Combinatorial Abstractions and Tropicalization

Eric Katz (University of Waterloo)

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Hypersurfaces

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The hypersurface $V(f) \subset \mathbb{C}^n$ is the zero locus of f.

Example:

1. $x + y + 1 = 0$ is a line.
2. $y^2 - x^3 - x - 1 = 0$ is an elliptic curve.
3. $z^2 - x^2 - y^2 - 1 = 0$ is a conic surface.

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Bézout's Theorem: Let f, g be generic polynomials of two variables of degrees d and e respectively. Then $V(f), V(g) \subset \mathbb{P}^2_{\mathbb{C}}$ intersect in $d \cdot e$ points.
Here, generic means, for generic choice of coefficients. This theorem has a generalization for intersecting n hypersurfaces in $\mathbb{P}^n_{\mathbb{C}}$.

Newton polytope

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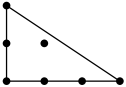
The Newton polytope of $y^2 - x^3 - x - 1$ is



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In the two-dimensional case, for two generic 2-variable polynomials f, g with given Newton polytopes, the intersection number of $V(f)$ and $V(g)$ in $(\mathbb{C}^*)^2$ is

$$\mathrm{Vol}(P(f) + P(g)) - \mathrm{Vol}(P(f)) - \mathrm{Vol}(P(g))$$

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By results of Danilov-Khovanskii, one can compute the Euler characteristic $\chi_c(V(f))$ for generic hypersurfaces for a given Newton polytope. More specifically, one can compute the Hodge polynomial for the mixed Hodge structure on $H^*_c(V(f))$.

Projective Subspaces

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Let $\mathbb{P}^n = \mathbb{P}(\mathbb{C}^{n+1})$ be projective space with a choice of basis $\vec{e}_0, \ldots, \vec{e}_n \in \mathbb{C}^{n+1}$. Let $V^r \subset \mathbb{P}^n$ be a projective subspace not contained in any coordinate subspace. Consider the hyperplane arrangement complement

$$V \setminus (H_0 \cup \cdots \cup H_n),$$

where $H_0, \ldots, H_n$ are the coordinate hyperplanes. We may want to compute its Euler characteristic or some of its Hodge-theoretic invariants. The compactly supported cohomology of this space is determined by a combinatorial encoding of the projective subspace called a matroid.

Matroids

Let L_I be the coordinate subspace given by

$$L_I = \{x_{i_1} = x_{i_2} = \cdots = x_{i_l} = 0\}$$

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4. $\rho(\{0, \ldots, n\}) = r + 1.$

Matroids

Note: Item (3) abstracts

$$\operatorname{codim}(((V \cap L_I) \cap (V \cap L_J)) \subset (V \cap L_{I \cap J})) \leq$$

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This is one of the definitions of matroids. There are many others.

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2. Over $\mathbb{Q}$, an algorithm to determine representability is equivalent to Diophantine decidability algorithm over $\mathbb{Q}$ which is open but thought to be impossible.
3. It is a conjecture of Rota to characterize $\mathbb{F}_q$-representable matroids in terms of forbidden minors ($\mathbb{F}_2$ due to Tutte; $\mathbb{F}_3$ due to Seymour; $\mathbb{F}_4$ due to Geelen-Gerards-Kapoor).

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The set $\mathrm{Log}(X)$ is said to be the amoeba of X.

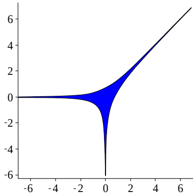

Figure: The amoeba of the line $\{z_1 + z_2 - 1 = 0\} \subset (\mathbb{C}^*)^2$.

The tentacles correspond to

1. $z_1 \to 0, \ z_2 \to 1$,
2. $z_2 \to 0, \ z_1 \to 1$,
3. $|z_1| \to \infty$.

Tropicalizations

To get something combinatorial, we need to look at the tropicalization which is the limit set

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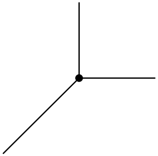

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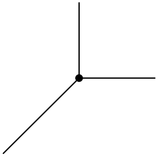

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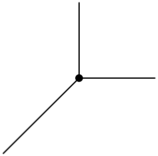

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In practice, the logarithmic limit set definition is mostly unusable, and it's more pleasant to use a purely algebraic definition.

Tropicalizations of Families

We may also consider the tropicalization of a family of varieties X_t parameterized by $t \in \mathbb{C} \setminus \{0\}$. In this case,

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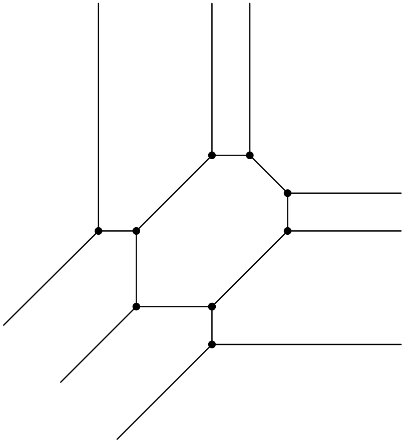
Example: Consider a family of cubic curves $V(f_t) \subset (\mathbb{C}^*)^2$ where

$$f_t = \sum_{\substack{0 \le i,j \le 3 \\ i+j \le 3}} a_{ij} x^i y^j$$

for $a_{ij} \in \mathbb{C}[t, t^{-1}] \setminus \{0\}$.
The limit may have many different combinatorial types but below is one possibility.

A cubic curve in the plane



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Integral: Each polyhedral cell is cut out by linear inequalities with rational coefficients.
Weighted: Each top-dimensional cell has a weight $w(P) \in \mathbb{N}$. (in almost all of our examples, it will be 1.)

In general (cont'd)

Balanced: For 1-dimensional varieties, it's easy to state For v, a vertex of Σ and adjacent edges $E_1, \ldots, E_k$ in primitive $\mathbb{Z}^n$ directions, $\vec{u}_1, \ldots, \vec{u}_k$ then

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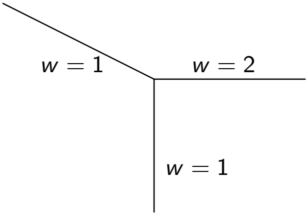

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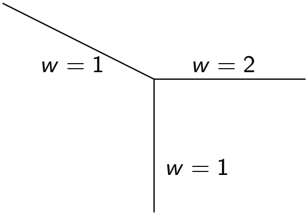

For higher dimensions, the balancing condition is analogous.

Tropicalization compared to Newton polytope

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The normal fan is made up of cones dual to the faces of the polytope. A cone dual to a face F is the set of all linear functionals on $\mathbb{R}^n$ that achieve their minimum on F. The codimension 1 skeleton means that we look at cones dual to positive dimensional faces.

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Theorem (Sturmfels, Ardila-Klivans): Let $V \subset \mathbb{P}^n$ be a projective subspace. Then $\text{Trop}(V \cap (\mathbb{C}^*)^n)$ is determined by the matroid $\mathbb{M}$ of V.

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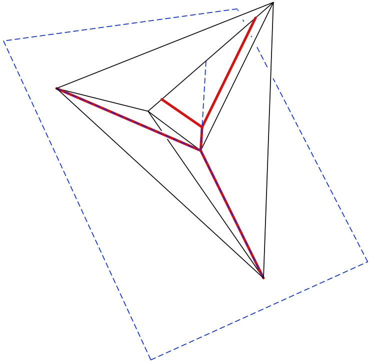
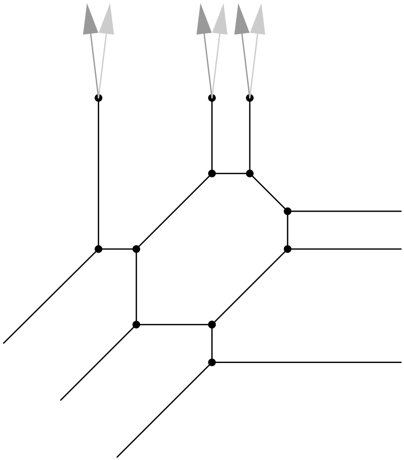
Now let's look at some pictures.

Tropicalization of a family of lines in the tropicalization of a plane in space



An elliptic curve in a plane in space



All multiplicities are 1. There are arrows pointing into and out of the screen to ensure balancing.

Properties encoded in tropicalization

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Some Intersection Theory:
It knows the degree of the variety.
Given two varieties $X, Y \subset (\mathbb{C}^*)^n$ with $\dim(X) + \dim(Y) = n$, we can also read off an expected intersection number under genericity assumptions.
This is a generalization of Bernstein's theorem due to K., Osserman-Payne, Rabinoff in different degrees of generality.

Properties encoded in tropicalization (cont'd)

Some Hodge Theory: For $X \subset (\mathbb{C}^*)^n$ satisfying genericity assumptions, we can look at $H^*(X)$. This has a mixed Hodge structure. The lowest weight bit is described by $H^*(\mathrm{Trop}(X))$ by a theorem of Hacking. For families, the analogous result is due to Helm-K.

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Under certain assumptions, the tropical variety knows much much more about the original variety. This is when the tropical variety locally looks like the tropicalization of a linear subspace. These are the so-called smooth tropical varieties. Results due to Itenberg-Kazarkov-Mikhalkin-Zharkov and K.-Stapledon.

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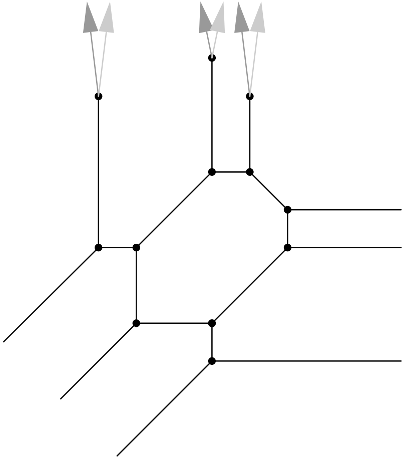
Here is an example of a non-liftable graph due to Mikhalkin and Speyer.

Example of non-liftable curve

Change the length of a bounded edge in the spatial elliptic curve so that it does not lie on the tropicalization of any plane (possible by dimension counting).



Example of non-liftable curve (cont'd)

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no lift of the curve can be planar or genus 0, so the curve does not lift.

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4. There's an interesting example due to Vigeland of a curve C and a surface S in $(\mathbb{C}^*)^3$ where $\text{Trop}(C) \subset \text{Trop}(S)$ but it's impossible to change C, S to ensure $C \subset S$ without changing the tropicalizations. This makes enumerating curves on surfaces through tropical geometry tricky. This class of examples has been studied by Bogart-K., Brugallé-Shaw, Gathmann-Winstel.

Pathological curve in a surface