A curve that is smooth at every point is called a *smooth curve* (or *non-singular curve*). A point p of C is smooth if and only if there exists a unique normal vector to C at p. The *tangent line* to C at p is given by the equation $\nabla f(a, b) \cdot (x - a, y - b) = 0$.

1.2 Examples.

(i) The parabola $\mathrm{V}(y - x^2)$ is non-singular since its gradient is $(-2x, 1)$, which is never $(0, 0)$.

(ii) The cusp $\mathrm{V}(y^2 - x^3)$ has a singular point at $(0, 0)$ since its gradient is $(-3x^2, 2y)$ which is $(0, 0)$ at $(0, 0)$.

(iii) The alpha curve $\mathrm{V}(y^2 - x^3 - x^2)$ has a singular point at $(0, 0)$.

1.3 Definition. The *Zariski tangent space* to C at $p = (a, b)$ is

$$\mathrm{T}_p(C) = \{v \in \mathbb{A}^2 \mid \nabla f(a, b) \cdot v = 0\}.$$

Therefore, $\dim_{\Bbbk}(\mathrm{T}_p(C)) = 1$ if and only if p is a smooth point of C and $\mathrm{T}_p = \mathbb{A}^2$ if and only if p is singular. Clearly, $\mathrm{T}_p(C)$ is a $\Bbbk$-vector space. The $\Bbbk$-vector space dual $(\mathrm{T}_p(C))^*$ is called the *Zariski cotangent space* of C at p. Note that $(\mathrm{T}_p(C))^*$ is considered as a subspace of $(\mathbb{A}^2)^*$, whose elements are 1×2 matrices. Recall the differential map $\mathrm{d}_p : \Bbbk[x, y] \to (\mathbb{A}^2)^*$ defined by

$$\mathrm{d}_p\, g = \begin{bmatrix} \frac{\partial g}{\partial x}(p) & \frac{\partial g}{\partial y}(p) \end{bmatrix} = \mathrm{Jac}(g)(p).$$

Without loss of generality, we can assume that $p = (0, 0)$, as all notions under discussion are invariant under translations. Then, if C is given by $f \in \Bbbk[x, y]$, $\mathrm{T}_p(C) = \ker(\mathrm{d}_p\, f)$. We can also define a differential map on $\mathrm{M}_p(C)$.

1.4 Definition. Let C be an affine plane curve. If $p \in C$, the map $\mathrm{d}_p : \mathrm{M}_p(C) \to (\mathrm{T}_p(C))^*$ defined by

$$\mathrm{d}_p \frac{\overline{a}}{\overline{b}} = \left.\frac{\mathrm{d}_p\, a}{b(p)}\right|_{\mathrm{T}_p(C)}$$

is called the *differential map at* p. Moreover, given $g \in \mathrm{M}_p(C)$, $\mathrm{d}_p\, g$ is the *differential of* g *at* p.

1.5 Proposition. *Let C be an affine plane curve. If $p \in C$, the differential map $\mathrm{d}_p : \mathrm{M}_p(C) \to (\mathrm{T}_p(C))^*$ is a well-defined surjective linear map whose kernel is $\mathrm{M}_p^2(C)$.*

PROOF: As usual, without loss of generality we can assume that $p = (0,0)$, as all notions under discussion are invariant under translations. Let $f \in \Bbbk[x,y]$ be such that $C = \mathrm{V}(f)$. Suppose $g \in \mathrm{M}_p(C)$ is such that $g = \overline{a}/\overline{b} = \overline{c}/\overline{d}$, where $\overline{a}, \overline{b}, \overline{c}, \overline{d} \in \Bbbk[C]$ are such that $a(p) = c(p) = 0$ and $b(p), d(p) \neq 0$. Then $\overline{a}\overline{d} - \overline{c}\overline{b} = 0$, so $ad - cb = hf$ for some $h \in \Bbbk[x,y]$. Thus,

$$\mathrm{d}_p(ad - cb) = \mathrm{d}_p(hf).$$

Expanding both sides of this equation gives

$$a(p)\,\mathrm{d}_p\,d + d(p)\,\mathrm{d}_p\,a - c(p)\,\mathrm{d}_p\,b - b(p)\,\mathrm{d}_p\,c = h(p)\,\mathrm{d}_p\,f + f(p)\,\mathrm{d}_p\,h.$$

But $a(p) = c(p) = f(p) = 0$, so

$$d(p)\,\mathrm{d}_p\,a - b(p)\,\mathrm{d}_p\,c = h(p)\,\mathrm{d}_p\,f.$$

Since $\mathrm{d}_p\,f|_{\mathrm{T}_p(C)} = 0$,

$$d(p)\,\mathrm{d}_p\,a - b(p)\,\mathrm{d}_p\,c|_{\mathrm{T}_p(C)} = 0,$$

i.e.

$$\frac{\mathrm{d}_p\,a}{b(p)}\Big|_{\mathrm{T}_p(C)} = \frac{\mathrm{d}_p\,c}{d(p)}\Big|_{\mathrm{T}_p(C)}.$$

Therefore, $\mathrm{d}_p : \mathrm{M}_p(C) \to (\mathrm{T}_p(C))^*$ is well-defined. Clearly, d_p is linear. If $\varphi \in (\mathrm{T}_p(C))^*$, then φ is the restriction to $\mathrm{T}_p(C)$ of some linear function f on $\mathbb{A}^2$, so $\mathrm{d}_p\,f = \varphi$. Let $M = \langle x, y\rangle$. Let $\delta : \Bbbk[x,y] \to (\mathbb{A}^2)$ be the map

$$\delta(h) = \mathrm{d}_p(h)|_{\mathrm{T}_p(C)}.$$

Then, it is easy to see that $\ker(\delta|_M) = M^2$. Since $\mathrm{M}_p(C) = \overline{M}\mathcal{O}_p(C)$,

$$\ker(\mathrm{d}_p) = \overline{M}^2\mathcal{O}_p(C) = \mathrm{M}_p^2(C). \qquad \square$$

1.6 Corollary. *Let C be an affine plane curve. Then* $\mathrm{M}_p(C)/(\mathrm{M}_p(C))^2 \cong (\mathrm{T}_p(C))^*$.

PROOF: This is immediate from the previous proposition and the First Isomorphism Theorem. $\square$

1.7 Examples.

(i) Let C be the parabola $\mathrm{V}(y - x^2)$. C is smooth at every point and $\Bbbk[C] = \Bbbk[t]$. At $p = (0,0)$, $\mathrm{M}_p(C) = \langle x\rangle$ and $\mathrm{M}_p(C)/(\mathrm{M}_p(C))^2 = \{\lambda x \mid \lambda \in \Bbbk\}$.

(ii) Let C be $\mathrm{V}(y^2 - x^3)$, which is singular at the origin $p = (0,0)$. Here $\Bbbk[C] = \Bbbk[x,y]/\langle y^2 - x^3\rangle$ and $\mathrm{M}_p(C) = \langle \overline{x}, \overline{y}\rangle \subseteq \Bbbk(C)$. We have

$$(\mathrm{M}_p(C))^2 = \langle \overline{x}^2, \overline{xy}, \overline{y}^2\rangle = \langle \overline{x}^2, \overline{xy}\rangle,$$

so $\mathrm{M}_p(C)/(\mathrm{M}_p(C))^2 = \{a\overline{x} + b\overline{y} \mid a, b \in \Bbbk\}$. Since this has dimension two, we know that p is a singular point.

1.8 Proposition. *Let C be an affine plane curve given by $f \in \Bbbk[x,y]$. Then $p \in C$ is a smooth point of C if and only if $\mathrm{M}_p(C)$ is principal. In this case, $M_p(C) = \langle t \rangle$, where $\mathrm{V}(t)$ is any line through p that is not $\mathrm{T}_p(C)$.*

PROOF: Suppose that C is smooth at p. By making an appropriate affine transformation, we may assume that $p = (0,0)$ and that the tangent line at p is $y = 0$. We will show that M_p is generated by $\overline{x}$; the proof that M is generated by any line that is not the tangent line is similar. By the above assumptions $f(0,0) = 0$, so it has no constant term. That the tangent line at $(0,0)$ is $y = 0$ implies that f has no linear term in x. Therefore

$$f(x,y) = y + \text{higher order terms.}$$

Grouping the terms with y, we get $f = yg - x^2h$, where g is a unit in $\mathcal{O}_p(C)$ and $h \in \Bbbk[x]$. Taking residue classes, we get that

$$0 = \overline{f} = \overline{yg} - \overline{x}^2\overline{h},$$

so $\overline{y} = \overline{g}^{-1}\overline{h}\overline{x}^2$. Therefore $\overline{y} \in \langle \overline{x} \rangle$, so $\mathrm{M}_p = \langle \overline{x}, \overline{y} \rangle = \langle \overline{x} \rangle$ is principal.

Conversely, if the maximal ideal is principal, say $\mathrm{M}_p = \langle t \rangle$, then $M_p^2 = \langle t^2 \rangle$ and $\mathrm{M}_p \,/\, \mathrm{M}_p^2 = \{at \mid a \in \Bbbk\}$, a one dimensional $\Bbbk$-vector space. This implies that C is smooth at p by Corollary 1.6 and the remark after the definition of the tangent space. □

1.9 Corollary. *Let C be an affine plane curve. Then $p \in C$ is a smooth point of C if and only if $\mathcal{O}_p(C)$ is a DVR.*

PROOF: This follows from the preceding proposition and the fact that $\mathcal{O}_p(C)$ is a Noetherian local ring. □

1.10 Corollary. *Let C be an affine plane curve. Then C is smooth everywhere if and only if $\Bbbk[C]$ is a Dedekind domain.*

PROOF: This follows from the preceding corollary and the characterization of Dedekind domains as the one-dimensional integral domains whose localizations at all maximal ideals are DVRs. □