Lifting Tropical Curves and Linear Systems on Graphs

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1. Usual answer: geometry over the tropical semifield.
2. My answer: the combinatorial study of degenerations and stratifications of algebraic varieties.

I will not precisely define all the terms in my answer but I will give you an example of it.

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I begin with tropical algebraic geometry where the algebraic varieties are piecewise-linear objects.

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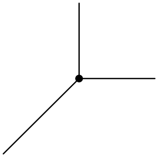

Valuation-theoretic approach

There is an algebraic approach to tropical geometry due to Kapranov. Let $\mathbb{K} = \mathbb{C}\{\{t\}\} = \overline{\mathbb{C}((t))}$, the field of formal Puiseux series. It is the algebraic closure of the field of formal Laurent series.

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Elements of $\mathbb{K}$ are of the form

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Non-Archimedean: $v(x+y) \geq \min(v(x), v(y))$, $v(xy) = v(x) + v(y)$.

Tropicalization

The Cartesian product $(\mathbb{K}^*)^n$ is called an algebraic torus. (In complex case, $(\mathbb{C}^*)^n$ is the natural analog of $(S^1)^n$.) An algebraic variety in $(\mathbb{K}^*)^n$ is the common zero locus of a system of Laurent polynomials in n variables with coefficients in $\mathbb{K}$.

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Tropicalization is a procedure that takes subvarieties of an algebraic torus to polyhedral complexes. The tropicalization of a variety $X \subset (\mathbb{K}^*)^n$ is defined to be

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Question: Why is this even reasonable?

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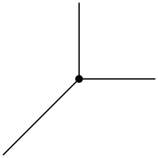

and, in fact, is equal by a theorem due to Kapranov.

Kapranov's theorem

Theorem (Kapranov) If f is a Laurent polynomial in $x_1, \ldots, x_n$ with support set $\mathcal{A} \subset \mathbb{Z}^n$,

$$\begin{aligned} f &= \sum_{\omega \in \mathcal{A}} a_\omega x^\omega \\ \mathrm{trop}(f) &= \bigoplus_{\omega \in \mathcal{A}} v(a_\omega) \odot x^{\odot \omega}. \end{aligned}$$

Let $Z(f) \subset (\mathbb{K}^*)^n$ be the zero-locus of f. Then $\mathrm{Trop}(Z(f))$ is equal to the tropical zero-locus of $\mathrm{trop}(f)$.

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So the valuation definition generalizes the min-plus definition in the case of hypersurfaces. This lets you talk about the tropicalization of higher codimensional subvarieties.

Tropicalization of curves

Tropicalization map:

$$\text{Trop} : \{\text{curves } C \subset (\mathbb{K}^*)^n\} \to \{\text{tropical graphs } \Sigma = \text{Trop}(C) \subset \mathbb{R}^n\}$$

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Integral: Each edge is a line-segment or a ray parallel to $\vec{u} \in \mathbb{Z}^n$.

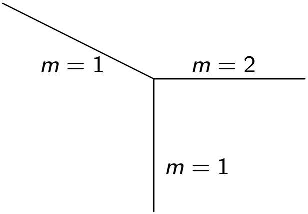
Weighted: Each edge has a weight (multiplicity) $m(E) \in \mathbb{N}$.

Tropicalization

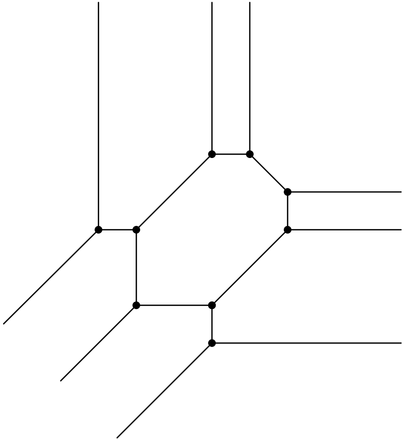
Balanced: For v, a vertex of Σ and adjacent edges $E_1, \ldots, E_k$ in primitive $\mathbb{Z}^n$ directions, $\vec{u}_1, \ldots, \vec{u}_k$ then

$$\sum m(E_i)\vec{u}_i = \vec{0}.$$

Example:



An elliptic curve in the plane



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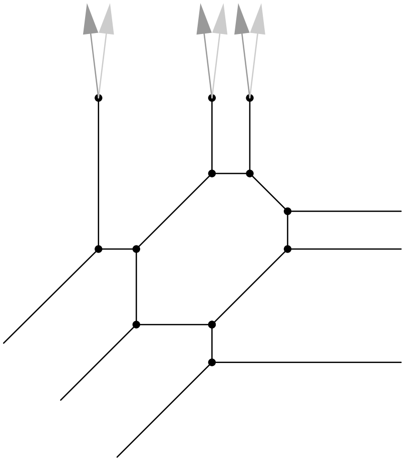
An elliptic curve in space



All multiplicities are 1. Note that the cycle in the graph is contained in the plane of the screen.

More generally...

Tropicalizations of general subvarieties are balanced, weighted, integral polyhedral complexes (by results of Bieri-Groves and Speyer).

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Ignoring issues of convergence, if we fix a particular value of u, we get a variety in $(\mathbb{C}^*)^n$. So by including all values of u in a punctured neighborhood of $u = 0$, we get a family of varieties in $(\mathbb{C}^*)^n$ over a punctured disc. So in a certain sense we are tropicalizing a family of varieties.

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A: Today's talk.

Statement of lifting problem for curves

Lifting Problem: Which tropical (that is, balanced, weighted, integral) graphs are tropicalizations of curves?

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The condition we'll talk about today implies the necessity of these previously known conditions.

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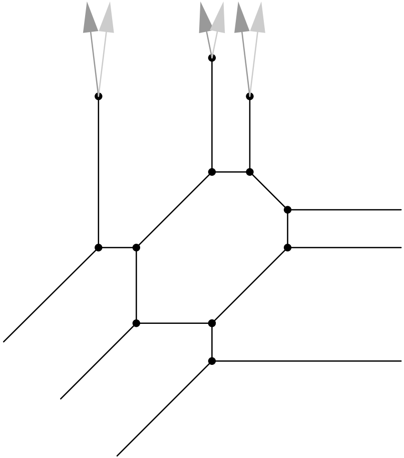
- The problem is combinatorial, but what kind of combinatorics even encodes this?
- Closely tied to deformation theory which is often grungy, maybe there's a combinatorial approach.

Example of non-liftable curve

Change the length of a bounded edge in the spatial elliptic curve so that it does not lie on the tropicalization of any plane (possible by dimension counting).



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no lift of the curve can be planar or genus 0, so the curve does not lift.

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Note: If all the multiplicities of Σ are 1 and all vertices are trivalent, then the only parameterization of Σ is the identity. In fact, the only parameterization used in explicit examples will be the identity.

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$\tilde{\Sigma}$ has canonical divisor:

$$K_{\tilde{\Sigma}} = \sum_{v} (\deg(v) - 2)(v)$$

Main theorem

Theorem: If $\Sigma \subset \mathbb{R}^n$ is a tropicalization of a curve then there exists $p : \tilde{\Sigma} \to \Sigma$ and for all $m \in \mathbb{Z}^n$ (which will be the normal vector to a plane), there is a piecewise-linear function $\varphi_m : \tilde{\Sigma}_l \to \mathbb{R}_{\geq 0}$ ($\tilde{\Sigma}_l$ is the l-fold subdivision of $\tilde{\Sigma}$) with $\mathbb{Z}$-slopes such that

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4. φ_m obeys the cycle-ampleness condition.

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Set $h = \min_{v \in \Gamma} (\varphi_m(v))$ then,

$$D_{\varphi_m} \equiv \sum_{v \in \Gamma | \varphi_m(v) = h} \left(\sum_{E \notin \Gamma | s(v,E) < 0} (-s(v, E)) \right) \geq 2.$$

"sum of positive slopes coming into the cycle at min's of φ_m must be at least 2."

Sections of canonical bundle

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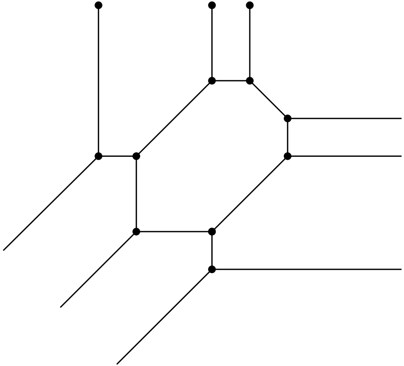
If $\deg(v) = 2$, then the slope is non-increasing through v (φ_m is concave at v).

Elliptic curve example



Note: This is $p^{-1}(H)$ where H is the plane of the screen.

Elliptic curve example (cont'd)

Need to pay attention to positive incoming slope coming into the cycle.

1. Direct edges towards cycle.

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7. For $\deg(D_{\varphi_m}) \geq 2$, the minimum distance must be achieved at least twice.

Elliptic curve example (concluded)

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Also get generalization to higher genus as given by Nishinou and Brugallé-Mikhalkin. This requires strong conditions on combinatorics of Σ.

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if $H \subset \mathbb{R}^n$ is a hyperplane and Γ' is any component of $p^{-1}(H) \subset \tilde{\Sigma}$ with $h^1(\Gamma') > 0$ then $\partial\Gamma'$ is not a single trivalent vertex of $\tilde{\Sigma}$.

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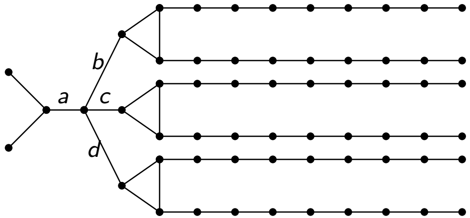

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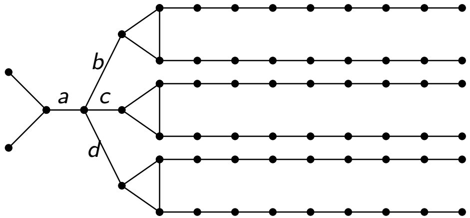

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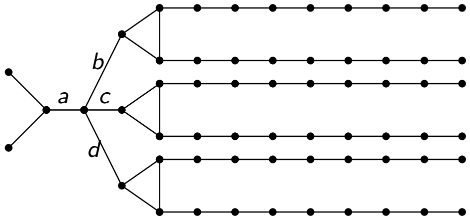

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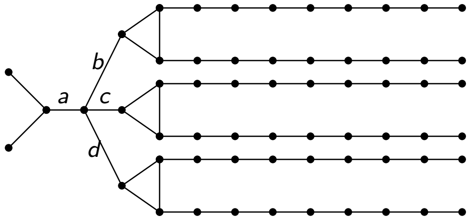

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There does not exist the desired φ_m, so it does not lift.

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7. $\deg(D_{\varphi_m}) \leq 1$ on one cycle.

Quick Outline of proof

1. Suppose Σ lifts. By Nishinou-Siebert, $C \hookrightarrow (\mathbb{K}^*)^n$ extends to a stable map $f : \mathcal{C} \to \mathcal{P}$ from a complete semi-stable curve to a toric scheme. These are families of object over an unpunctured disc.

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4. φ_m is a combinatorial shadow of ω_m measuring the vanishing of ω_m on components of the central fiber.
5. Cycle-ampleness condition comes from ω_m being "almost" exact on the cycle and the fact that a non-constant rational function on a (possibly degenerate) elliptic curve must have (counted with multiplicity) at least two poles.

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6. Possible applications to number theory? Further refinement of Chabauty in bad reduction case?

Thanks!

K, *Lifting Tropical Curves in Space and Linear Systems on Graphs*, **arXiv**:1009.1783, Adv. Math., to appear.

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