The Hodge theory of hypersurfaces

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Hypersurfaces

Let $f \in \mathbb{C}[z_1, \ldots, z_n]$ be a *generic* (to be defined later) polynomial in n variables. We can define the hypersurface $Z_f \subset (\mathbb{C}^*)^n$ cut out by $f = 0$.

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It is useful to phrase the decomposition in terms of a decreasing filtration

$$F^0 = H^k \supset F_1 \supset \cdots \supset F_k$$

such that

$$h^{p,q} = \dim \mathrm{Gr}^p_F(H^k).$$

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This implies that there is an increasing filtration W and a decreasing filtration F on H^k such that the associated gradeds with respect to W have a pure Hodge structure. We define

$$h^{p,q}(H^k(Z)) = \dim \operatorname{Gr}_F^p \operatorname{Gr}_{p+q}^W (H_c^k(Z)).$$

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$$h^{p,q}(H^k(Z)) = \dim \operatorname{Gr}^p_F \operatorname{Gr}^W_{p+q}(H^k_c(Z)).$$

Warning: Note that we may have $h^{p,q}(H^k(Z)) \neq 0$ even though $p + q \neq k$. So there's a lot more data.

Danilov-Khovanskii's approach

To throw out some of the excess data, we take the Hodge-Deligne numbers

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Therefore one may compactify $(\mathbb{C}^*)^n$ to the toric variety X_P given by the Newton polytope of f. Let $\overline{Z}$ be the closure of Z in X_P. One can remove the stuff that we added later. Now, we can define the genericity of f which means that f is generic among polynomials with Newton polytope P so that the strata of $\overline{Z}$ are smooth.

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Secondly, one has a Lefschetz hyperplane theorem: for $p, q > n-1$,

$$e^{p,q}(Z) = e^{p+1,q+1}((\mathbb{C}^*)^n).$$

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The specialization $E(Z;u,1)$ can be computed by taking the Euler characteristic of an ideal sheaf sequence (twisted by differentials) together with an adjunction exact sequence.
We end up getting

$$uE(V(P)^\circ;u,1)=(u-1)^{\dim P}+(-1)^{\dim P+1}h^*_P(u).$$

Batyrev-Borisov formula

Danilov-Khovanskii provide an algorithm for finding $e^{p,q}$. Much later, Batyrev-Borisov gave an explicit formula (inspired by intersection cohomology) in terms of the face-poset of P:

$$\begin{aligned} E(Z;u,v) &= (1/uv)[(uv-1)^{d+1} \\ &\quad +(-1)^d \sum_{Q\subseteq P} u^{\dim Q+1}\tilde{S}(Q,u^{-1}v)G([Q,P]^*,uv)]. \end{aligned}$$

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Naive Question: Is the machinery of $\tilde{S}$ a combinatorial abstraction of the resolution of singularities for the dual fan of P?

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If we make the consider $tf(t^{-1}x_1, t^{-1}x_2) = t + x_1 + x_2 + x_1x_2$ and set $t = 0$, we get $f_2(x_1, x_2) = x_1 + x_2 + x_1x_2$, so a line in a different $\mathbb{P}^2$.

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Now, the ambient $\mathbb{P}^1 \times \mathbb{P}^1$ degenerates to two $\mathbb{P}^2$'s joined along a line. Our curve degenerates into two twice-punctured lines joined along a node.

Monodromy Filtration

In general, if we have a family Z_t, there is an additional filtration on the cohomology. View the family over the punctured disc. The cohomology $H^*(Z_t)$ gives a locally trivial fiber bundle over the punctured disc. Consequently, one can parallel transport around the puncture. This gives a monodromy operation $T : H^*(Z_t) \to H^*(Z_t)$. By possibly replacing T by T^m for some $m \in \mathbb{Z}_{\geq 1}$, we can suppose T is unipotent. Set $N = \log(T)$ which is nilpotent. There is an additional filtration coming from the Jordan decomposition of N.

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If Z_t were compact, then one could put an increasing monodromy filtration M on $H^k(Z_t)$,

$$0 \subseteq M_0 \subseteq M_1 \subseteq \cdots \subseteq M_{2k} = H^k(Z),$$

with associated graded pieces $\mathrm{Gr}_l^M := M_l/M_{l-1}$, satisfying the following properties for any non-negative integer l,

1. $N(M_l) \subseteq M_{l-2}$,
2. the induced map $N^l : \mathrm{Gr}_{k+l}^M \to \mathrm{Gr}_{k-l}^M$ is an isomorphism.

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(A twist of) the monodromy filtration has the above properties on the W-associated gradeds.

This gives us tons of structure. We can refine the Hodge numbers even further:

$$h^{p,q,r}(Z)_k = \dim(\mathrm{Gr}^p_F \, \mathrm{Gr}^{M(r)}_{p+q} \, \mathrm{Gr}^W_r \, H^k(Z)).$$

and form refined Hodge-Deligne numbers:

$$e^{p,q,r}(Z) = \sum (-1)^k h^{p,q,r}(Z)_k.$$

Specializations

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Observation: $E(Z_{\text{gen}}; u, 1) = E(Z_\infty; u, 1)$ since this forgets both M and W.

Degeneration formula

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Now we need to introduce the Newton subdivision associated to a degenerating hypersurface. Let $f \in \mathbb{C}((t))[x_1, \ldots, x_n]$. Write

$$f = \sum a_{\mathbf{u}} x^{\mathbf{u}}.$$

For $a_{\mathbf{u}} \in \mathbb{C}((t))$, let $\mathrm{val}(\mathbf{u})$ be the smallest exponent of t with non-zero coefficient. Consider the function

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The upper hull is the convex hull of all points lying above the graph of this function. Its lower faces induces a subdivision of P.

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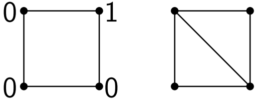

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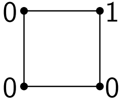

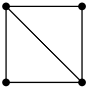

Now, we have the following degeneration formula which follows from the spectral sequence of Steenbrink or the motivic nearby fiber of Bittner:

Theorem (K-Stapledon)

$$E((Z_P)_\infty; u, v) = \sum_{\mathrm{Int}(Q) \subseteq \mathrm{Int}(P)} E(Z_Q; u, v)(1 - uv)^{\mathrm{codim}\, Q}.$$

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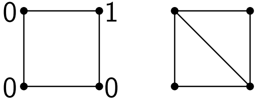

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This gives the specialization

$$E(Z_P; u, 1) = \sum_{\mathrm{Int}(Q) \subseteq \mathrm{Int}(P)} E(Z_Q; u, 1)(1 - u)^{\mathrm{codim}\, Q}.$$

Determining $E(Z_P; u, 1)$

This formula lets us identify $E(Z_P; u, 1)$

Definition: Let $\mathcal{P}_{\mathbb{Z}^n}$ be the set of convex lattice polytopes in $\mathbb{Z}^n$. A unimodular valuation on $\mathcal{P}_{\mathbb{Z}^n}$ is a map $\phi : \mathcal{P}_{\mathbb{Z}^n} \to \mathbb{R}$ satisfying

1. $\phi(P \cup Q) + \phi(P \cap Q) = \phi(P) + \phi(Q)$ whenever $P, Q, P \cup Q, P \cap Q \in \mathcal{P}_{\mathbb{Z}^n}$,
2. $\phi(\emptyset) = 0$, and
3. $\phi(P) = \phi(UP + u)$ for $P \in \mathcal{P}_{\mathbb{Z}^n}$, $U \in \mathrm{Sl}_n(\mathbb{Z})$, $u \in \mathbb{Z}^n$.

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The degeneration formula allows us to verify

Lemma The following function is a unimodular valuation

$$P \mapsto \frac{E(Z_P; u, 1)}{(u-1)^{\dim P + 1}}$$

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3. $\phi(P) = \phi(UP + u)$ for $P \in \mathcal{P}_{\mathbb{Z}^n}$, $U \in \mathrm{Sl}_n(\mathbb{Z})$, $u \in \mathbb{Z}^n$.

Valuations are determined by their values on unimodular simplices.

The degeneration formula allows us to verify

Lemma The following function is a unimodular valuation

$$P \mapsto \frac{E(Z_P; u, 1)}{(u-1)^{\dim P + 1}}$$

We obtain Danilov-Khovanskii's formula by checking that the right-hand side is a unimodular valuation and showing that the formula is true for unimodular simplices by (easy) explicit computation.

Refining $\tilde{S}(P)$

We may also use this machinery to refine $\tilde{S}(P,t)$. By Batyrev-Borisov's formula, we have the following formula for coefficients of $\tilde{S}(P,t)$:

$$\tilde{S}(P)_{p+1} = h^{p,n-1-p}(H^{n-1}_{c,\mathrm{na}}((Z_P)_{\mathrm{gen}}))$$

where na refers to the non-ambient cohomology, the cokernel of the map

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We have

$$\tilde{S}(P)_{p+1} = \sum_q h^{p,q,n-1}(H^{n-1}_{c,\text{na}}(Z_f)).$$

Note that the right-hand side depends on the Newton subdivision of P.

Structure of $\tilde{S}(P)$

Now, by the structure of the monodromy filtration, the sequence $\{h^{l+i,i,k}(H^{n-1}_{c,\text{na}}(Z_P)) | 0 \leq i \leq k-l\}$ is symmetric and unimodal. This decomposes the coefficients of $\tilde{S}(P)$ into the sum of symmetric and unimodal sequences. If we can show that some of them vanish, then we can get inequalities for $\tilde{S}$.

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For example, if P admits a regular, unimodalar lattice triangulation, then the refined limit mixed Hodge numbers are concentrated in (p,p). In this case $\tilde{S}(P)$ is symmetric and unimodal.

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Question: Are local h-vectors a combinatorial abstraction of semistable reduction?

Thanks!

Vladimir Danilov and Askold Khovanskii, *Newton polyhedra and an algorithm for calculating Hodge-Deligne numbers*.

K. and Alan Stapledon, *The tropical motivic nearby fiber and the Hodge theory of hypersurfaces.* in preparation.