Tropical Laplacians and Curve Arrangements on Surfaces

Eric Katz (University of Waterloo)
joint with June Huh (IAS)

January 24, 2015

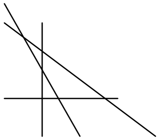
Incidence graphs

Let X be a smooth projective complex surface. Let $C_1, C_2, \ldots C_n$ be a set of disjoint smooth curves on X. Let Γ be the incidence graph whose vertices v_i correspond to the curves C_i and where there are k edges between v_i and v_j if and only if there are k points of $C_i \cap C_j$ (counted with multiplicity).

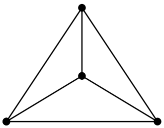
Incidence graphs

Let X be a smooth projective complex surface. Let $C_1, C_2, \ldots C_n$ be a set of disjoint smooth curves on X. Let Γ be the incidence graph whose vertices v_i correspond to the curves C_i and where there are k edges between v_i and v_j if and only if there are k points of $C_i \cap C_j$ (counted with multiplicity).

For four generic lines in the plane we have



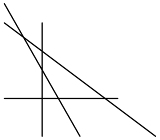
$\mapsto$



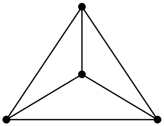
Incidence graphs

Let X be a smooth projective complex surface. Let $C_1, C_2, \ldots C_n$ be a set of disjoint smooth curves on X. Let Γ be the incidence graph whose vertices v_i correspond to the curves C_i and where there are k edges between v_i and v_j if and only if there are k points of $C_i \cap C_j$ (counted with multiplicity).

For four generic lines in the plane we have



$\mapsto$



In general, m generic lines becomes K_m.

Natural question

Question: Which finite graphs occur as incidence graphs.

Natural question

Question: Which finite graphs occur as incidence graphs.

Answer 1 Any.

Natural question

Question: Which finite graphs occur as incidence graphs.

Answer 1 Any.

This is an observation from a paper of Kollar. Let $K_n^{d^2}$ be K_n with each edge replaced by d^2 parallel edges. This is the incidence graph of n generically chosen degree d curves. Any two such curves intersect in d^2 points. Now, let Γ be a subgraph of $K_n^{d^2}$.

Natural question

Question: Which finite graphs occur as incidence graphs.

Answer 1 Any.

This is an observation from a paper of Kollar. Let $K_n^{d^2}$ be K_n with each edge replaced by d^2 parallel edges. This is the incidence graph of n generically chosen degree d curves. Any two such curves intersect in d^2 points. Now, let Γ be a subgraph of $K_n^{d^2}$.

We need to delete some edges which corresponds to removing the intersection points. We can do this by blowing up the corresponding intersection points. Locally this replaces the intersection point $C_i \cap C_j$ with a line. Then C_i and C_j now intersect this line in distinct points. Our surface is now a blow-up of $\mathbb{P}^2$. That line, by the way, is called an exceptional divisor.

Natural question

Question: Which finite graphs occur as incidence graphs.

Answer 1 Any.

This is an observation from a paper of Kollar. Let $K_n^{d^2}$ be K_n with each edge replaced by d^2 parallel edges. This is the incidence graph of n generically chosen degree d curves. Any two such curves intersect in d^2 points. Now, let Γ be a subgraph of $K_n^{d^2}$.

We need to delete some edges which corresponds to removing the intersection points. We can do this by blowing up the corresponding intersection points. Locally this replaces the intersection point $C_i \cap C_j$ with a line. Then C_i and C_j now intersect this line in distinct points. Our surface is now a blow-up of $\mathbb{P}^2$. That line, by the way, is called an exceptional divisor.

Answer 2 You asked a dumb question.

A better question

Recall that a curve C on a surface X is said to be very ample if there is an embedding

$$i : X \hookrightarrow \mathbb{P}^N$$

into some big projective space such that $C = i^{-1}(H)$ where H is a hyperplane. Any projective surface has a very ample divisor. Very ample divisors intersect every curve on X.

A better question

Recall that a curve C on a surface X is said to be very ample if there is an embedding

$$i : X \hookrightarrow \mathbb{P}^N$$

into some big projective space such that $C = i^{-1}(H)$ where H is a hyperplane. Any projective surface has a very ample divisor. Very ample divisors intersect every curve on X.

Definition: A curve arrangement $C_1, \ldots, C_n \subset X$ is said to be *sufficient* if some integer combination $\sum a_i C_i$ is very ample.

A better question

Recall that a curve C on a surface X is said to be very ample if there is an embedding

$$i : X \hookrightarrow \mathbb{P}^N$$

into some big projective space such that $C = i^{-1}(H)$ where H is a hyperplane. Any projective surface has a very ample divisor. Very ample divisors intersect every curve on X.

Definition: A curve arrangement $C_1, \ldots, C_n \subset X$ is said to be *sufficient* if some integer combination $\sum a_i C_i$ is very ample.

Better Question: Which graphs occur as the incidence graph of a sufficient curve arrangement?

A better question

Recall that a curve C on a surface X is said to be very ample if there is an embedding

$$i : X \hookrightarrow \mathbb{P}^N$$

into some big projective space such that $C = i^{-1}(H)$ where H is a hyperplane. Any projective surface has a very ample divisor. Very ample divisors intersect every curve on X.

Definition: A curve arrangement $C_1, \ldots, C_n \subset X$ is said to be *sufficient* if some integer combination $\sum a_i C_i$ is very ample.

Better Question: Which graphs occur as the incidence graph of a sufficient curve arrangement?

Answer It's harder. The argument we used before doesn't work because blow-ups change which curves are very ample. But we'll give some necessary conditions.

Intersections of divisors

To phrase the necessary conditions, we need to talk about divisors. Recall that a divisor on a surface is a finite integer sum of curves. There is an intersection product

$$\operatorname{Div} X \times \operatorname{Div} X \to \mathbb{Z}$$

extending the usual intersection product of curves. Two divisors D_1, D_2 are said to be numerically equivalent if

$$(D_1 - D_2) \cdot C = 0$$

for all curves C on X. The group of numerical equivalence classes of divisors is denoted Num.

Intersections of divisors

To phrase the necessary conditions, we need to talk about divisors. Recall that a divisor on a surface is a finite integer sum of curves. There is an intersection product

$$\operatorname{Div} X \times \operatorname{Div} X \to \mathbb{Z}$$

extending the usual intersection product of curves. Two divisors D_1, D_2 are said to be numerically equivalent if

$$(D_1 - D_2) \cdot C = 0$$

for all curves C on X. The group of numerical equivalence classes of divisors is denoted Num.

Hodge Index Theorem: The intersection product on $\operatorname{Num}(X)$ has a single positive eigenvalue.

Intersections of divisors

To phrase the necessary conditions, we need to talk about divisors. Recall that a divisor on a surface is a finite integer sum of curves. There is an intersection product

$$\operatorname{Div} X \times \operatorname{Div} X \to \mathbb{Z}$$

extending the usual intersection product of curves. Two divisors D_1, D_2 are said to be numerically equivalent if

$$(D_1 - D_2) \cdot C = 0$$

for all curves C on X. The group of numerical equivalence classes of divisors is denoted Num.

Hodge Index Theorem: The intersection product on $\operatorname{Num}(X)$ has a single positive eigenvalue.

Alternative formulation: Let H be a very ample divisor on X. Let $D \in \operatorname{Num}(X)$ be nonzero with $D \cdot H = 0$. Then $D^2 < 0$.

Moving the goalposts

To exploit the Hodge index theorem, we need to put some additional structure on Γ.

Moving the goalposts

To exploit the Hodge index theorem, we need to put some additional structure on Γ.

Definition The multiplicity of a vertex v_i is $d(v_i) = -D_i^2$, the opposite of the self-intersection of D_i. Let us also collapse Γ to a simple graph and weight the edge v_i, v_j with a multiplicity $m(v_I, v_j)$ remembering how many edges we collapsed together.

Moving the goalposts

To exploit the Hodge index theorem, we need to put some additional structure on Γ.

Definition The multiplicity of a vertex v_i is $d(v_i) = -D_i^2$, the opposite of the self-intersection of D_i. Let us also collapse Γ to a simple graph and weight the edge v_i, v_j with a multiplicity $m(v_I, v_j)$ remembering how many edges we collapsed together.

The tropical Laplacian of (Γ, m, d) is

$$L = \begin{bmatrix} d(v_1) & -m(v_1, v_2) & \dots & -m(v_1, v_m) \\ -m(v_1, v_1) & d(v_2) & \dots & -m(v_2, v_m) \\ \vdots & & \ddots & \vdots \\ -m(v_m, v_1) & -m(v_m, v_2) & \dots & d(v_m) \end{bmatrix}.$$

This is just the intersection matrix on the vector space generated by the D_i's, but you should think of it as being like a graph Laplacian.

Examples

Four generic lines in $\mathbb{P}^2$ gives

$$L = \begin{bmatrix} -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 \end{bmatrix}.$$

Examples

Four generic lines in $\mathbb{P}^2$ gives

$$L = \begin{bmatrix} -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 \end{bmatrix}.$$

Note that the tropical Laplacian looks a lot like the usual graph Laplacian except the diagonal entries are different.

The Hodge index theorem constraint

Hodge Index Theorem: If Γ comes from a sufficient curve arrangement then L has a single negative eigenvalue.

This puts significant combinatorial constraints on Γ.

The Hodge index theorem constraint

Hodge Index Theorem: If Γ comes from a sufficient curve arrangement then L has a single negative eigenvalue.

This puts significant combinatorial constraints on Γ.

A divisor is said to be irrelevant if it is in the kernel of L. For a divisor $D = \sum a_i D_i$, write $D(v_i) = a_i$. Let K be subspace of irrelevant divisors.

The Hodge index theorem constraint

Hodge Index Theorem: If Γ comes from a sufficient curve arrangement then L has a single negative eigenvalue.

This puts significant combinatorial constraints on Γ.

A divisor is said to be irrelevant if it is in the kernel of L. For a divisor $D = \sum a_i D_i$, write $D(v_i) = a_i$. Let K be subspace of irrelevant divisors.

Let $C\Gamma$ be the cone on Γ, considered as a 2-dimensional simplicial complex, called a fan. We identify v_i with a point on its corresponding ray. There is a piecewise linear map:

$$u : C\Gamma \to K^\vee$$

given by

$$v_i \mapsto (D \mapsto D(v_i)).$$

The Hodge index theorem constraint

Hodge Index Theorem: If Γ comes from a sufficient curve arrangement then L has a single negative eigenvalue.

This puts significant combinatorial constraints on Γ.

A divisor is said to be irrelevant if it is in the kernel of L. For a divisor $D = \sum a_i D_i$, write $D(v_i) = a_i$. Let K be subspace of irrelevant divisors.

Let $C\Gamma$ be the cone on Γ, considered as a 2-dimensional simplicial complex, called a fan. We identify v_i with a point on its corresponding ray. There is a piecewise linear map:

$$u : C\Gamma \to K^\vee$$

given by

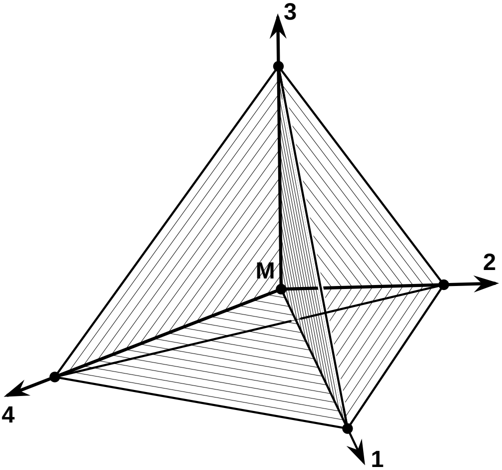
$$v_i \mapsto (D \mapsto D(v_i)).$$

The map obeys the balacning condition:

$$d(v_i)u(v_i) = \sum m(v_i, v_j)u(v_j).$$

The image of the map

For four lines in $\mathbb{P}^2$, we get the much-celebrated tropical plane in $\mathbb{R}^3$. It's the complex consisting of all six cones between four rays:



The natural example

The natural example of this map comes from tropical geometry. Suppose our surface X is a subvariety of a toric variety $X(\Delta)$. There's a very strong smoothness condition called *schönness*. This essentially means that the surface is smooth and (after possibly blowing-up $X(\Delta)$), X intersects every toric stratum transversely.

The natural example

The natural example of this map comes from tropical geometry. Suppose our surface X is a subvariety of a toric variety $X(\Delta)$. There's a very strong smoothness condition called *schönness*. This essentially means that the surface is smooth and (after possibly blowing-up $X(\Delta)$), X intersects every toric stratum transversely.

Then the intersection of X with the toric divisors gives a curve arrangement on X. Then there's a parameterizing complex $\Sigma = C\Gamma$ and a map $p : \Sigma \to \mathrm{Trop}(X) \subset \mathbb{R}^d$ factoring through u:

$$\begin{array}{ccc} \Sigma & \xrightarrow{\;u\;} & \mathrm{im}(u) \\ & \searrow^{p} & \downarrow{\scriptstyle \pi} \\ & & \mathrm{Trop}(X) \end{array}$$

where π is some linear projection.

Tropical map

In general, we cannot suppose the map is an embedding. By the way, the data of the map and the $m(v_i, v_j)$'s lets us recover the $d(v_i)$'s.

Tropical map

In general, we cannot suppose the map is an embedding. By the way, the data of the map and the $m(v_i, v_j)$'s lets us recover the $d(v_i)$'s.

Now, let's cut up the image by hyperplanes.

We view $D \in K$ as cutting out a hyperplane

$$H_D = \{\ell \in K^\vee | D(\ell) = 0\}$$

with open half-spaces H_D^+, H_D^-. Note here that $v_i \in H_D^+$ means $D(v_i) > 0$. Write $\Gamma_D^+, \Gamma_D^-, \Gamma_D^0$ for the subgraphs induced by vertices lying in H_D^+, H_D^-, H_D.

Half-space constraints

We have some cool results coming out of the theory of Colin de Verdiere numbers.

Lemma (Lovasz, Schrijver, van der Holst) If X_1 and X_2 are disjoint components of Γ_D^+ then there is $D' \in K$ such that $X_1 = \Gamma_{D'}^+, X_2 = \Gamma_{D'}^-$.

Half-space constraints

We have some cool results coming out of the theory of Colin de Verdiere numbers.

Lemma (Lovasz, Schrijver, van der Holst) If X_1 and X_2 are disjoint components of Γ_D^+ then there is $D' \in K$ such that $X_1 = \Gamma_{D'}^+, X_2 = \Gamma_{D'}^-$.

This is just a theorem in linear algebra.

Half-space constraints

We have some cool results coming out of the theory of Colin de Verdiere numbers.

Lemma (Lovasz, Schrijver, van der Holst) If X_1 and X_2 are disjoint components of Γ_D^+ then there is $D' \in K$ such that $X_1 = \Gamma_{D'}^+, X_2 = \Gamma_{D'}^-$.

This is just a theorem in linear algebra.

From this van der Holst showed:

Proposition: If Γ is planar than $\dim K \leq 3$.

More constraints

We can do even better!

Proposition (Lovasz and Scrijver): Let $D \in K$. If Γ_D^+ is disconnected then

1. there are no edges connecting Γ_D^+ and Γ_D^-,
2. Any component Y of Γ_D^+ or Γ_D^- satisfies $N(Y) = N(\Gamma_D^+ \cup \Gamma_D^-)$.

Here, N denotes the set of neighbours. This puts strong conditions on the graph if one side of it (wrt a hyperplane) is disconnected.

Huh's counterexample

June Huh was able to cook up a fan $\mathcal{F}$ embedded in $\mathbb{R}^4$ that violated the above proposition. Its underlying graph is a subgraph of $K_{4,4}$. Therefore, it cannot come from a surface. Moreover, one can view this fan as a tropical variety. That's a special kind of polyhedral complex. These polyhedral complexes often arise from tropicalizing algebraic varieties in $(\mathbb{C}^*)^n$.

Huh's counterexample

June Huh was able to cook up a fan $\mathcal{F}$ embedded in $\mathbb{R}^4$ that violated the above proposition. Its underlying graph is a subgraph of $K_{4,4}$. Therefore, it cannot come from a surface. Moreover, one can view this fan as a tropical variety. That's a special kind of polyhedral complex. These polyhedral complexes often arise from tropicalizing algebraic varieties in $(\mathbb{C}^*)^n$.

Proposition (Huh): The fan $\mathcal{F}$ is not the tropicalization of any surface in $(\mathbb{C}^*)^4$.

By examining the class, one shows it cannot be decomposed into smaller balanced fans with non-negative weights. Therefore, it must be the tropicalization of a reducible surface. Yet any irreducible surface would obey the Hodge index theorem. So it isn't a tropicalization!

Huh's counterexample (continued)

Using ideas from tropical geometry, one can interpret $\mathcal{F}$ as a cohomology class in a toric 4-fold. It is integral and of $(2,2)$-type. The same argument shows that the class can not be expressed as the non-negative sum

$$\sum \lambda_i X_i$$

where $\lambda_i \in \mathbb{R}^{\geq 0}$ and the X_i's are irreducible surfaces.

Huh's counterexample (continued)

Using ideas from tropical geometry, one can interpret $\mathcal{F}$ as a cohomology class in a toric 4-fold. It is integral and of $(2,2)$-type. The same argument shows that the class can not be expressed as the non-negative sum

$$\sum \lambda_i X_i$$

where $\lambda_i \in \mathbb{R}^{\geq 0}$ and the X_i's are irreducible surfaces.

Moreover, Babaee and Huh have recently related such cohomology classes to currents and used this to produce a counterexample to Demailly's strongly positive Hodge conjecture.

A tropical Hodge index theorem?

Question: Given an embedded fan $\mathcal{F}$ with weights m, when can we be sure that the tropical Laplacian has only one positive eigenvalue?

A tropical Hodge index theorem?

Question: Given an embedded fan $\mathcal{F}$ with weights m, when can we be sure that the tropical Laplacian has only one positive eigenvalue?

This is the tropical Hodge index theorem. The Lovasz-Schrijver-van der Holst results give necessary conditions. We have no idea what the sufficient conditions arel.

A tropical Hodge index theorem?

Question: Given an embedded fan $\mathcal{F}$ with weights m, when can we be sure that the tropical Laplacian has only one positive eigenvalue?

This is the tropical Hodge index theorem. The Lovasz-Schrijver-van der Holst results give necessary conditions. We have no idea what the sufficient conditions arel.

By some involved arguments, the tropical Hodge index theorem (with weak enough conditions) would imply the log-concavity of the characteristic polynomial of a matroid, resolving the Rota-Heron-Welsh conjecture. My undergraduate research student, Theo Belaire verified the tropical Hodge index theorem for certain fans coming from matroids of up to 9 elements. So maybe it's true. We're trying to prove it a different way.

A tropical Hodge index theorem?

Question: Given an embedded fan $\mathcal{F}$ with weights m, when can we be sure that the tropical Laplacian has only one positive eigenvalue?

This is the tropical Hodge index theorem. The Lovasz-Schrijver-van der Holst results give necessary conditions. We have no idea what the sufficient conditions arel.

By some involved arguments, the tropical Hodge index theorem (with weak enough conditions) would imply the log-concavity of the characteristic polynomial of a matroid, resolving the Rota-Heron-Welsh conjecture. My undergraduate research student, Theo Belaire verified the tropical Hodge index theorem for certain fans coming from matroids of up to 9 elements. So maybe it's true. We're trying to prove it a different way.

The difficulty is that the curve arrangement does not let us test a divisor well enough to tell that it is ample. This makes it difficult to port over a classical proof of the Hodge index theorem unless you have some additional structure.

A fun aside or a wasted year of my life

Just as one can build the Baker-Norine theory of linear systems on graphs (chip-firing) out of the usual graph Laplacian, one can build a theory of linear systems on tropical fan surfaces out of the tropical Laplacian.

A fun aside or a wasted year of my life

Just as one can build the Baker-Norine theory of linear systems on graphs (chip-firing) out of the usual graph Laplacian, one can build a theory of linear systems on tropical fan surfaces out of the tropical Laplacian.

I spent a year thinking about this in order to prove the tropical Hodge index theorem by first proving the tropical Riemann-Roch theorem for surfaces. This is WAY beyond our current technology.

A fun aside or a wasted year of my life

Just as one can build the Baker-Norine theory of linear systems on graphs (chip-firing) out of the usual graph Laplacian, one can build a theory of linear systems on tropical fan surfaces out of the tropical Laplacian.

I spent a year thinking about this in order to prove the tropical Hodge index theorem by first proving the tropical Riemann-Roch theorem for surfaces. This is WAY beyond our current technology.

Still, in this theory, the specialization lemma holds and you can bound the dimensions of linear systems on surfaces in terms of combinatorics. Because of the nature of the tropical Laplacian, this has a very different flavour from chip-firing. In fact, the number of chips isn't preserved but their centre of mass is.

Colin de Verdiere number

There is an invariant of graphs that is determined by looking at the maximum of dimension of kernels of matrices similar to the tropical Laplacian. It's called the Colin de Verdiere number, $\mu(\Gamma)$.

1. It is minor monotone,
2. By the Robertson-Seymour Graph Minor theorem, there are a finite list of forbidden minors for $\mu(\Gamma) \leq d$,

Colin de Verdiere number

There is an invariant of graphs that is determined by looking at the maximum of dimension of kernels of matrices similar to the tropical Laplacian. It's called the Colin de Verdiere number, $\mu(\Gamma)$.

1. It is minor monotone,
2. By the Robertson-Seymour Graph Minor theorem, there are a finite list of forbidden minors for $\mu(\Gamma) \leq d$,

For low values, we have nice descriptions.

1. $\mu(\Gamma) \leq 1$ if and only Γ is a path,
2. $\mu(\Gamma) \leq 2$ if and only Γ is a outerplanar,
3. $\mu(\Gamma) \leq 3$ if and only Γ is a planar,
4. $\mu(\Gamma) \leq 4$ if and only Γ is linklessly embeddable in $\mathbb{R}^3$.

If the tropical Laplacian satisfies a technical condition called the strong Arnold hypothesis, then L is such a matrix and $\dim K \leq \mu(\Gamma)$.

Lines in the plane

In the case of lines in the plane, we can say a lot more.

Here we understand the strong Arnold hypothesis. A line configuration satisfies SAH if and only no point is contained in more than 3 lines.

Lines in the plane

In the case of lines in the plane, we can say a lot more.

Here we understand the strong Arnold hypothesis. A line configuration satisfies SAH if and only no point is contained in more than 3 lines.

So we have a very complicated proof of

Theorem: The incidence graph of a line arrangement with $n+1$ lines such that each point is contained in no more than 3 lines satisfies $\mu(\Gamma) \geq n$.

Lines in the plane

In the case of lines in the plane, we can say a lot more.

Here we understand the strong Arnold hypothesis. A line configuration satisfies SAH if and only no point is contained in more than 3 lines.

So we have a very complicated proof of

Theorem: The incidence graph of a line arrangement with $n+1$ lines such that each point is contained in no more than 3 lines satisfies $\mu(\Gamma) \geq n$.

Moreover, the tropical Hodge index theorem is known for the fans coming from rank 3 matroids (which abstract line arrangements), so the analogue of this theorem is true for matroids. I suspect that this theorem can be proven in a combinatorial fashion by a density argument or by $\Delta - Y$ moves.