Matroids in Tropical Geometry

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Matroids

What are matroids?

1. Combinatorial abstractions of linear subspaces or of point configurations or of hyperplane arrangements
2. Generalization of graphs
3. A source of bizarre counterexamples in studying moduli spaces
4. A testing ground for theorems about representability of cohomology classes.

Linear Subspaces

Let $\mathbf{k}$ be a field. Consider the vector space k^{n+1} with a choice of basis $\vec{e}_0, \ldots, \vec{e}_n \in \mathbf{k}^{n+1}$. Let V^{r+1} be a linear subspace not contained in any coordinate hyperplane.

We can encode the linear subspace combinatorially by the use of a rank function.

Let L_I be the coordinate subspace given by

$$L_I = \{x_{i_1} = x_{i_2} = \cdots = x_{i_l} = 0\}$$

for $I = \{i_1, i_2, \ldots, i_l\} \subset \{0, \ldots, n\}$.

The rank of a subset is defined to be

$$\rho(I) = \operatorname{codim}(V \cap L_I \subset V).$$

Matroids

We may abstract the linear space to a rank function

$$\rho : 2^{\{0,\dots,n\}} \to \mathbb{Z}$$

satisfying

1. $0 \leq \rho(I) \leq |I|$
2. $I \subset J$ implies $\rho(I) \leq \rho(J)$
3. $\rho(I \cup J) + \rho(I \cap J) \leq \rho(I) + \rho(J)$
4. $\rho(\{0, \dots, n\}) = r + 1.$

Note: Item (3) abstracts

$$\operatorname{codim}(((V \cap L_I) \cap (V \cap L_J)) \subset (V \cap L_{I \cap J})) \leq$$

$$\operatorname{codim}((V \cap L_I) \subset (V \cap L_{I \cap J})) + \operatorname{codim}((V \cap L_J) \subset (V \cap L_{I \cap J})).$$

This is one of the definitions of matroids. There are **many** others.

Matroids and Graphs

Matroids are generalization of graphs in the following sense:

Let Γ be a loop-free graph. Then we have the complex of simplicial chains

$$C_1(\Gamma) \xrightarrow{\partial} C_0(\Gamma)$$

inducing

$$C^0(\Gamma) \xrightarrow{d} C^1(\Gamma).$$

So we consider the subspace $dC^0(\Gamma) \subseteq C^1(\Gamma)$ and form the matroid. By Whitney's isomorphism theorem, the matroid of this subspace encodes the graph up to two well-understood moves.

Representability

Not every matroid comes from a subspace. One can construct matroids corresponding to impossible arrangements of hyperplanes. If a matroid comes from a subspace, then it is said to be representable.

Representability in general is hard over infinite fields by Mnëv's theorem. Consequences:

1. One can construct matroids that are only representable over fields in which certain algebraic equations have solutions.
2. Over $\mathbb{Q}$, an algorithm to determine representability is equivalent to Diophantine decidability algorithm over $\mathbb{Q}$ which is open but thought to be impossible.
3. One can consider the space of all representations of a given matroid. This space is a thin Schubert cell and was much studied by Lafforgue and others. They are arbitrarily singular by Mnëv. Vakil constructed arbitrarily pathological moduli spaces from them to prove "Murphy's Law."

Structure theory

Matroids have a good structure theory over finite fields (note: not finite characteristic!). There is a notion of deletion and restriction for matroids similar to deletion and contraction for graphs that allows one to define minors.

There are matroids that are representable over every field. They are called "regular" matroids, but maybe they should be thought of as matroids over $\mathbb{F}_1$, the mythical field of one element. They have a forbidden minor characterization.

It is a conjecture of Rota to characterize $\mathbb{F}_q$-representable matroids in terms of a finite number of forbidden minors ($\mathbb{F}_2$ due to Tutte; $\mathbb{F}_3$ due to Seymour; $\mathbb{F}_4$ due to Geelen-Gerards-Kapoor).

As an aside, the combinatorial theory of representability should be tied into deformation theory and descent in algebraic geometry. There are no results in this direction. It'd be fun to make the connection.

Tropicalization of linear subspaces

The tropicalization of linear subspaces is controlled by the underlying matroid.

If $V \subseteq \mathbf{k}^{n+1}$ is a linear subspace, we may projectivize to $\mathbb{P}(V) \subseteq \mathbb{P}^n$. We use the coordinates on $\mathbb{P}^n$ to identify a copy of the algebraic torus $(\mathbf{k}^*)^n$. We will tropicalize $\mathbb{P}(V) \cap (\mathbf{k}^*)^n$.

Now, to tropicalize V, we need to introduce the notion of flat:

Definition For $I \subset \{0, \ldots, n\}$, let the interior of the coordinate subspace be

$$L_I^* = \{x | x_i = 0 \text{ iff } i \in I\}.$$

I is a flat if $V \cap L_I^* \neq \emptyset$.

This can be defined in terms of the matroid: I is a flat if and only if for any $J \supsetneq I$, $\rho(J) > \rho(I)$.

Think: We've got a stratification of $\mathbb{P}^n$ by L_I^*'s and we're recording which open strata $\mathbb{P}(V)$ intersects. Also because everything is linear, if it intersects in a zero dimensional set, it intersects in a single point.

Flags of flats

Definition: A flag of flats is a chain

$$\mathcal{F} = \{\emptyset \subset F_1 \subset F_2 \subset \cdots \subset F_k \subset \{0, \ldots, n\}\}$$

where each F_i is a flat.

We will suppress $\emptyset, \{0, \ldots, n\}$ below. A flag is said to be full if it has $k = r$.

Let $e_1, \ldots, e_n$ be a basis for $\mathbb{R}^n$. Set

$$e_0 = -e_1 - \cdots - e_n.$$

For a flat F, set

$$e_F = \sum_{i \in F} e_i.$$

For a flag of flats $\mathcal{F}$, define a cone

$$\sigma_{\mathcal{F}} = \mathrm{Span}_+(e_{F_1}, \ldots, e_{F_k}).$$

Ardila-Klivans description

Now, the tropicalization of $\mathbb{P}(V) \cap (\mathbf{k}^*)^n$ is described by Ardila-Klivans based on work of Sturmfels and collaborators on Bergman fans.

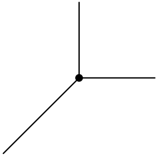
The tropicalization is the union of $\sigma_{\mathcal{F}}$ for each flag of flats in the matroid $\mathbb{M}$. The top-dimensional cones correspond to full flags of flats. They are given weight 1.

There is a sort of converse to this description saying that if the tropicalization of a variety looks like the tropicalization of a subspace, then the variety is a subspace. I like calling it the duck theorem.

This theorem can be used to come up with nice counterexamples to questions about tropical lifting. By starting with an appropriately pathological matroid, one can produce a balanced weighted tropical fan Δ that is not the tropicalization of a variety or is only the tropicalization of a variety over a field of characteristic 2 or ..., etc.

Example 1

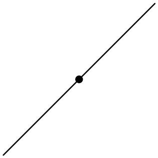
Let us first consider a generic line L in $\mathbb{P}^2$. It intersects each of the coordinate lines in a point in the interior. So its flats are $\{0\}, \{1\}, \{2\}$. Therefore, its Ardila-Klivans class takes the value 1 on each of the rays through $e_1, e_2, -e_1 - e_2$. You can visualize it as:



This is the generic tropical line in a plane. By law, it is required to show up at least once in every tropical geometry talk.

Example 2

Now, let us consider the line in $\mathbb{P}^2$ given by $x_1 = x_2$. It intersects the line $x_0 = 0$ in the point $[0 : 1 : 1]$ while it intersects the line $x_1 = 0$ in the point $[1 : 0 : 0]$ which is also contained in the line $x_2 = 0$. Therefore, the flats are $\{0\}, \{1, 2\}$. Therefore, its Ardila-Klivans class takes the value 1 on the rays through $e_1 + e_2$ and $-e_1 - e_2$. You can visualize it as:



This is a less exciting tropical line.

Example 3

Now, let us consider the generic plane in $\mathbb{P}^3$. It intersects each of the coordinate hyperplanes in a line. It also intersects each of the coordinate lines ($x_i = x_j = 0$, $i \neq j$ in a point). Therefore, its flats are of the form $\{i\}$ and $\{i, j\}$ for $i, j \in \{0, 1, 2, 3\}$, $i \neq j$.

The Minkowski weight is 1 on the 12 cones of the form:

$$\mathrm{Span}(e_i, e_i + e_j).$$

The support of the Minkowski weight is a polyhedral complex made up of 10 rays and 12 cones. It is a subdivision of the 2-skeleton of the normal fan of the standard simplex.

This is a generic tropical plane in 3-space.

Characteristic Polynomial

Now I will discuss an application of tropical geometry to matroids due to Huh-k.

Consider the Grothendieck ring of varieties over $\mathbf{k}$, $K_0(\mathrm{Var}_{\mathbf{k}})$. This is the ring of varieties under disjoint union and Cartesian product subject to the scissors relation: for $Z \subset X$,

$$[X] = [Z] + [X \setminus Z].$$

There is a homomorphism

$$e : K_0(\mathrm{Var}_k) \to \mathbb{Z}[q]$$

such that

$$e(\mathbb{A}^1_{\mathbf{k}}) = q.$$

Think: You're counting $\mathbb{F}_q$-points. Alternatively, can think in terms of Poincaré polynomial.

Characteristic Polynomial (cont'd)

Definition: For $V^{r+1} \subseteq \mathbf{k}^{n+1}$, the characteristic polynomial of V is

$$\chi_V(q) \equiv e([V \cap (\mathbf{k}^*)^{n+1}]).$$

Example: By inclusion/exclusion, in the generic subspace case, we have

$$\chi_V(q) = q^{r+1} - \binom{r+1}{1} q^r + \binom{r+1}{2} q^{r-1} - \cdots + (-1)^{r+1} \binom{r+1}{0}.$$

Think: Start with V, remove $r+1$ coordinate hyperplanes, add back in $\binom{r+1}{2}$ codimension 2 coordinate flats, and so on.

The characteristic polynomial is of degree equal to the dimension of V. It has alternating coefficients.

Characteristic Polynomial (concluded)

We can write for some choice of $\nu_I \in \mathbb{Z}$,

$$[V \cap (\mathbf{k}^*)^{n+1}] = \sum_{\text{flats } I} \nu_I [V \cap L_I].$$

Fact: $(-1)^{\rho(I)}\nu_V$ is always positive.

Then, the characteristic polynomial of V is

$$\begin{aligned} \chi_V(q) &= \sum_{i=0}^{r+1} \left(\sum_{\substack{\text{flats } I \\ \rho(I)=i}} \nu_I \right) q^{r+1-i} \\ &\equiv \mu_0 q^{r+1} - \mu_1 q^r + \cdots + (-1)^{r+1} \mu_{r+1} \end{aligned}$$

Rota-Heron-Welsh Conjecture (in the realizable case) (Huh-k '11):
$\chi_V(q)$ is log-concave.

Definition: A polynomial

$$\mu_0 + \mu_1 q + \cdots + \mu_{r+1} q^{r+1}$$

is said to be log-concave if for all i,

$$|\mu_{i-1}\mu_{i+1}| \leq \mu_i^2.$$

(so log of coefficients is a concave sequence.)

Note: Log concavity implies unimodality,

Definition: The polynomial above is said to be unimodal if the coefficients are unimodal in absolute value, i.e. there is a j such that

$$|\mu_0| \leq |\mu_1| \leq \cdots \leq |\mu_j| \geq |\mu_{j+1}| \geq \cdots \geq |\mu_{r+1}|.$$

Motivation:Chromatic Polynomials of Graphs

Original Motivation: Let Γ be a loop-free graph. Define the chromatic function χ_Γ by setting $\chi_\Gamma(q)$ to be the number of colorings of Γ with q colors such that no edge connects vertices of the same color.

Fact: $\chi_\Gamma(q)$ is a polynomial of degree equal to the number of vertices with alternating coefficients.

Read's Conjecture '68 (Huh '10): $\chi_\Gamma(q)$ is unimodal.

It is not very hard to show that

$$\chi_\Gamma(q) = q^c \cdot \chi_{dC^0(\Gamma)}(q)$$

where c is the number of components of Γ.

So Rota-Heron-Welsh is a generalization of Read's conjecture. In fact, Huh proved Rota-Heron-Welsh in the characteristic 0 case.

Rota-Heron-Welsh Conjecture

For matroids, $\chi_{\mathbb{M}}(q)$ can be defined combinatorially by Möbius inversion without reference to any linear space. This leads us to

Rota-Heron-Welsh Conjecture '71: For any matroid, $\chi(q)$ is log-concave.

This is still open, and it's very hard.

Outline of Proof

Our proof is very close to Huh's original proof. We replace singularity theory in the original proof with some toric intersection theory.

Step 1: Use the reduced characteristic polynomial, that is, projectivize.

From the fact $\chi(1) = 0$, we can set

$$\overline{\chi}(q) = \frac{\chi(q)}{q-1}.$$

Motivically, this follows from the fact that $\mathbf{k}^*$ acts freely on $V \cap (\mathbf{k}^*)^{n+1}$ so

$$e(V \cap (\mathbf{k}^*)^{n+1}) = e((V \cap (\mathbf{k}^*)^{n+1})/\mathbf{k}^*)e(\mathbf{k}^*).$$

The log-concavity of $\overline{\chi}$ implies the log-concavitiy of χ.

Write

$$\overline{\chi}_V(q) = \mu^0 q^r - \mu^1 q^{r-1} + \cdots + (-1)^r \mu^r q^0.$$

Outline of Proof

Step 2: Identify μ^i with intersection numbers.

We will define a r-dimensional variety $\widetilde{V} \subset \mathbb{P}^n \times \mathbb{P}^n$ called the total transform.

Lemma $\mu^i = \deg((p_1^* H^{r-i})(p_2^* H^i) \cap [\widetilde{V}])$ where H is a hyperplane class on $\mathbb{P}^n$ and the p_i's are projections.

We can use tropical or toric intersection theory (they're the same thing!) to compute these intersection numbers.

Outline of Proof

Step 3: Apply Khovanskii-Teissier inequality.

Let X be a complete irreducible r-dimensional variety, and let α, β be nef divisors on X.

Then

$$a_i = \deg((\alpha^i \beta^{r-i}) \cap [X])$$

is a log-concave sequence.

To get the conclusion, we set $X = \widetilde{V}$, $\alpha = p_1^* H$, $\beta = p_2^* H$.

This inequality is proved by reducing to the case of surfaces where it follows from the Hodge index theorem.

Total Transform

I want to say some words about the setup for step 2.

We have $\mathbb{P}(V) \subset \mathbb{P}^n$.

Let Crem : $\mathbb{P}^n \dashrightarrow \mathbb{P}^n$ be the generalized Cremona transform

$$[X_0 : X_1 : \cdots : X_n] \mapsto [\frac{1}{X_0} : \frac{1}{X_1} : \cdots : \frac{1}{X_r}].$$

Caution: This is indeterminate on coordinate subspaces. It is a rational map.

Let $\widetilde{\mathbb{P}}$ be the closure of the graph of Crem. The effect of the indeterminacy is to iteratively blow-up strata of $\mathbb{P}^n$. First, we blow-up the $n+1$ fixed points of the torus action, then we blow-up the proper transforms of the lines between the fixed points, then the proper transform of the planes through three of the fixed points, etc.

$\widetilde{\mathbb{P}}$ is a lovely toric variety. It is associated to a lattice polytope called the permutohedron.

Total Transform

Let $\widetilde{V} \subset \widetilde{\mathbb{P}}$ be the closure of the graph of $\mathrm{Crem}|_{\mathbb{P}(V)}$. We have

$$\widetilde{V} \subseteq \widetilde{\mathbb{P}} \subset \mathbb{P}^n \times \mathbb{P}^n.$$

Now, toric varieties have nice stratifications that generalize the coordinate stratification of $\mathbb{P}^n$. It is a theorem of Fulton-MacPherson-Sottile-Sturmfels that Chow cohomology classes of complete toric varieties are determined by their values on closed strata.

Upshot: The cohomology class Poincaré-dual to $\widetilde{V}$ in $\widetilde{\mathbb{P}}$ is determined by intersection numbers of $\widetilde{V}$ with closed torus strata of $\widetilde{\mathbb{P}}$. These strata are labeled by full flags of flats in $\mathbb{M}$. The tropicalization of $V \cap (\mathbf{k}^*)^n$ is exactly the same thing as the Poincaré dual of $\widetilde{V} \subseteq \widetilde{\mathbb{P}}$.

Intersection Theory computation

We will need to show

Lemma $\mu^i = \deg((p_1^* c_1(\mathcal{O}(1)))^{r-i}(p_2^* c_1(\mathcal{O}(1)))^i \cap [\widetilde{V}])$ where $p_j : \mathbb{P}^n \times \mathbb{P}^n \to \mathbb{P}^n$ are the projections.

Now, it seems plausible that these intersection numbers should have something to do with the reduced characteristic polynomial since you are blowing up coordinate subspaces which makes it harder for varieties to intersect on them.

Set $\alpha = p_1^* H, \beta = p_2^* H$.

We will call α the truncation operator and β the cotruncation operator.

Toric Varieties

A toric variety $Y(\Delta)$ is a certain abstract algebraic variety with a $(\mathbf{k}^*)^n$-action associated to a rational polyhedral fan $\Delta \subset \mathbb{R}^n$. Toric varieties are normal and have a dense $(\mathbf{k}^*)^n$-orbit. In fact, they are characterized by those properties.

If Δ is a complete fan then $Y(\Delta)$ is complete.

$Y(\Delta)$ has a stratification by torus orbits $\mathcal{O}_\sigma$ which are indexed by cones in Δ. For $\sigma \in \Delta^{(k)}$ (the set of codimension k cones in Δ), we let $V(\sigma)$ denote the closure of the corresponding orbit. It is a k-dimensional subvariety of $Y(\Delta)$.

Think: This is a generalization of the stratification of $\mathbb{P}^n$ by L_I^*'s.

Intersection Theory on Toric Varieties

I will need to review intersection theory on complete toric varieties. The theorem that makes intersection theory combinatorial is

Theorem (Fulton-MacPherson-Sottile-Sturmfels) Let $Y(\Delta)$ be a complete toric variety. Let $c \in A^k(Y(\Delta))$. Then c is determined by $c([V(\sigma)])$ for all $\sigma \in \Delta^{(k)}$.

To completely understand the cohomology class c, you only need to evaluate it on very special cycles.

Minkowski Weights

Definition A Minkowski weight of codimension k is a function

$$c : \Delta^{(k)} \to \mathbb{Z}$$

such that for all $\tau \in \Delta^{(k+1)}$,

$$\sum_{\sigma \supset \tau} c(\sigma) u_{\sigma/\tau} = 0 \text{ in } N/N_\sigma$$

where $u_{\sigma/\tau} \in N/N_\tau$ (positive integrally) spans $(\sigma + N_\tau)/N_\tau$.

Theorem (Fulton-Sturmels) $A^k(Y(\Delta)) \cong \mathrm{MW}^k(\Delta)$.

The Minkowski weight condition ensures that c is constant on linear-equivalence classes (this is a more sensitive algebraic geometric analog of homological equivalence).

We will let $Y(\Delta)$ be the permutohedral variety $\widetilde{\mathbb{P}}$, the closure of the graph of

$$\mathrm{Crem} : \mathbb{P}^n \dashrightarrow \mathbb{P}^n.$$

To compute $\alpha^{r-i}\beta^i \cap [\widetilde{V}]$, we will find a Poincare-dual $c \in A^{n-r}(\widetilde{\mathbb{P}})$ to $[\widetilde{V}]$.

Then

$$\deg(\alpha^{r-i}\beta^i \cap [\widetilde{V}]) = \deg(\alpha^{r-i}\beta^i \cup c).$$

Finding the Poincare-dual

To find the Poincare-dual, we use the following

Lemma Let $X \subset Y(\Delta)$ be an r-dimensional subvariety that intersects every orbit closure $V(\tau)$ of $Y(\Delta)$ in the expected dimension. Define

$$c : \Delta^{(r)} \to \mathbb{Z}$$

by

$$c(\sigma) = \deg(X \cdot V(\sigma)).$$

Then $c \cap [Y(\Delta)] = [X]$.

To understand the subvariety X in intersection theory, you only need to know how it intersects the orbit closures. These intersection numbers describe a Minkowski weight.

So c acts like a Poincare-dual to X. If you're a tropical person, $\mathrm{Trop}(X)$ is the union of closures of cones on which c is non-zero. The weight on σ in $\mathrm{Trop}(X)$ is $c(\sigma)$.

Ardila-Klivans class

$\widetilde{V} \subset Y(\Delta) = \widetilde{\mathbb{P}}$ has a well-understood Poincare-dual given by the Ardila-Klivans construction.

Definition Let $c \in A^{n-r}(\widetilde{\mathbb{P}}))$, the Ardila-Klivans class be defined to be non-zero only on r-dimensional cones of the form $\sigma_{\mathcal{F}}$ for $\mathcal{F}$ a full flag of flats. The value (weight) on $\sigma_{\mathcal{F}}$ is 1.

This is indeed the operational Poincare-dual:

$$c \cap [\widetilde{\mathbb{P}}] = [\widetilde{V}].$$

We had to pass from $\mathbb{P}^n$ to the permutohedral variety $\widetilde{\mathbb{P}}$ to ensure that $\tilde{V}$ intersects the orbit closures in the expected dimensions.

α, β as operators

We will view $\alpha\cup, \beta\cup$ as operators $\mathrm{MW}^k(\Delta) \to \mathrm{MW}^{k+1}(\Delta)$.

For any class $d \in A^*(Y(\Delta))$, to give a description of $\alpha \cup d, \beta \cup d$, I only need to tell you its values on the appropriate dimensional cones.

$\alpha\cup$: the truncation operator

$\alpha\cup$ is like intersecting with a generic hyperplane. It replaces $\widetilde{V}$ with $\widetilde{V \cap H}$ where H is a generic hyperplane. It lowers the possible codimension of the flats that V can intersect.

Top-dimensional cones on which $\alpha \cup c$ are non-zero are $\sigma_{\mathcal{F}}$ for which

$$\mathcal{F} = \{F_1 \subset F_2 \subset \cdots \subset F_{r-1}\}$$

where $\rho(F_j) = j$. All weights are 1.

This is still an Ardila-Klivans class of a linear subspace. So we can iterate.

Top-dimensional cones on which $\alpha^{r-i} \cup c$ are non-zero are $\sigma_{\mathcal{F}}$ for which

$$\mathcal{F} = \{F_1 \subset F_2 \subset \cdots \subset F_i\}$$

where $\rho(F_j) = j$. All weights are 1.

$\beta\cup$: the cotruncation operator

$\beta\cup$ is more mysterious. Its action can be computed using intersection theory and Weisner's theorem.

$\beta \cup c$ is non-zero on cones of the form $\sigma_{\mathcal{F}}$ for

$$\mathcal{F} = \{F_2 \subset \cdots \subset F_r\}$$

for $\rho(F_j) = j$. The weight on such a cone is ν_{F_2}. So $\beta\cup$ removes smallest rank flats.

For $i < r$, $\beta^i \cup c$ is non-zero on cones of the form $\sigma_{\mathcal{F}}$ for

$$\mathcal{F} = \{F_{i+1} \subset \cdots \subset F_r\}$$

for $\rho(F_j) = j$. The weight on such a cone is $(-1)^{i+1}\nu_{F_{i+1}}$.

To prove this, we iterated the following formula for ν_F: for any $a \in F$,

$$\nu_F = -\sum_{a \notin F' \lessdot F} \nu_{F'}$$

where $A \lessdot B$ means that $A \subset B$ and $\rho(A) = \rho(B) - 1$.

$\beta\cup$: the cotruncation operator (cont'd)

So $\alpha^{r-i}\beta^{i-1} \cup c$ is non-zero on cones $\sigma_{\mathcal{F}}$ for

$$\mathcal{F} = \{F_i\}$$

with weight $(-1)^i \nu_{F_{i+1}}$.

Apply β to $\alpha^{r-i}\beta^{i-1} \cup c$, get a weight on origin equal to

$$\sum_{0 \notin F_i} (-1)^i \nu_{F_i} = \mu^i.$$

This is the degree of the intersection product

$$\deg(\alpha^{r-i}\beta^i \cap [\widetilde{V}]).$$

Q.E.D.

General Rota-Heron-Welsh Conjecture

Why doesn't this prove the general conjecture for matroids? The intersection theory was entirely combinatorial.

We used algebraic geometry to establish the Khovanskii-Teissier inequality. If it could be established combinatorially for Ardila-Klivans classes, then we could prove the general conjecture.

Unfortunately, the Khovanskii-Teissier inequality is deeply algebraic geometric in nature.

Khovanskii-Teissier inequality

The Khovanskii-Teissier inequality only involves comparing 3 intersection numbers,

$$(\alpha^{i+1}\beta^{r-i-1}) \cap [X], (\alpha^{i}\beta^{r-i}) \cap [X], (\alpha^{i-1}\beta^{r-i+1}) \cap [X].$$

By Kleiman's criterion, we can approximate α and β by ample classes. By homogeneity, we can replace them by positive multiples, hence very ample classes. Now we can consider the class

$$Z = (\alpha^{i-1}\beta^{r-i-1}) \cap [X]$$

which by Kleiman-Bertini is represented by an irreducible surface.

Now we only need to consider

$$\alpha^2 \cap [Z], \alpha\beta \cap [Z], \beta^2 \cap [Z].$$

Then the inequality follows from the Hodge index theorem.

Class of $\widetilde{V}$

Now, step 2 is equivalent to proving the following:

Lemma The homology class of $[\widetilde{V}]$ in $\mathbb{P}^n \times \mathbb{P}^n$ is

$$\mu^0[\mathbb{P}^r \times \mathbb{P}^0] + \mu^1[\mathbb{P}^{r-1} \times \mathbb{P}^1] + \cdots + \mu^r[\mathbb{P}^0 \times \mathbb{P}^r].$$

Except for some easy border cases, some multiple of the homology class like that above is representable by an irreducible algebraic variety if and only if the μ^i's are log-concave and zero-free by the work of Huh.

Representability questions

So this is interesting and fun:

1. You have a homology class in $\widetilde{\mathbb{P}}$ corresponding to the matroid. Its representability by an irreducible algebraic variety is equivalent to the representability of the matroid.
2. When you push this class forward to $\mathbb{P}^n \times \mathbb{P}^n$, the representability of a multiple of that class is equivalent to numerical conditions, log-concavity and zero-freeness, that are conjectured always to hold.

Could it always be true that some multiple of the homology class of the matroid is representable in $\widetilde{\mathbb{P}}$? If true, this would prove the Rota-Heron-Welsh conjecture in general.

Thanks!

Huh, June and K, *Log-concavity of characteristic polynomials and the Bergman fan of matroids*. **arXiv**:arXiv:1104.2519

Huh, June. *Milnor numbers of projective hypersurfaces and the chromatic polynomial of graphs*. **arXiv**:1008.4749