Hodge theory in combinatorics

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"But Hodge shan't be shot; no, no, Hodge shall not be shot."
– Samuel Johnson

The characteristic polynomial of a subspace

Let $\mathbf{k}$ be a field. Let $V \subset \mathbf{k}^{n+1}$ be an $(r+1)$-dim linear subspace not contained in any coordinate hyperplane. Would like to use inclusion/exclusion to express $[V \cap (\mathbf{k}^*)^{n+1}]$ as a linear combination of $[V \cap L_I]$'s where L_I is the coordinate subspace given by

$$L_I = \{x_{i_1} = x_{i_2} = \cdots = x_{i_l} = 0\}$$

for $I = \{i_1, i_2, \ldots, i_l\} \subset \{0, \ldots, n\}$.

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Example: Let V be a generic subspace (intersecting every coordinate subspace in the expected dimension). Then

$$[V \cap ((\mathbf{k}^*)^{n+1})] = [V \cap L_\emptyset] - \sum_i [V \cap L_i] + \sum_{\substack{I \\ |I|=2}} [V \cap L_I] - \sum_{\substack{I \\ |I|=3}} [V_\cap L_I] + \ldots.$$

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If you're fancy, you can say that this is a motivic expression.

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We can now write for some choice of $\nu_I \in \mathbb{Z}$,

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Fact: $(-1)^{\rho(I)} \nu_V$ is always positive.

Characteristic Polynomial

Definition

The characteristic polynomial of V is

$$\begin{aligned} \chi_V(q) &= \sum_{i=0}^{r+1} \left(\sum_{\substack{\text{flats } I \\ \rho(I)=i}} \nu_I \right) q^{r+1-i} \\ &\equiv \mu_0 q^{r+1} - \mu_1 q^r + \cdots + (-1)^{r+1} \mu_{r+1} \end{aligned}$$

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We can think of χ as an evaluation of the classes $[V \cap L_I]$ of the form

$$[V \cap L_I] \mapsto q^{r+1-\rho(I)}$$

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Example: In the generic case subspace case, we have

$$\chi_V(q) = q^{r+1} - \binom{r+1}{1} q^r + \binom{r+1}{2} q^{r-1} - \cdots + (-1)^{r+1} \binom{r+1}{r+1}.$$

Rota-Heron-Welsh Conjecture

Theorem (Rota-Heron-Welsh Conjecture (in the realizable case) (Huh-k '11))

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A polynomial with coefficients $\mu_0, \ldots, \mu_{r+1}$ is said to be unimodal if the coefficients are unimodal in absolute value, i.e. there is a j such that

$$|\mu_0| \leq |\mu_1| \leq \cdots \leq |\mu_j| \geq |\mu_{j+1}| \geq \cdots \geq |\mu_{r+1}|.$$

Motivation:Chromatic Polynomials of Graphs

Original Motivation: Let Γ be a loop-free graph. Define the chromatic function χ_Γ by setting $\chi_\Gamma(q)$ to be the number of colorings of Γ with q colors such that no edge connects vertices of the same color.

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Read's Conjecture '68 (Huh '10): $\chi_\Gamma(q)$ is unimodal.

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Note: Item (3) abstracts

$$\operatorname{codim}(((V \cap L_I) \cap (V \cap L_J)) \subset (V \cap L_{I \cap J})) \leq$$

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This is one of the definitions of matroids.

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Conjecture: For any matroid, $\chi(q)$ is log-concave.

We think we have it! We're writing it up now.

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We make a linear change of variables for the packaging of the f_k's: define h_k by

$$\sum_{i=0}^{d} f_{i-1}(t-1)^{d-i} = \sum_{k=0}^{d} h_k t^{d-k}.$$

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Here the Dehn-Sommerville relations say that the h_k's form a symmetric sequence:

$$h_k = h_{d-k}.$$

Stanley-Reisner rings

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This statement is implied by a statement in commutative algebra about Stanley-Reisner rings. Let Δ be the boundary of P, considered as a simplicial complex. Let $v_1, \ldots, v_n$ be the vertices of P. Introduce variables $x_1, \ldots, x_n$. For a field $\mathbf{k}$, let

$$I_\Delta \subset \mathbf{k}[x_1, \ldots, x_n]$$

be the non-face ideal. This is defined as follows: for $S \subset \{1, \ldots, n\}$ let

$$x^S = \prod_{i \in S} x_i,$$

then

$$I_\Delta = \langle x^S \mid S \text{ is not a face of } P \rangle.$$

Lefschetz elements

The Stanley-Reisner ring is

$$\mathbf{k}[\Delta] = \mathbf{k}[x_1, \ldots, x_n]/I_\Delta.$$

Because I_Δ is a homogeneous ideal, $\mathbf{k}[\Delta]$ is a graded ring. Now let $l_1, \ldots, l_d$ be generic degree 1 elements of $\mathbf{k}[\Delta]$. Then

$$\dim(\mathbf{k}[\Delta]/(l_1, \ldots, l_d))_i = h_i.$$

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The lower bound theorem is reduced to the existence of a weak Lefschetz element $\omega \in \mathbf{k}[\Delta]$ for which the multiplication map

$$\cdot\omega : (\mathbf{k}[\Delta]/(l_1, \ldots, l_d))_{i-1} \to (\mathbf{k}[\Delta]/(l_1, \ldots, l_d))_i$$

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Note here that the unimodality of h_i's is different from the unimodality of the characteristic polynomial as the characteristic polynomial is not symmetric. We have no idea where the mode is supposed to be.

Hard algebraic geometry but...

The existence of the Lefschetz element comes form identifying the quotient $\mathbf{k}[\Delta]/(l_1, \ldots, l_d)$ with the cohomology of a projective algebraic variety $X \subset \mathbb{P}^n$, that is $h_i = \dim H^{2i}(X)$. This variety, a toric variety, is mildly singular, but the Hard Lefschetz theorem gives a Lefschetz element. So the result relies on hard algebraic geometry, but

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McMullen's proof uses an alternative presentation of the Stanley-Reisner ring. Then, he applies flip moves to transform P into a simplex where the Hard Lefschetz theorem is known to hold, checking that the Hard Leschetz theorem is preserved by these moves.

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Incidentally, the presentations should be thought of in the following way: the Stanley-Reisner presentation is homology under intersection product; the Minkowski weight ring (used by McMullen) is cohomology; the conewise polynomial ring (used by Karu) is a quotient of equivariant cohomology.

Related work

I should mention that there is recent, related work by Ben Elias and Geordie Williamson proving the Hard Lefschetz theorem in a synthetic context. They are interested in questions involving the positivity of Kazhdan-Lusztig polynomials and the Kazhdan-Lusztig conjecture in the context of Coxeter systems.

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These theorems were proven in the case of Weyl groups by studying the intersection cohomology of a Schubert variety.

In general, there may be no Schubert variety, so certain modules act as an abstract avatar. They prove that these modules have the required Hodge theoretic properties.

Now some hard algebraic geometry

Let us delve into the hard algebraic geometry. I will discuss two theorems, the Hard Lefschetz theorem, and the Hodge Index theorem, and will explain how they are implied by an even deeper theorem, the Hodge-Riemann-Minkowski relations.

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Let $X \subset \mathbb{P}^n$ be a smooth projective d-dimensional algebraic variety. The cohomology ring $H^*(X)$ is a graded ring in degrees $0, 1, \ldots, 2d$. It's an algebra over $\mathbb{C}$. We think of $H^i(X)$ as the group of codimension i cycles in X. Now $H^{2d}(X) \cong \mathbb{C}$ is generated by the class of a point.

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There is a Hodge decomposition:

$$H^k(X) = \bigoplus_{p+q=k} H^{p,q}(X)$$

Hard Lefschetz theorem

If H is a generic hyperplane in $\mathbb{P}^n$, $H \cap X$ gives a codimenison 2 cycle in X, hence an element of $H^2(X)$. The Hard Lefschetz Theorem shows that H is a strong Lefschetz element:

Theorem (Hodge)

Let $L : H^k(X) \to H^{k+2}(X)$ be given by multiplication by H. Then for all $k \leq d$,

$$L^{d-k} : H^k(X) \to H^{2d-k}(X)$$

is an isomorphism.

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This implies the unimodality of h_{2i}'s.

Lefschetz decomposition

The Hard Lefschetz theorem gives the Lefschetz decomposition of cohomology: define primitive cohomology $P^k \subset H^k(X)$ by

$$P^k = \ker(L^{d-k+1} : H^k(X) \to H^{2d-k+2}(X)).$$

Then

$$H^k(X) = P^k \oplus LP^{k-2} \oplus L^2P^{k-4} \oplus \dots.$$

The Hodge index theorem

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Let X be a projective complex surface (2 complex dimensions, 4 real dimensions). Consider $H^2(X)$ equipped with intersection product

$$H^2(X) \otimes H^2(X) \to H^4(X) \cong \mathbb{C}.$$

Theorem (Hodge)

The intersection product restricted to $H^{1,1}(X)$ is non-degenerate with a single positive eigenvalue.

The Hodge inequality

This implies the Hodge inequality:

Corollary

Let $\alpha, \beta \in H^{1,1}(X)$ be given by pulling back a hyperplane class from two embeddings $i_1, i_2 : X \to \mathbb{P}^{n_i}$. Then

$$(\alpha^2)(\beta^2) \leq (\alpha \cdot \beta)^2.$$

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This comes from the intersection product being indefinite on $\mathrm{Span}(\alpha, \beta)$ so the discriminant is negative. Note we can replace α and β by positive multiples (ample classes). Or look at classes that can be approximated by hyperplane classes (nef).

Hodge-Riemann-Minkowski Relations

An even stronger theorem holds for algebraic varieties in all dimensions.

Theorem

Let α be an ample class. Let P^* be the primitive cohomology with respect to α. Then the pairing $Q_{p,q}$ on

$$H^{p,q}_{\text{prim}} = P^{p+q}(X) \cap H^{p,q}(X)$$

given by

$$Q_{p,q}(\beta, \gamma) = (-1)^{\frac{(p+q)(p+q-1)}{2}} i^{p-q-k} (\beta \cdot \gamma \cdot \alpha^{d-(p+q)})$$

is positive definite.

This is deep and analytic.

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Theorem

Let α be an ample class. Let P^* be the primitive cohomology with respect to α. Then the pairing $Q_{p,q}$ on

$$H^{p,q}_{\text{prim}} = P^{p+q}(X) \cap H^{p,q}(X)$$

given by

$$Q_{p,q}(\beta,\gamma) = (-1)^{\frac{(p+q)(p+q-1)}{2}} i^{p-q-k} (\beta \cdot \gamma \cdot \alpha^{d-(p+q)})$$

is positive definite.

This is deep and analytic.

In the sequel, we will restrict to $H^{p,p}$ so

$$Q_{p,p}(\beta,\gamma) = (-1)^p (\beta \cdot \gamma \cdot \alpha^{d-2p}).$$

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The Hodge-Riemann-Minkowski relations immediately imply the Hard Lefschetz theorem. They also imply the Hodge index theorem:

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$$H^{1,1}(X) = LH^{0,0}_{\text{prim}}(X) \oplus H^{1,1}_{\text{prim}}(X).$$

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More generally, we get the Khovanskii-Teissier inequality: for α, β nef

$$(\alpha^{r-i+1}\beta^{i-1})(\alpha^{r-i-1}\beta^{i+1}) \leq (\alpha^{r-i}\beta^{i})^2.$$

Proof of Log-concavity

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Coefficients of $\overline{\chi}$ have a combinatorial description:

$$\overline{\chi}_V(q) = \mu^0 q^r - \mu^1 q^{r-1} + \cdots + (-1)^r \mu^r q^0.$$

Then

$$\mu^i = (-1)^i \sum_{\substack{\text{flats } I \\ \rho(I)=i \\ 0 \notin I}} \nu_I.$$

A new Stanley-Reisner ring

We define a Stanley-Reisnerish ring attached to the matroid:

Definition

Let x_F be indeterminates indexed by proper flats. Let I_M be the ideal in $\mathbf{k}[x_F]$ generated by

1. For each $i, j \in \{0, 1, \ldots, n\}$,

$$\sum_{F \ni i} x_F - \sum_{F \ni j} x_F,$$

2. For incomparable flats F, F',

$$x_F x'_F.$$

Let $R_M = \mathbf{k}[x_F]/I_M$.

This is the Stanley-Reisner ring of the order complex of the lattice of flats of the matroid quotiented by a linear ideal. Henceforth, let us take $\mathbf{k} = \mathbb{C}$.

Properties of the ring

There is a canonical isomorphism

$$\deg : (R_M)_r \to \mathbb{C}$$

that takes the value 1 on an ascending chain of flats $x_{F_1} \ldots x_{F_r}$.

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$$\begin{aligned} \alpha &= \sum_{F \ni i} x_F \\ \beta &= \sum_{F \not\ni i} x_F . \end{aligned}$$

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We have the equality

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Aside: We proved this using tropical intersection theory. You can give a direct proof in this presentation.

Hodge-Riemann-Minkowski holds

Theorem

If M is realizable over $\mathbb{C}$, there is an algebraic variety $\widetilde{V}$ with $H^{2*}(\widetilde{V}) = R_M$. The classes α and β are nef on $\widetilde{V}$ and the Hodge-Riemann-Minkowski relations hold for suitably perturbed α and β.

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So HRM implies the log-concavity of the μ^i's by the Hodge inequality. This implies the log-concavity of the μ_i's.

The same argument holds over fields besides $\mathbb{C}$. One has to use a different derivation of the Khovanskii-Teissier inequality making use of Kleiman's transversality.

The space $\widetilde{V}$

The space $\widetilde{V}$ is natural. Start with $V \subset \mathbb{C}^{n+1}$. Projectivize to get $\mathbb{P}(V) \subset \mathbb{P}^n$. The coordinate hyplerplanes of $\mathbb{P}^n$ induce a hyperplane arrangement on $\mathbb{P}(V)$. We blow-up the 0-dimensional strata, and then the proper transforms of the 1-dimensional strata, and so on to produce $\widetilde{V}$.

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The space $\widetilde{V}$ lives in a blown-up projective space $\widetilde{\mathbb{P}}^n$ which has two natural maps to $\pi_1, \pi_2 : \widetilde{\mathbb{P}}^n \to \mathbb{P}^n$. Think: it resolves a Cremona transform. Then $\alpha = \pi_1^* H,\ \beta = \pi_2^* H$.

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We perturb α and β so that they are ample. We get an inequality and then take limits.

We made this argument combinatorial!

Every time I've given a talk about log-concavity, I've asked if this result can be made purely combinatorial and thus prove Rota-Heron-Welsh. Every time, I've suggested some approach. I've even made jokes about the failures of these approaches.

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Our idea is to start with projective space and do each blow-up one-by-one in a purely combinatorial fashion to produce intermediate Stanley-Reisner rings. We also have intermediate analogues of α, β. We have to show that the Hodge-Riemann-Minkowski relations (with respect to a "combinatorial ample cone") are preserved by our blow-ups. We have a geometric picture in mind of slicing faces off of a simplex to get a permutohedron.

Outline of proof

The proof has several steps making use of an inductive argument used by McMullen and Karu and elevated to a cornerstone of Hodge theory by de Cataldo and Migliorini:

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Outline of proof (cont'd)

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It is exactly as difficult as giving a purely (linear) algebraic proof of the following:

Theorem

Let X is a smooth projective variety with ample divisor H. Let Z be a smooth subvariety. Suppose that X and Z satisfy the Hodge-Riemann-Minkowski relations. Then $\mathrm{Bl}_Z X$ satisfies the Hodge-Riemann-Minkowski relations with respect to $H - \epsilon E$ where E is the exceptional divisor and $\epsilon > 0$.

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Here, a perturbation argument suffices.

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A Stanley-Reisner ring modulo a linear ideal, $\mathbb{R}[\Delta]/(l_1, \ldots, l_d)$ is said to have an r-dimensional fundamental class if there an isomorphism

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To every degree 1 generator is associated a ray. To every square-free monomial not in I_Δ (thus a face) is associated a cone. The top-dimensional cones are given a weight by looking at the value of their corresponding monomial under deg. The linear ideal generated an embedding into $\mathbb{R}^d$ for which the fan is balanced.

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This procedures produces the face fan from the S-R ring of a polytope. It produces the Bergman fan from the S-R ring of a matroid.

Thanks!

Huh, June and K, *Log-concavity of characteristic polynomials and the Bergman fan of matroids*.

Huh, June. *Milnor numbers of projective hypersurfaces and the chromatic polynomial of graphs*.