Log-Concavity of Characteristic Polynomials and Toric Intersection Theory

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Inclusion/exclusion

Let $\mathbf{k}$ be a field. Let $V \subset \mathbf{k}^{n+1}$ be an $(r+1)$-dim linear subspace not contained in any coordinate hyperplane. Would like to use inclusion/exclusion to express $[V \cap (\mathbf{k}^*)^{n+1}]$ as a linear combination of $[V \cap L_I]$'s where L_I is the coordinate subspace given by

$$L_I = \{x_{i_1} = x_{i_2} = \cdots = x_{i_l} = 0\}$$

for $I = \{i_1, i_2, \ldots, i_l\} \subset \{0, \ldots, n\}$.

Example: Let V be a generic subspace (intersecting every coordinate subspace in the expected dimension). Then

$$[V \cap ((\mathbf{k}^*)^{n+1})] = [V \cap L_\emptyset] - \sum_i [V \cap L_i] + \sum_{\substack{I \\ |I|=2}} [V \cap L_I] - \sum_{\substack{I \\ |I|=3}} [V_\cap L_I] + \ldots.$$

If you're fancy, you can say that this is a motivic expression.

Flats

In general, you may have to be a little more careful as there may be $I, J \subseteq \{0, \ldots, n\}$ with $V \cap L_I = V \cap L_J$. Need to make sure we do not overcount.

Definition: A subset $I \subset \{0, \ldots, n\}$ is said to be a flat if for any $J \supset I$, $V \cap L_J \neq V \cap L_I$.

The rank of a flat is

$$\rho(I) = \operatorname{codim}(V \cap L_I \subset V).$$

We can now write for some choice of $\nu_I \in \mathbb{Z}$,

$$[V \cap (\mathbf{k}^*)^{n+1}] = \sum_{\text{flats } I} \nu_I [V \cap L_I].$$

Fact: $(-1)^{\rho(I)} \nu_V$ is always positive.

Characteristic Polynomial

Definition: The characteristic polynomial of V is

$$\begin{aligned}\chi_V(q) &= \sum_{i=0}^{r+1}\left(\sum_{\substack{\text{flats } I\\ \rho(I)=i}} \nu_I\right) q^{r+1-i}\\ &\equiv \mu_0 q^{r+1} - \mu_1 q^r + \cdots + (-1)^{r+1}\mu_{r+1}\end{aligned}$$

We can think of χ as an evaluation of the classes $[V \cap L_I]$ of the form

$$[V \cap L_I] \mapsto q^{r+1-\rho(I)}$$

so the characteristic polynomial is the image of $[V \cap (k^*)^{n+1}]$ under this evaluation.

Example: In the generic case subspace case, we have

$$\chi_V(q) = q^{r+1} - \binom{r+1}{1} q^r + \binom{r+1}{2} q^{r-1} - \cdots + (-1)^{r+1}\binom{r+1}{0}.$$

Rota-Heron-Welsh Conjecture

Rota-Heron-Welsh Conjecture (in the realizable case) (Huh-k '11):
$\chi_V(q)$ is log-concave.

Definition: A polynomial with coefficients $\mu_0, \ldots, \mu_{r+1}$ is said to be log-concave if for all i,

$$|\mu_{i-1}\mu_{i+1}| \leq \mu_i^2.$$

(so log of coefficients is a concave sequence.)

Note: Log concavity is a more robust form of unimodality...

Definition: A polynomial with coefficients $\mu_0, \ldots, \mu_{r+1}$ is said to be unimodal if the coefficients are unimodal in absolute value, i.e. there is a j such that

$$|\mu_0| \leq |\mu_1| \leq \cdots \leq |\mu_j| \geq |\mu_{j+1}| \geq \cdots \geq |\mu_{r+1}|.$$

Motivation:Chromatic Polynomials of Graphs

Original Motivation: Let Γ be a loop-free graph. Define the chromatic function χ_Γ by setting $\chi_\Gamma(q)$ to be the number of colorings of Γ with q colors such that no edge connects vertices of the same color.

Fact: $\chi_\Gamma(q)$ is a polynomial of degree equal to the number of vertices with alternating coefficients.

Read's Conjecture '68 (Huh '10): $\chi_\Gamma(q)$ is unimodal.

Graphs and Subspaces

The connection between graphs and subspaces is as follows

$$C_1(\Gamma) \xrightarrow{\partial} C_0(\Gamma)$$

induces

$$C^0(\Gamma) \xrightarrow{d} C_1(\Gamma).$$

So $dC^0(\Gamma) \subseteq C^1(\Gamma)$. It can be shown

$$\chi_\Gamma(q) = q^c \cdot \chi_{dC^0(\Gamma)}(q).$$

In fact, Huh proved the Rota-Heron-Welsh conjecture when the characteristic of $\mathbf{k}$ is 0.

Matroids

We may abstract the linear space to a rank function

$$\rho : 2^{\{0,\ldots,n\}} \to \mathbb{Z}$$

satisfying

1. $0 \leq \rho(I) \leq |I|$
2. $I \subset J$ implies $\rho(I) \leq \rho(J)$
3. $\rho(I \cup J) + \rho(I \cap J) \leq \rho(I) + \rho(J)$
4. $\rho(\{0, \ldots, n\}) = r + 1.$

Note: Item (3) abstracts

$$\operatorname{codim}(((V \cap L_I) \cap (V \cap L_J)) \subset (V \cap L_{I\cap J})) \leq$$

$$\operatorname{codim}((V \cap L_I) \subset (V \cap L_{I\cap J})) + \operatorname{codim}((V \cap L_J) \subset (V \cap L_{I\cap J})).$$

This is one of the definitions of matroids.

Rota-Heron-Welsh Conjecture

For matroids, ν_I and hence $\chi(q)$ can be defined combinatorially by Möbius inversion without reference to any linear space. This leads us to

Rota-Heron-Welsh Conjecture '71: For any matroid, $\chi(q)$ is log-concave.

This is still open, but I'll explain some approaches to it at the end of the talk.

Outline of Proof

Our proof is very close to Huh's original proof. We replace singularity theory in the original proof with some toric intersection theory.

Step 1: Use the reduced characteristic polynomial.

From the fact $\chi(1) = 0$, we can set

$$\overline{\chi}(q) = \frac{\chi(q)}{q-1}.$$

The log-concavity of $\overline{\chi}$ implies the log-concavitiy of χ.

Coefficients of $\overline{\chi}$ have a combinatorial description:

$$\overline{\chi}_V(q) = \mu^0 q^r - \mu^1 q^{r-1} + \cdots + (-1)^r \mu^r q^0.$$

Then

$$\mu^i = (-1)^i \sum_{\substack{\text{flats } I \\ \rho(I)=i \\ 0 \notin I}} \nu_I.$$

Outline of Proof

Step 2: Identify μ^i with intersection numbers.

We will define a r-dimensional variety $\widetilde{V} \subset \mathbb{P}^n \times \mathbb{P}^n$ called the total transform.

Lemma $\mu^i = \deg((p_1^* c_1(\mathcal{O}(1)))^{r-i}(p_2^* c_1(\mathcal{O}(1)))^i \cap [\widetilde{V}])$.

Outline of Proof

Step 3: Apply Khovanskii-Teissier inequality.

Let X be a complete irreducible r-dimensional variety, and let α, β be nef divisors on X.

Then

$$a_i = (\alpha^i \beta^{r-i}) \cap [X]$$

is a log-concave sequence.

Total Transform

We have $\mathbb{P}(V) \subset \mathbb{P}^n$.

Let Crem : $\mathbb{P}^n \dashrightarrow \mathbb{P}^n$ be the generalized Cremona transform

$$[X_0 : X_1 : \cdots : X_n] \mapsto [\frac{1}{X_0} : \frac{1}{X_0} : \cdots : \frac{1}{X_r}].$$

Caution: This is indeterminate on coordinate subspaces. It is a rational map.

Let $\widetilde{V} \subset \mathbb{P}^n \times \mathbb{P}^n$ be the closure of the graph of $\mathbb{P}(V)$.

Then $\tilde{V} \to \mathbb{P}(V)$ is an iterated blow-up of $\mathbb{P}(V)$ at subvarieties of the form $\mathbb{P}(V \cap L_I)$.

Intersection Theory computation

We will need to show

Lemma $\mu^i = \deg((p_1^* c_1(\mathcal{O}(1)))^{r-i}(p_2^* c_1(\mathcal{O}(1)))^i \cap [\widetilde{V}])$ where $p_j : \mathbb{P}^n \times \mathbb{P}^n$ are the projections.

Now, it seems plausible that these intersection numbers should have something to do with the reduced characteristic polynomial since you are blowing up coordinate subspaces which makes it harder for varieties to intersect on them. I do not have a wholly geometric proof of this fact.

Set $\alpha = p_1^* c_1(\mathcal{O}(1)), \beta = p_2^* c_1(\mathcal{O}(1))$.

We will call α the truncation operator and β the cotruncation operator.

Toric Varieties

A toric variety $Y(\Delta)$ is a certain abstract algebraic variety with a $(\mathbb{C}^*)^n$-action associated to a rational polyhedral fan $\Delta \subset \mathbb{R}^n$. Toric varieties are normal and have a dense $(\mathbb{C}^*)^n$-orbit. In fact, they are characterized by those properties.

If Δ is a complete fan then $Y(\Delta)$ is complete.

$Y(\Delta)$ has a stratification by torus orbits which are indexed by cones in Δ. For $\sigma \in \Delta^{(k)}$ (the set of codimension k cones in Δ), we let $V(\sigma)$ denote the closure of the corresponding orbit. It is a k-dimensional subvariety of $Y(\Delta)$.

Intersection Theory on Toric Varieties

I will need to review intersection theory on complete toric varieties. The theorem that makes intersection theory combinatorial is

Theorem (Fulton-MacPherson-Sottile-Sturmfels) Let $Y(\Delta)$ be a complete toric variety. Let $c \in A^k(Y(\Delta))$. Then c is determined by $c([V(\sigma)])$ for all $\sigma \in \Delta^{(k)}$.

To completely understand the cohomology class c, you only need to evaluate it on very special cycles.

Minkowski Weights

Definition A Minkowski weight of codimension k is a function

$$c : \Delta^{(k)} \to \mathbb{Z}$$

such that for all $\tau \in \Delta^{(k+1)}$,

$$\sum_{\sigma \supset \tau} c(\sigma) u_{\sigma/\tau} = 0 \text{ in } N/N_\sigma$$

where $u_{\sigma/\tau} \in N/N_\tau$ (positive integrally) spans $(\sigma + N_\tau)/N_\tau$.

Theorem (Fulton-Sturmels) $A^k(Y(\Delta)) \cong \mathrm{MW}^k(\Delta)$.

The Minkowski weight condition ensures that c is constant on linear-equivalence classes (this is a more sensitive algebraic geometric analog of homological equivalence).

Intersection theory set-up

We will let $Y(\Delta)$ be the closure of the graph of

$$\mathrm{Crem} : \mathbb{P}^n \dashrightarrow \mathbb{P}^n.$$

To compute $\alpha^{r-i}\beta^i \cap [\widetilde{V}]$, we will find a Poincare-dual $c \in A^{n-r}(Y(\Delta))$ to $[\widetilde{V}]$.

Then

$$\deg(\alpha^{r-i}\beta^i \cap [\widetilde{V}]) = \deg(\alpha^{r-i}\beta^i \cup c).$$

Finding the Poincare-dual

To find the Poincare-dual, we use the following

Lemma Let $X \subset Y(\Delta)$ be an r-dimensional subvariety that intersects every orbit closure $V(\tau)$ of $Y(\Delta)$ in the expected dimension. Define

$$c : \Delta^{(r)} \to \mathbb{Z}$$

by

$$c(\sigma) = \deg(X \cdot V(\sigma)).$$

Then $c \cap [Y(\Delta)] = [X]$.

So c acts like a Poincare-dual to X. If you're a tropical person, $\mathrm{Trop}(X)$ is the union of closures of cones on which c is non-zero. The weight on σ in $\mathrm{Trop}(X)$ is $c(\sigma)$.

Flags of flats

I need some notation to describe the dual to $\widetilde{V}$.

Definition: A flag of flats is a chain

$$\mathcal{F} = \{\emptyset \subset F_1 \subset F_2 \subset \cdots \subset F_k \subset \{0, \ldots, n\}\}$$

where each F_i is a flat.

We will suppress $\emptyset, \{0, \ldots, n\}$ below. A flag is said to be full if it has $k = r$.

Let $e_1, \ldots, e_n$ be a basis for $\mathbb{R}^n$. Set

$$e_0 = -e_1 - \cdots - e_n.$$

For a flat F, set

$$e_F = \sum_{i \in F} e_i.$$

For a flag of flats $\mathcal{F}$, let

$$\sigma_{\mathcal{F}} = \mathrm{Span}_+(e_{F_1}, \ldots, e_{F_k}).$$

Ardila-Klivans class

$\widetilde{V} \subset Y(\Delta)$ has a well-understood Poincare-dual described by Ardila-Klivans based on work of Sturmfels and collaborators on Bergman fans.

Definition Let $c \in A^{n-r}(Y(\Delta))$, the Ardila-Klivans class be defined to be non-zero only on r-dimensional cones of the form $\sigma_{\mathcal{F}}$ for $\mathcal{F}$ a full flag of flats. The value (weight) on $\sigma_{\mathcal{F}}$ is 1.

This is indeed the operational Poincare-dual:

$$c \cap [Y(\Delta)] = [\widetilde{V}].$$

α, β as operators

We will view $\alpha\cup, \beta\cup$ as operators $\mathrm{MW}^k(\Delta) \to \mathrm{MW}^{k+1}(\Delta)$.

For any class $d \in A^*(Y(\Delta))$, to give a description of $\alpha \cup d, \beta \cup d$, I only need to tell you its values on the appropriate dimensional cones.

$\alpha\cup$: the truncation operator

$\alpha\cup$ is like intersecting with a generic hyperplane. It replaces $\widetilde{V}$ with $\widetilde{V \cap H}$ where H is a generic hyperplane. It lowers the possible codimension of the flats that V can intersect.

Top-dimensional cones on which $\alpha \cup c$ are non-zero are $\sigma_{\mathcal{F}}$ for which

$$\mathcal{F} = \{F_1 \subset F_2 \subset \cdots \subset F_{r-1}\}$$

where $\rho(F_j) = j$. All weights are 1.

This is still an Ardila-Klivans class of a linear subspace. So we can iterate.

Top-dimensional cones on which $\alpha^{r-i} \cup c$ are non-zero are $\sigma_{\mathcal{F}}$ for which

$$\mathcal{F} = \{F_1 \subset F_2 \subset \cdots \subset F_i\}$$

where $\rho(F_j) = j$. All weights are 1.

$\beta\cup$: the cotruncation operator

$\beta\cup$ is more mysterious. Its action can be computed using intersection theory and Weisner's theorem.

$\beta \cup c$ is non-zero on cones of the form $\sigma_{\mathcal{F}}$ for

$$\mathcal{F} = \{F_2 \subset \cdots \subset F_r\}$$

for $\rho(F_j) = j$. The weight on such a cone is ν_{F_2}. So $\beta\cup$ removes smallest rank flats.

For $i < r$, $\beta^i \cup c$ is non-zero on cones of the form $\sigma_{\mathcal{F}}$ for

$$\mathcal{F} = \{F_{i+1} \subset \cdots \subset F_r\}$$

for $\rho(F_j) = j$. The weight on such a cone is $(-1)^{i+1}\nu_{F_{i+1}}$.

To prove this, we iterated the following formula for ν_F: for any $a \in F$,

$$\nu_F = -\sum_{a \notin F' \lessdot F} \nu_{F'}$$

where $A \lessdot B$ means that $A \subset B$ and $\rho(A) = \rho(B) - 1$.

$\beta\cup$: the cotruncation operator (cont'd)

So $\alpha^{r-i}\beta^{i-1} \cup c$ is non-zero on cones $\sigma_{\mathcal{F}}$ for

$$\mathcal{F} = \{F_i\}$$

with weight $(-1)^i \nu_{F_{i+1}}$.

Apply β to $\alpha^{r-i}\beta^{i-1} \cup c$, get a weight on origin equal to

$$\sum_{0 \notin F_i} (-1)^i \nu_{F_i} = \mu^i.$$

This is the degree of the intersection product

$$\deg(\alpha^{r-i}\beta^i \cap [\widetilde{V}]).$$

Q.E.D.

General Rota-Heron-Welsh Conjecture

Why doesn't this prove the general conjecture for matroids? The intersection theory was entirely combinatorial.

We used algebraic geometry to establish the Khovanskii-Teissier inequality. If it could be established combinatorially for Ardila-Klivans classes, then we could prove the general conjecture.

We have an approach to the general conjecture.

Okounkov bodies for matroids

Khovanskii-Teissier can also be established using Okounkov bodies. If V is a k-dimensional variety, F is a flag of irreducible subvarieties on V, and L is a line-bundle on V, then the Okounkov body $\Delta_F(L) \subset \mathbb{R}^k$ is a convex set.

Okounkov bodies obey Minkowski subadditivity: if $\Delta_F(L), \Delta_F(M) \neq \emptyset$ then

$$\Delta_F(L) + \Delta_F(M) \subseteq \Delta_F(L \otimes M).$$

If L is big line-bundle then the volume of the Okounkov body is equal (up to a normalizing factor) to the degree of L.

These facts together with the Brunn-Minkowski inequality establish the Khovanskii-Teissier inequality.

Problem: Okounkov bodies are often non-polyhedral, mysterious, sensitive invariants.

Okounkov bodies for surfaces associated to matroids

Lucky coincidence Log-concavity only requires understanding three consecutive intersection numbers of the form $\alpha^{r-i}\beta^i \cap [\widetilde{V}]$. These can be computed on the (almost-)surface $\alpha^{r-i-1}\beta^{i-1} \cap [\widetilde{V}]$. On surfaces, Okounkov bodies are not so bad.

Let $S = \alpha^{r-i-1}\beta^{i-1} \cap [\widetilde{V}]$ which we pretend is a surface. Now we can try to examine the Okounkov body $\Delta_F(\beta)$ where the flag F is given by a curve in class α and a generic point on the curve.

The Okounkov body only cares about the Zariski decomposition of $\beta - t\alpha$ for $t \geq 0$. The Zariski decomposition is a certain way of writing a divisor as the sum of a nef and effective divisor.

Problem: nef divisors are not really visible in tropical geometry. We do not have enough curves to test nefness.

Lazy solution: Maybe we could just use curves corresponding to rays of tropicalization. Then nef is very different from ample and a lot of things break. Still, this gives a combinatorial Okounkov body.

Okounkov bodies for surfaces associated to matroids

Then the log-concavity conjecture reduces to the following bigness conjecture for combinatorial Okounkov bodies:

$$\mathrm{Area}(\Delta_F(\beta)) \geq \frac{1}{2}\beta^2?$$

This is true for realizable matroids by a sort of specialization lemma. The specialization lemma says that the combinatorial Okounkov body contains the classical Okounkov body. Computing the volume of the classical Okounkov body requires the Riemann-Roch for surfaces.

I have no idea what sort of invariant the combinatorial Okounkov body is. If it has an easy combinatorial structure, maybe we can establish the bigness conjecture by hand. I'm going to have an undergrad do some (thousands of) examples.

Thanks!

Huh, June and K, *Log-concavity of characteristic polynomials and the Bergman fan of matroids*. **arXiv**:arXiv:1104.2519

Huh, June. *Milnor numbers of projective hypersurfaces and the chromatic polynomial of graphs*. **arXiv**:1008.4749