Geometric Rank Functions and Rational Points on Curves

Eric Katz (University of Waterloo)
joint with David Zureick-Brown (Emory University)

April 22, 2015

Rational points on curves

Given an algebraic variety (a system of polynomial equations in many variables), one can ask how many rational points it has. The most significant theorem in this direction is Faltings's theorem that tells us:

Theorem (Faltings)

Let C be a curve defined over $\mathbb{Q}$. If $g(C) \geq 2$ then C has finitely many rational points.

Rational points on curves

Given an algebraic variety (a system of polynomial equations in many variables), one can ask how many rational points it has. The most significant theorem in this direction is Faltings's theorem that tells us:

Theorem (Faltings)

Let C be a curve defined over $\mathbb{Q}$. If $g(C) \geq 2$ then C has finitely many rational points.

This theorem is not effective. It does not tell how many rational points there are. However, there is an effective special case:

Theorem (Coleman)

Let C be a curve defined over $\mathbb{Q}$. Let J be the Jacobian of C, and let $r = \operatorname{rank}_{\mathbb{Z}} J(\mathbb{Q})$ be its Mordell-Weil rank. If $r < g$ then for $p > 2g$, a prime of good reduction of C,

$$|C(\mathbb{Q})| \leq |C(\mathbb{F}_p)| + 2g - 2.$$

Rational points on curves (cont'd)

Theorem (Coleman)

Let C be a curve defined over $\mathbb{Q}$. Let J be the Jacobian of C, and let $r = \operatorname{rank}_{\mathbb{Z}} J(\mathbb{Q})$ be its Mordell-Weil rank. If $r < g$ then for $p > 2g$, a prime of good reduction of C,

$$|C(\mathbb{Q})| \leq |C(\mathbb{F}_p)| + 2g - 2.$$

For $p \leq 2g$, there is a small correction term.

Rational points on curves (cont'd)

Theorem (Coleman)

Let C be a curve defined over $\mathbb{Q}$. Let J be the Jacobian of C, and let $r = \mathrm{rank}_{\mathbb{Z}} J(\mathbb{Q})$ be its Mordell-Weil rank. If $r < g$ then for $p > 2g$, a prime of good reduction of C,

$$|C(\mathbb{Q})| \leq |C(\mathbb{F}_p)| + 2g - 2.$$

For $p \leq 2g$, there is a small correction term.

Note that this bound depends on the first prime of good reduction. However, $|C(\mathbb{F}_p)|$ can be controlled by the Hasse-Weil bounds.

Rational points on curves (cont'd)

Theorem (Coleman)

Let C be a curve defined over $\mathbb{Q}$. Let J be the Jacobian of C, and let $r = \mathrm{rank}_{\mathbb{Z}} J(\mathbb{Q})$ be its Mordell-Weil rank. If $r < g$ then for $p > 2g$, a prime of good reduction of C,

$$|C(\mathbb{Q})| \leq |C(\mathbb{F}_p)| + 2g - 2.$$

For $p \leq 2g$, there is a small correction term.

Note that this bound depends on the first prime of good reduction. However, $|C(\mathbb{F}_p)|$ can be controlled by the Hasse-Weil bounds.

The Mordell-Weil rank is very computable. There are a large number of implemented algorithms.

Rational points on curves (cont'd)

Theorem (Coleman)

Let C be a curve defined over $\mathbb{Q}$. Let J be the Jacobian of C, and let $r = \operatorname{rank}_{\mathbb{Z}} J(\mathbb{Q})$ be its Mordell-Weil rank. If $r < g$ then for $p > 2g$, a prime of good reduction of C,

$$|C(\mathbb{Q})| \leq |C(\mathbb{F}_p)| + 2g - 2.$$

For $p \leq 2g$, there is a small correction term.

Note that this bound depends on the first prime of good reduction. However, $|C(\mathbb{F}_p)|$ can be controlled by the Hasse-Weil bounds.

The Mordell-Weil rank is very computable. There are a large number of implemented algorithms.

This bound does not tell you the height of the rational points, so if the bound is not sharp, it does not let you know if you've found all the rational points.

Today's Goal

Today's goal: Tighter bounds coming from primes of bad reduction.

Today's Goal

Today's goal: Tighter bounds coming from primes of bad reduction.

Let p be some prime. Let $\mathcal{C}$ be a regular minimal model of C over $\mathbb{Z}_p$. This implies that the total space is regular. They can be worse than nodes. Our main result is a combination of improvements due to Stoll, McCallum-Poonen, and Lorenzini-Tucker.

Today's Goal

Today's goal: Tighter bounds coming from primes of bad reduction.

Let p be some prime. Let $\mathcal{C}$ be a regular minimal model of C over $\mathbb{Z}_p$. This implies that the total space is regular. They can be worse than nodes. Our main result is a combination of improvements due to Stoll, McCallum-Poonen, and Lorenzini-Tucker.

Theorem (K-Zureick-Brown)

Let p be a prime with $p > 2g(C)$. Suppose $r < g$ then

$$|C(\mathbb{Q})| \leq |\mathcal{C}_0^{\mathrm{sm}}(\mathbb{F}_p)| + 2r$$

Today's Goal

Today's goal: Tighter bounds coming from primes of bad reduction.

Let p be some prime. Let $\mathcal{C}$ be a regular minimal model of C over $\mathbb{Z}_p$. This implies that the total space is regular. They can be worse than nodes. Our main result is a combination of improvements due to Stoll, McCallum-Poonen, and Lorenzini-Tucker.

Theorem (K-Zureick-Brown)

Let p be a prime with $p > 2g(C)$. Suppose $r < g$ then

$$|C(\mathbb{Q})| \leq |\mathcal{C}_0^{\mathrm{sm}}(\mathbb{F}_p)| + 2r$$

This bound can be sharp! Here, the proof depends on the number of smooth points of the closed fiber of regular minimal model. This bound depends on the curve and can be arbitrarily large.

Today's Goal

Today's goal: Tighter bounds coming from primes of bad reduction.

Let p be some prime. Let $\mathcal{C}$ be a regular minimal model of C over $\mathbb{Z}_p$. This implies that the total space is regular. They can be worse than nodes. Our main result is a combination of improvements due to Stoll, McCallum-Poonen, and Lorenzini-Tucker.

Theorem (K-Zureick-Brown)

Let p be a prime with $p > 2g(C)$. Suppose $r < g$ then

$$|C(\mathbb{Q})| \leq |\mathcal{C}_0^{\mathrm{sm}}(\mathbb{F}_p)| + 2r$$

This bound can be sharp! Here, the proof depends on the number of smooth points of the closed fiber of regular minimal model. This bound depends on the curve and can be arbitrarily large.

However, next week David Zureick-Brown will talk about making this bound uniform in genus for a more restrictive class of curves.

Outline of Chabauty's proof

Coleman's approach is based on an earlier ineffective method of Chabauty. If C has a rational point x_0 , use it for the base-point of the Abel-Jacobi map $i : C \to J$. So the identity of J corresponds to a rational point of C. Now, intuitively, we have a

Outline of Chabauty's proof

Coleman's approach is based on an earlier ineffective method of Chabauty. If C has a rational point x_0 , use it for the base-point of the Abel-Jacobi map $i : C \to J$. So the identity of J corresponds to a rational point of C. Now, intuitively, we have a

Naive hope: If $r < g$, then the rational point $J(\mathbb{Q})$ are contained in an Abelian subvariety $A \subset J$.

Outline of Chabauty's proof

Coleman's approach is based on an earlier ineffective method of Chabauty. If C has a rational point x_0 , use it for the base-point of the Abel-Jacobi map $i : C \to J$. So the identity of J corresponds to a rational point of C. Now, intuitively, we have a

Naive hope: If $r < g$, then the rational point $J(\mathbb{Q})$ are contained in an Abelian subvariety $A \subset J$.

If this were true, we could intersect C with A in J. We know that C is not contained in a proper Abelian subvariety of J. So, as algebraic subvarieties, C and A can only intersect in finitely many points.

Outline of Chabauty's proof

Coleman's approach is based on an earlier ineffective method of Chabauty. If C has a rational point x_0 , use it for the base-point of the Abel-Jacobi map $i : C \to J$. So the identity of J corresponds to a rational point of C. Now, intuitively, we have a

Naive hope: If $r < g$, then the rational point $J(\mathbb{Q})$ are contained in an Abelian subvariety $A \subset J$.

If this were true, we could intersect C with A in J. We know that C is not contained in a proper Abelian subvariety of J. So, as algebraic subvarieties, C and A can only intersect in finitely many points.

Unfortunately, the naive hope does not hold.

Outline of Chabauty's proof (cont'd)

Fortunately, the naive hope holds p-adically.

Outline of Chabauty's proof (cont'd)

Fortunately, the naive hope holds p-adically.

There is a globally defined p-adic logarithm,

$$\mathrm{Log} : J(\mathbb{Q}_p) \to \mathrm{Lie}(J)(\mathbb{Q}_p) = \mathbb{Q}_p^g.$$

This is very strange if you think about it.

Outline of Chabauty's proof (cont'd)

Fortunately, the naive hope holds p-adically.

There is a globally defined p-adic logarithm,

$$\mathrm{Log} : J(\mathbb{Q}_p) \to \mathrm{Lie}(J)(\mathbb{Q}_p) = \mathbb{Q}_p^g.$$

This is very strange if you think about it.

By arguments involving p-adic Lie groups, $\mathrm{Log}(J(\mathbb{Q}))$ is contained in a proper subspace V of $\mathrm{Lie}(J)$. By a p-adic analysis argument, $C \cap J(\mathbb{Q})$ is finite.

Coleman's proof

To make this proof effective, Coleman needed a genuinely new idea.

Coleman's proof

To make this proof effective, Coleman needed a genuinely new idea.

Coleman's amazing insight: the composition of Abel-Jacobi and logarithm $\mathrm{Log} \circ i$ can be computed locally on the curve.

Coleman's proof

To make this proof effective, Coleman needed a genuinely new idea.

Coleman's amazing insight: the composition of Abel-Jacobi and logarithm $\mathrm{Log} \circ i$ can be computed locally on the curve.

Specifically, we note $\mathrm{Lie}(J) = \Omega(C)^\vee$. We pick a 1-form $\omega \in \Omega(C)$ vanishing on the subspace V containing the logarithms of the rational points of J. Then the composition

$$C(\mathbb{Q}_p) \xrightarrow{i} J(\mathbb{Q}_p) \xrightarrow{\mathrm{Log}} \mathrm{Lie}(J) \xrightarrow{\omega} \mathbb{Q}_p$$

vanishes on $C(\mathbb{Q})$.

Coleman's proof

To make this proof effective, Coleman needed a genuinely new idea.

Coleman's amazing insight: the composition of Abel-Jacobi and logarithm $\mathrm{Log} \circ i$ can be computed locally on the curve.

Specifically, we note $\mathrm{Lie}(J) = \Omega(C)^\vee$. We pick a 1-form $\omega \in \Omega(C)$ vanishing on the subspace V containing the logarithms of the rational points of J. Then the composition

$$C(\mathbb{Q}_p) \xrightarrow{i} J(\mathbb{Q}_p) \xrightarrow{\mathrm{Log}} \mathrm{Lie}(J) \xrightarrow{\omega} \mathbb{Q}_p$$

vanishes on $C(\mathbb{Q})$.

It turns out that this composition can be written as a p-adic integral

$$f_\omega : x \mapsto \int_{x_0}^{x} \omega.$$

Coleman's proof (cont'd)

The function f_ω is a p-adic integral as defined by Coleman. It is characterized by two properties:

1. in residue discs, it can be computed by antidifferentiating a power series for ω, and
2. it obeys a change of variables formula with respect to any lift of Frobenius (the Dwork principle).

Coleman's proof (cont'd)

The function f_ω is a p-adic integral as defined by Coleman. It is characterized by two properties:

1. in residue discs, it can be computed by antidifferentiating a power series for ω, and
2. it obeys a change of variables formula with respect to any lift of Frobenius (the Dwork principle).

Here, a residue disc means $\rho^{-1}(Q)$ for the specialization map $\rho : C(\mathbb{Q}_p) \to \mathcal{C}_0(\mathbb{F}_p)$ given by

$$\rho(x) = \overline{\{x\}} \cap \mathcal{C}_0(\mathbb{F}_p)$$

and $Q \in \mathcal{C}_0(\mathbb{F}_p)$. In other words, all points specializing to the same point. Around a smooth point in $\mathcal{C}_0(\mathbb{F}_p)$, they look like open discs p-adically.

Coleman's proof (cont'd)

The function f_ω is a p-adic integral as defined by Coleman. It is characterized by two properties:

1. in residue discs, it can be computed by antidifferentiating a power series for ω, and
2. it obeys a change of variables formula with respect to any lift of Frobenius (the Dwork principle).

Here, a residue disc means $\rho^{-1}(Q)$ for the specialization map $\rho : C(\mathbb{Q}_p) \to \mathcal{C}_0(\mathbb{F}_p)$ given by

$$\rho(x) = \overline{\{x\}} \cap \mathcal{C}_0(\mathbb{F}_p)$$

and $Q \in \mathcal{C}_0(\mathbb{F}_p)$. In other words, all points specializing to the same point. Around a smooth point in $\mathcal{C}_0(\mathbb{F}_p)$, they look like open discs p-adically.

Now, to bound the number of rational points, we work residue disc by residue disc. For each residue point $Q \in C(\mathbb{F}_p)$, we concede that there might be one rational point x_Q with $\rho(x_Q) = Q$. Could there be more?

Coleman's proof (cont'd)

We pick a uniformizer t at x_Q and write

$$\omega = \sum_{n=0}^{\infty} a_n t^n dt$$

in the residue disc. Then,

$$f_\omega = \sum_{n=0}^{\infty} \frac{a_n}{n+1} t^{n+1}.$$

Coleman's proof (cont'd)

We pick a uniformizer t at x_Q and write

$$\omega = \sum_{n=0}^{\infty} a_n t^n dt$$

in the residue disc. Then,

$$f_\omega = \sum_{n=0}^{\infty} \frac{a_n}{n+1} t^{n+1}.$$

The Newton polygon for f_ω is very similar to that of ω. In fact, f_ω has at most one more zero in $\rho^{-1}(Q)$ than ω does. To get a handle on the number of zeroes, we restrict ω to the closed fiber. By multiplying by a power of p, can suppose that ω does not vanish on the closed fiber $\mathcal{C}_0$. Then the number of zeroes of ω in $\rho^{-1}(Q)$ is equal to the order of vanishing of $\omega|_{\mathcal{C}_0}$ at Q.

Coleman's proof (concluded)

Summing over residue classes $Q \in \mathcal{C}_0(\mathbb{F}_p)$, we get

$$
\begin{aligned}
|C(\mathbb{Q})| \leq |f_\omega^{-1}(0)| &= \sum_{Q \in \mathcal{C}_0(\mathbb{F}_p)} (1 + \mathrm{ord}_Q(\omega|_{\mathcal{C}_0})) \\
&= |\mathcal{C}_0(\mathbb{F}_p)| + \deg(\omega) \\
&= |\mathcal{C}_0(\mathbb{F}_p)| + 2g - 2.
\end{aligned}
$$

Stoll's improvement

Coleman's bound was improved by Stoll:

Theorem (Stoll)

If $r < g$ *then* $|C(\mathbb{Q})| \leq |\mathcal{C}_0(\mathbb{F}_p)| + 2r$.

Stoll's improvement

Coleman's bound was improved by Stoll:

Theorem (Stoll)

If $r < g$ then $|C(\mathbb{Q})| \leq |\mathcal{C}_0(\mathbb{F}_p)| + 2r$.

This improvement is important! A sharper bound means less searching for a rational point that may not exist.

Proof of Stoll's improvement

Stoll improved the bound by picking a good choice of ω for each residue disc.

Proof of Stoll's improvement

Stoll improved the bound by picking a good choice of ω for each residue disc.

Let $\Lambda \subset \Gamma(J_{\mathbb{Q}_p}, \Omega^1)$ be the 1-forms vanishing on $\overline{J(\mathbb{Q})}$. For each residue class $Q \in \mathcal{C}_0(\mathbb{F}_p)$, let

$$n(Q) = \min\{\mathrm{ord}_Q(\omega|_{\mathcal{C}_0}) | 0 \neq \omega \in \Lambda\}.$$

Let the Chabauty divisor on $\mathcal{C}_0$ be

$$D_0 = \sum_Q n(Q)(Q).$$

Proof of Stoll's improvement

Stoll improved the bound by picking a good choice of ω for each residue disc.

Let $\Lambda \subset \Gamma(J_{\mathbb{Q}_p}, \Omega^1)$ be the 1-forms vanishing on $\overline{J(\mathbb{Q})}$. For each residue class $Q \in \mathcal{C}_0(\mathbb{F}_p)$, let

$$n(Q) = \min\{\mathrm{ord}_Q(\omega|_{\mathcal{C}_0}) | 0 \neq \omega \in \Lambda\}.$$

Let the Chabauty divisor on $\mathcal{C}_0$ be

$$D_0 = \sum_Q n(Q)(Q).$$

Note that by Coleman's argument,

$$|C(\mathbb{Q}) \cap \rho^{-1}(Q)| \leq 1 + n(Q).$$

Proof of Stoll's improvement

Stoll improved the bound by picking a good choice of ω for each residue disc.

Let $\Lambda \subset \Gamma(J_{\mathbb{Q}_p}, \Omega^1)$ be the 1-forms vanishing on $\overline{J(\mathbb{Q})}$. For each residue class $Q \in \mathcal{C}_0(\mathbb{F}_p)$, let

$$n(Q) = \min\{\mathrm{ord}_Q(\omega|_{\mathcal{C}_0}) | 0 \neq \omega \in \Lambda\}.$$

Let the Chabauty divisor on $\mathcal{C}_0$ be

$$D_0 = \sum_Q n(Q)(Q).$$

Note that by Coleman's argument,

$$|C(\mathbb{Q}) \cap \rho^{-1}(Q)| \leq 1 + n(Q).$$

By summing over residue classes, one gets

$$|C(\mathbb{Q})| \leq |\mathcal{C}_0(\mathbb{F}_p)| + \deg(D_0).$$

Proof of Stoll's improvement (concluded)

Now, we just need to bound $\deg(D_0)$. Every $\omega \in \Lambda$ extends (up to a multiple by a power of p) to a regular 1-form vanishing on D_0.

Proof of Stoll's improvement (concluded)

Now, we just need to bound $\deg(D_0)$. Every $\omega \in \Lambda$ extends (up to a multiple by a power of p) to a regular 1-form vanishing on D_0.

By a semi-continuity argument and using Clifford's theorem, one gets

$$\dim \Lambda \leq \dim H^0(\mathcal{C}_0, K_{\mathcal{C}_0} - D_0) \leq g - \frac{\deg(D_0)}{2}.$$

Proof of Stoll's improvement (concluded)

Now, we just need to bound $\deg(D_0)$. Every $\omega \in \Lambda$ extends (up to a multiple by a power of p) to a regular 1-form vanishing on D_0.

By a semi-continuity argument and using Clifford's theorem, one gets

$$\dim \Lambda \leq \dim H^0(\mathcal{C}_0, K_{\mathcal{C}_0} - D_0) \leq g - \frac{\deg(D_0)}{2}.$$

Since $\dim \Lambda = g - r$, $\deg(D_0) \leq 2r$.

Proof of Stoll's improvement (concluded)

Now, we just need to bound $\deg(D_0)$. Every $\omega \in \Lambda$ extends (up to a multiple by a power of p) to a regular 1-form vanishing on D_0.

By a semi-continuity argument and using Clifford's theorem, one gets

$$\dim \Lambda \leq \dim H^0(\mathcal{C}_0, K_{\mathcal{C}_0} - D_0) \leq g - \frac{\deg(D_0)}{2}.$$

Since $\dim \Lambda = g - r$, $\deg(D_0) \leq 2r$.

Therefore, we get

$$|C(\mathbb{Q})| \leq |\mathcal{C}_0(\mathbb{F}_p)| + 2r.$$

Bad reduction case

The bad reduction case of Coleman's bound was proved independently by Lorenzini-Tucker and McCallum-Poonen. The bad reduction case of the Stoll bound was proved for hyperelliptic curves by Stoll and the general case was posed as a question in a paper of McCallum-Poonen.

Bad reduction case

The bad reduction case of Coleman's bound was proved independently by Lorenzini-Tucker and McCallum-Poonen. The bad reduction case of the Stoll bound was proved for hyperelliptic curves by Stoll and the general case was posed as a question in a paper of McCallum-Poonen.

The set-up for the bad reduction case is where $\mathcal{C}$ is a regular minimal model over $\mathbb{Z}_p$. This means that the total space is regular, but there are no conditions on the types of singularities on the closed fiber. They can be worse than nodes.

Bad reduction case

The bad reduction case of Coleman's bound was proved independently by Lorenzini-Tucker and McCallum-Poonen. The bad reduction case of the Stoll bound was proved for hyperelliptic curves by Stoll and the general case was posed as a question in a paper of McCallum-Poonen.

The set-up for the bad reduction case is where $\mathcal{C}$ is a regular minimal model over $\mathbb{Z}_p$. This means that the total space is regular, but there are no conditions on the types of singularities on the closed fiber. They can be worse than nodes.

Theorem:(Lorenzini-Tucker,McCallum-Poonen) Suppose $r < g$ then

$$|C(\mathbb{Q})| \leq |\mathcal{C}_0^{\mathrm{sm}}(\mathbb{F}_p)| + 2g - 2.$$

Bad reduction case

The bad reduction case of Coleman's bound was proved independently by Lorenzini-Tucker and McCallum-Poonen. The bad reduction case of the Stoll bound was proved for hyperelliptic curves by Stoll and the general case was posed as a question in a paper of McCallum-Poonen.

The set-up for the bad reduction case is where $\mathcal{C}$ is a regular minimal model over $\mathbb{Z}_p$. This means that the total space is regular, but there are no conditions on the types of singularities on the closed fiber. They can be worse than nodes.

Theorem:(Lorenzini-Tucker,McCallum-Poonen) Suppose $r < g$ then

$$|C(\mathbb{Q})| \leq |\mathcal{C}_0^{\mathrm{sm}}(\mathbb{F}_p)| + 2g - 2.$$

The reason why we only need to look at the smooth points is that any rational point of C specializes to a smooth point of $\mathcal{C}_0$. Therefore, we need only consider residue classes in $\mathcal{C}_0^{\mathrm{sm}}(\mathbb{F}_p)$.

Extending Stoll's bound

Stoll's proof does not extend to the bad reduction case! It breaks in a way that a lot of semicontinuity arguments break. We can proceed as before to get

$$\dim \Lambda \leq \dim H^0(\mathcal{C}_0, K_{\mathcal{C}_0} - D_0).$$

Extending Stoll's bound

Stoll's proof does not extend to the bad reduction case! It breaks in a way that a lot of semicontinuity arguments break. We can proceed as before to get

$$\dim \Lambda \leq \dim H^0(\mathcal{C}_0, K_{\mathcal{C}_0} - D_0).$$

Unfortunately, Clifford's theorem does not hold and we do not get a bound on the right-hand side. This should probably be expected because the divisor $K_{\mathcal{C}_0} - D_0$ could be really negative on a component of the closed fiber and then the section just vanishes on the component. But there could be lots of sections on other components. The space of sections is just too big and cannot be bounded in the usual way.

A new question

We need to do something different. Perhaps we want to think in the following direction. Let D be a divisor on C supported on $C(\mathbb{Q}_p^{\mathrm{unr}})$. Let F_0 be a divisor on $\mathcal{C}_0^{\mathrm{sm}}(\overline{\mathbb{F}_p})$. Let

$$|D|_{F_0} = \{D' \in |D| \;\big|\; F_0 \subset \overline{D'}\}$$

where $|D|$ means the set of all divisors linearly equivalent to D.

A new question

We need to do something different. Perhaps we want to think in the following direction. Let D be a divisor on C supported on $C(\mathbb{Q}_p^{\mathrm{unr}})$. Let F_0 be a divisor on $\mathcal{C}_0^{\mathrm{sm}}(\overline{\mathbb{F}_p})$. Let

$$|D|_{F_0} = \{D' \in |D| \bigm| F_0 \subset \overline{D'}\}$$

where $|D|$ means the set of all divisors linearly equivalent to D.

Definition: We say the *rank* $r(D, F_0)$ is greater than or equal to r if for any rank r effective divisor E supported on $C(\mathbb{K})$, $|D - E|_{F_0} \neq \emptyset$.

A new question

We need to do something different. Perhaps we want to think in the following direction. Let D be a divisor on C supported on $C(\mathbb{Q}_p^{\mathrm{unr}})$. Let F_0 be a divisor on $\mathcal{C}_0^{\mathrm{sm}}(\overline{\mathbb{F}_p})$. Let

$$|D|_{F_0} = \{D' \in |D| \mid F_0 \subset \overline{D'}\}$$

where $|D|$ means the set of all divisors linearly equivalent to D.

Definition: We say the *rank* $r(D, F_0)$ is greater than or equal to r if for any rank r effective divisor E supported on $C(\mathbb{K})$, $|D - E|_{F_0} \neq \emptyset$.

For Stoll's bounds, we immediately have $\dim \Lambda - 1 < r(K_{\mathcal{C}}, D_0)$ because we can assign $\dim \Lambda - 1$ base-points on the 1-forms in V. We would need to prove $r(K_{\mathcal{C}}, D_0) < g - 1 - \frac{\deg(D_0)}{2}$

A new question

We need to do something different. Perhaps we want to think in the following direction. Let D be a divisor on C supported on $C(\mathbb{Q}_p^{\mathrm{unr}})$. Let F_0 be a divisor on $\mathcal{C}_0^{\mathrm{sm}}(\overline{\mathbb{F}_p})$. Let

$$|D|_{F_0} = \{D' \in |D| \mid F_0 \subset \overline{D'}\}$$

where $|D|$ means the set of all divisors linearly equivalent to D.

Definition: We say the *rank* $r(D, F_0)$ is greater than or equal to r if for any rank r effective divisor E supported on $C(\mathbb{K})$, $|D - E|_{F_0} \neq \emptyset$.

For Stoll's bounds, we immediately have $\dim \Lambda - 1 < r(K_{\mathcal{C}}, D_0)$ because we can assign $\dim \Lambda - 1$ base-points on the 1-forms in V. We would need to prove $r(K_{\mathcal{C}}, D_0) < g - 1 - \frac{\deg(D_0)}{2}$

Question: Can we bound $r(D, F_0)$ in terms of $\mathcal{C}_0$, $\deg(D)$ and F_0?

A new question

We need to do something different. Perhaps we want to think in the following direction. Let D be a divisor on C supported on $C(\mathbb{Q}_p^{\mathrm{unr}})$. Let F_0 be a divisor on $\mathcal{C}_0^{\mathrm{sm}}(\overline{\mathbb{F}_p})$. Let

$$|D|_{F_0} = \{D' \in |D| \mid F_0 \subset \overline{D'}\}$$

where $|D|$ means the set of all divisors linearly equivalent to D.

Definition: We say the *rank* $r(D, F_0)$ is greater than or equal to r if for any rank r effective divisor E supported on $C(\mathbb{K})$, $|D - E|_{F_0} \neq \emptyset$.

For Stoll's bounds, we immediately have $\dim \Lambda - 1 < r(K_{\mathcal{C}}, D_0)$ because we can assign $\dim \Lambda - 1$ base-points on the 1-forms in V. We would need to prove $r(K_{\mathcal{C}}, D_0) < g - 1 - \frac{\deg(D_0)}{2}$

Question: Can we bound $r(D, F_0)$ in terms of $\mathcal{C}_0$, $\deg(D)$ and F_0?

By the way, it suffices to consider only semistable curves, and we shall do so for the rest of the talk.

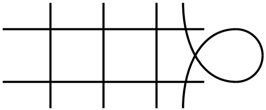
A more general framework

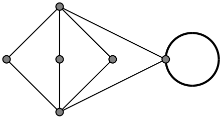
Let $\mathbb{K}$ be a discretely valued field with valuation ring $\mathcal{O}$ and residue field $\mathbf{k}$. Let C be a curve with semistable reduction over $\mathbb{K}$. In other words, C can be completed to a family of curves $\mathcal{C}$ over $\mathcal{O}$ such that the total space is regular and that the closed fiber $\mathcal{C}_0$ has ordinary double-points as singularities.

A more general framework

Let $\mathbb{K}$ be a discretely valued field with valuation ring $\mathcal{O}$ and residue field $\mathbf{k}$. Let C be a curve with semistable reduction over $\mathbb{K}$. In other words, C can be completed to a family of curves $\mathcal{C}$ over $\mathcal{O}$ such that the total space is regular and that the closed fiber $\mathcal{C}_0$ has ordinary double-points as singularities.

Here's a semistable curve and its dual graph.

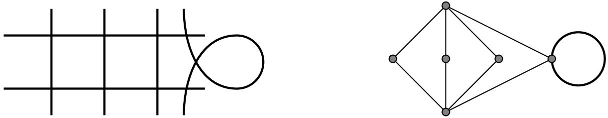




A more general framework

Let $\mathbb{K}$ be a discretely valued field with valuation ring $\mathcal{O}$ and residue field $\mathbf{k}$. Let C be a curve with semistable reduction over $\mathbb{K}$. In other words, C can be completed to a family of curves $\mathcal{C}$ over $\mathcal{O}$ such that the total space is regular and that the closed fiber $\mathcal{C}_0$ has ordinary double-points as singularities.

Here's a semistable curve and its dual graph.



Let D be a divisor on C, supported on $C(\mathbb{K})$. Would like to bound the dimension of $H^0(C, \mathcal{O}(D))$ by using the closed fiber.

Baker-Norine linear systems on graphs

The Baker-Norine theory of linear systems on graphs gives such bounds. Let the multi-degree $\underline{\deg}$ of a divisor D to be the formal sum

$$\underline{\deg}(D) = \sum_v \deg(\mathcal{O}(D)|_{C_v})(v)$$

where C_v are the components of $\mathcal{C}_0$.

Baker-Norine linear systems on graphs

The Baker-Norine theory of linear systems on graphs gives such bounds. Let the multi-degree $\underline{\deg}$ of a divisor D to be the formal sum

$$\underline{\deg}(D) = \sum_v \deg(\mathcal{O}(D)|_{C_v})(v)$$

where C_v are the components of $\mathcal{C}_0$.

Baker-Norine define a rank $r(\underline{\deg}(D))$ in terms of the combinatorics of the dual graph Γ of $\mathcal{C}_0$. I'll explain it in a minute.

Baker-Norine linear systems on graphs

The Baker-Norine theory of linear systems on graphs gives such bounds. Let the multi-degree $\underline{\deg}$ of a divisor D to be the formal sum

$$\underline{\deg}(D) = \sum_v \deg(\mathcal{O}(D)|_{C_v})(v)$$

where C_v are the components of $\mathcal{C}_0$.

Baker-Norine define a rank $r(\underline{\deg}(D))$ in terms of the combinatorics of the dual graph Γ of $\mathcal{C}_0$. I'll explain it in a minute.

The bound obeys the specialization lemma:

$$\dim(H^0(C, \mathcal{O}(D))) - 1 \leq r(\underline{\deg}(D)).$$

Baker-Norine linear systems on graphs

The Baker-Norine theory of linear systems on graphs gives such bounds. Let the multi-degree $\underline{\deg}$ of a divisor D to be the formal sum

$$\underline{\deg}(D) = \sum_v \deg(\mathcal{O}(D)|_{C_v})(v)$$

where C_v are the components of $\mathcal{C}_0$.

Baker-Norine define a rank $r(\underline{\deg}(D))$ in terms of the combinatorics of the dual graph Γ of $\mathcal{C}_0$. I'll explain it in a minute.

The bound obeys the specialization lemma:

$$\dim(H^0(C, \mathcal{O}(D))) - 1 \leq r(\underline{\deg}(D)).$$

These bounds are particularly nice in the case where all components of $\mathcal{C}_0$ are rational (the maximally degenerate case). Not so good in general.

Extension hierarchy for linear equivalence problem

To make sense of more interesting degenerations, we apply a certain extension hierarchy to this question. The steps have technical names which are inspired by the Néron model. Suppose I am given two divisors $\mathcal{D}_1$ and $\mathcal{D}_2$ of the same degree on $\mathcal{C}$. I want to know if they are linearly equivalent on C. In other words, does there exist a rational function s of $\mathcal{O}(\mathcal{D}_1 - \mathcal{D}_2)|_C$? Write D_1, D_2 for the generic fibers of $\mathcal{D}_1, \mathcal{D}_2$.

Extension hierarchy for linear equivalence problem

To make sense of more interesting degenerations, we apply a certain extension hierarchy to this question. The steps have technical names which are inspired by the Néron model. Suppose I am given two divisors $\mathcal{D}_1$ and $\mathcal{D}_2$ of the same degree on $\mathcal{C}$. I want to know if they are linearly equivalent on C. In other words, does there exist a rational function s of $\mathcal{O}(\mathcal{D}_1 - \mathcal{D}_2)|_C$? Write D_1, D_2 for the generic fibers of $\mathcal{D}_1, \mathcal{D}_2$.

1. Try to construct s_0 on the closed fiber such that $(s_0) = (\mathcal{D}_1)_0 - (\mathcal{D}_2)_0$.

Extension hierarchy for linear equivalence problem

To make sense of more interesting degenerations, we apply a certain extension hierarchy to this question. The steps have technical names which are inspired by the Néron model. Suppose I am given two divisors $\mathcal{D}_1$ and $\mathcal{D}_2$ of the same degree on $\mathcal{C}$. I want to know if they are linearly equivalent on C. In other words, does there exist a rational function s of $\mathcal{O}(\mathcal{D}_1 - \mathcal{D}_2)|_C$? Write D_1, D_2 for the generic fibers of $\mathcal{D}_1, \mathcal{D}_2$.

1. Try to construct s_0 on the closed fiber such that $(s_0) = (\mathcal{D}_1)_0 - (\mathcal{D}_2)_0$.
    1. numerical: Is there an extension $\mathcal{L}$ of $\mathcal{O}(D_1 - D_2)$ to $\mathcal{C}$ that has degree 0 on every component of the closed fiber?

Extension hierarchy for linear equivalence problem

To make sense of more interesting degenerations, we apply a certain extension hierarchy to this question. The steps have technical names which are inspired by the Néron model. Suppose I am given two divisors $\mathcal{D}_1$ and $\mathcal{D}_2$ of the same degree on $\mathcal{C}$. I want to know if they are linearly equivalent on C. In other words, does there exist a rational function s of $\mathcal{O}(\mathcal{D}_1 - \mathcal{D}_2)|_C$? Write D_1, D_2 for the generic fibers of $\mathcal{D}_1, \mathcal{D}_2$.

1. Try to construct s_0 on the closed fiber such that $(s_0) = (\mathcal{D}_1)_0 - (\mathcal{D}_2)_0$.
    1. numerical: Is there an extension $\mathcal{L}$ of $\mathcal{O}(D_1 - D_2)$ to $\mathcal{C}$ that has degree 0 on every component of the closed fiber?
    2. Abelian: For each component C_v of the closed fiber, is there a section s_v on C_v of $\mathcal{L}|_{C_v}$ with $(s_v) = ((\mathcal{D}_1)_0 - (\mathcal{D}_2))|_{C_v}$?

Extension hierarchy for linear equivalence problem

To make sense of more interesting degenerations, we apply a certain extension hierarchy to this question. The steps have technical names which are inspired by the Néron model. Suppose I am given two divisors $\mathcal{D}_1$ and $\mathcal{D}_2$ of the same degree on $\mathcal{C}$. I want to know if they are linearly equivalent on C. In other words, does there exist a rational function s of $\mathcal{O}(\mathcal{D}_1 - \mathcal{D}_2)|_C$? Write D_1, D_2 for the generic fibers of $\mathcal{D}_1, \mathcal{D}_2$.

1. Try to construct s_0 on the closed fiber such that $(s_0) = (\mathcal{D}_1)_0 - (\mathcal{D}_2)_0$.
   1. numerical: Is there an extension $\mathcal{L}$ of $\mathcal{O}(D_1 - D_2)$ to $\mathcal{C}$ that has degree 0 on every component of the closed fiber?
   2. Abelian: For each component C_v of the closed fiber, is there a section s_v on C_v of $\mathcal{L}|_{C_v}$ with $(s_v) = ((\mathcal{D}_1)_0 - (\mathcal{D}_2))|_{C_v}$?
   3. toric: Can the sections s_v be chosen to agree on nodes?

Extension hierarchy for linear equivalence problem

To make sense of more interesting degenerations, we apply a certain extension hierarchy to this question. The steps have technical names which are inspired by the Néron model. Suppose I am given two divisors $\mathcal{D}_1$ and $\mathcal{D}_2$ of the same degree on $\mathcal{C}$. I want to know if they are linearly equivalent on C. In other words, does there exist a rational function s of $\mathcal{O}(\mathcal{D}_1 - \mathcal{D}_2)|_C$? Write D_1, D_2 for the generic fibers of $\mathcal{D}_1, \mathcal{D}_2$.

1. Try to construct s_0 on the closed fiber such that $(s_0) = (\mathcal{D}_1)_0 - (\mathcal{D}_2)_0$.
    1. numerical: Is there an extension $\mathcal{L}$ of $\mathcal{O}(D_1 - D_2)$ to $\mathcal{C}$ that has degree 0 on every component of the closed fiber?
    2. Abelian: For each component C_v of the closed fiber, is there a section s_v on C_v of $\mathcal{L}|_{C_v}$ with $(s_v) = ((\mathcal{D}_1)_0 - (\mathcal{D}_2))|_{C_v}$?
    3. toric: Can the sections s_v be chosen to agree on nodes?
2. Use deformation theory to extend the glued together section s_0 to $\mathcal{C}$.

Extension hierarchy for linear equivalence problem

To make sense of more interesting degenerations, we apply a certain extension hierarchy to this question. The steps have technical names which are inspired by the Néron model. Suppose I am given two divisors $\mathcal{D}_1$ and $\mathcal{D}_2$ of the same degree on $\mathcal{C}$. I want to know if they are linearly equivalent on C. In other words, does there exist a rational function s of $\mathcal{O}(\mathcal{D}_1 - \mathcal{D}_2)|_C$? Write D_1, D_2 for the generic fibers of $\mathcal{D}_1, \mathcal{D}_2$.

1. Try to construct s_0 on the closed fiber such that $(s_0) = (\mathcal{D}_1)_0 - (\mathcal{D}_2)_0$.
    1. numerical: Is there an extension $\mathcal{L}$ of $\mathcal{O}(D_1 - D_2)$ to $\mathcal{C}$ that has degree 0 on every component of the closed fiber?
    2. Abelian: For each component C_v of the closed fiber, is there a section s_v on C_v of $\mathcal{L}|_{C_v}$ with $(s_v) = ((\mathcal{D}_1)_0 - (\mathcal{D}_2))|_{C_v}$?
    3. toric: Can the sections s_v be chosen to agree on nodes?
2. Use deformation theory to extend the glued together section s_0 to $\mathcal{C}$.

We will concentrate on the first step.

The rank hierarchy

This hierarchy lets us define new rank functions following Baker-Norine by asking how many base-points we can assign. We say a divisor D on C has i-rank $\geq r$ if for any effective divisor E on $C(\mathbb{K})$ of degree r, steps $(1) - (i)$ are satisfied for $\mathcal{D} = \overline{D}, \mathcal{E} = \overline{E}$:

The rank hierarchy

This hierarchy lets us define new rank functions following Baker-Norine by asking how many base-points we can assign. We say a divisor D on C has i-rank $\geq r$ if for any effective divisor E on $C(\mathbb{K})$ of degree r, steps $(1) - (i)$ are satisfied for $\mathcal{D} = \overline{D}, \mathcal{E} = \overline{E}$:

1. numerical: there is a divisor $\varphi = \sum_v a_v C_v$ supported on the closed fiber such that
$$\deg(\mathcal{O}(\mathcal{D} - \mathcal{E})(\varphi)|_{C_v}) \geq 0$$
for all v.

The rank hierarchy

This hierarchy lets us define new rank functions following Baker-Norine by asking how many base-points we can assign. We say a divisor D on C has i-rank $\geq r$ if for any effective divisor E on $C(\mathbb{K})$ of degree r, steps $(1) - (i)$ are satisfied for $\mathcal{D} = \overline{D}, \mathcal{E} = \overline{E}$:

1. numerical: there is a divisor $\varphi = \sum_v a_v C_v$ supported on the closed fiber such that
$$\deg(\mathcal{O}(\mathcal{D} - \mathcal{E})(\varphi)|_{C_v}) \geq 0$$
for all v.
2. Abelian: For each component C_v of the closed fiber, there is a non-vanishing section s_v on C_v of $\mathcal{O}(\mathcal{D} - \mathcal{E})(\varphi)|_{C_v}$.

The rank hierarchy

This hierarchy lets us define new rank functions following Baker-Norine by asking how many base-points we can assign. We say a divisor D on C has i-rank $\geq r$ if for any effective divisor E on $C(\mathbb{K})$ of degree r, steps $(1) - (i)$ are satisfied for $\mathcal{D} = \overline{D}, \mathcal{E} = \overline{E}$:

1. numerical: there is a divisor $\varphi = \sum_v a_v C_v$ supported on the closed fiber such that
$$\deg(\mathcal{O}(\mathcal{D} - \mathcal{E})(\varphi)|_{C_v}) \geq 0$$
for all v.
2. Abelian: For each component C_v of the closed fiber, there is a non-vanishing section s_v on C_v of $\mathcal{O}(\mathcal{D} - \mathcal{E})(\varphi)|_{C_v}$.
3. toric: The sections s_v be chosen to agree across nodes.

New rank functions

So we have rank functions $r_{\text{num}}, r_{\text{Ab}}, r_{\text{tor}}$.

New rank functions

So we have rank functions $r_{\mathrm{num}}, r_{\mathrm{Ab}}, r_{\mathrm{tor}}$.

1. $r_{\mathrm{num}}(D)$ depends only on the multi-degree of D, that is $\deg(D|_{C_v})$ for all v

New rank functions

So we have rank functions $r_{\mathrm{num}}, r_{\mathrm{Ab}}, r_{\mathrm{tor}}$.

1. $r_{\mathrm{num}}(D)$ depends only on the multi-degree of D, that is $\deg(D|_{C_v})$ for all v
2. $r_{\mathrm{Ab}}, r_{\mathrm{tor}}$ depend only on $\mathcal{D}_0$.

New rank functions

So we have rank functions $r_{\text{num}}, r_{\text{Ab}}, r_{\text{tor}}$.

1. $r_{\text{num}}(D)$ depends only on the multi-degree of D, that is $\deg(D|_{C_v})$ for all v
2. $r_{\text{Ab}}, r_{\text{tor}}$ depend only on $\mathcal{D}_0$.

The rank functions $r_{\text{Ab}}, r_{\text{tor}}$ are sensitive to the residue field $\mathbf{k}$ since bigger $\mathbf{k}$ allows for more divisors E. But they eventually stabilize.

Numerical rank and Baker-Norine rank

But $r_{\mathrm{num}}(D)$ is not new. In fact, it is the Baker-Norine rank of $\underline{\deg}(D)$. What is called here a *multi-degree* is what Baker and Norine call a divisor on a graph.

Numerical rank and Baker-Norine rank

But $r_{\text{num}}(D)$ is not new. In fact, it is the Baker-Norine rank of $\underline{\deg}(D)$. What is called here a *multi-degree* is what Baker and Norine call a divisor on a graph.

One observes that for $\varphi = \sum_v a_v C_v$, treated as a function on $V(\Gamma)$, we have

$$\underline{\deg}(\varphi) = -\Delta(\varphi)$$

where Δ is the graph Laplacian on the dual graph:

$$\Delta(\varphi)(v) = \sum_{e=vw} (\varphi(v) - \varphi(w))$$

where the sum is over edges containing v.

Numerical rank and Baker-Norine rank

But $r_{\text{num}}(D)$ is not new. In fact, it is the Baker-Norine rank of $\underline{\deg}(D)$. What is called here a *multi-degree* is what Baker and Norine call a divisor on a graph.

One observes that for $\varphi = \sum_v a_v C_v$, treated as a function on $V(\Gamma)$, we have

$$\underline{\deg}(\varphi) = -\Delta(\varphi)$$

where Δ is the graph Laplacian on the dual graph:

$$\Delta(\varphi)(v) = \sum_{e=vw} (\varphi(v) - \varphi(w))$$

where the sum is over edges containing v.

This statement makes use of the fact that

$$\deg(\mathcal{O}(C_w)|_{C_v}) = \begin{cases} |\{\text{edges between } v \text{ and } w\}| & \text{if } v \neq w \\ -|\{\text{non-loop edges at } v\}| & \text{if } v = w. \end{cases}$$

Also, after possible unramified field extension of $\mathbb{K}$ for any multi-degree, $\underline{E} = \sum a_v(v)$, there is a divisor E on C with $\underline{\deg}(E) = \underline{E}$.

Numerical rank and Baker-Norine rank (cont'd)

Also, after possible unramified field extension of $\mathbb{K}$ for any multi-degree, $\underline{E} = \sum a_v(v)$, there is a divisor E on C with $\underline{\deg}(E) = \underline{E}$.

Consequently, unpacking the definition of r_{num}, we see that it says $r_{\text{num}}(D) \geq r$ if and only if for any multi-degree $\underline{E} \geq 0$ with $\deg(\underline{E}) = r$, there is a $\varphi : V(\Gamma) \to \mathbb{Z}$ with

$$\underline{D} - \underline{E} - \Delta(\varphi) \geq 0.$$

Specialization lemma

These rank functions satisfy a specialization lemma. For D, a divisor supported on $C(\mathbb{K})$, set

$$r_C(D) = \dim H^0(C, \mathcal{O}(D)) - 1.$$

Specialization lemma

These rank functions satisfy a specialization lemma. For D, a divisor supported on $C(\mathbb{K})$, set

$$r_C(D) = \dim H^0(C, \mathcal{O}(D)) - 1.$$

Then

$$r_C(D) \leq r_{\text{tor}}(D) \leq r_{\text{Ab}}(D) \leq r_{\text{num}}(D).$$

Specialization lemma

These rank functions satisfy a specialization lemma. For D, a divisor supported on $C(\mathbb{K})$, set

$$r_C(D) = \dim H^0(C, \mathcal{O}(D)) - 1.$$

Then

$$r_C(D) \leq r_{\mathrm{tor}}(D) \leq r_{\mathrm{Ab}}(D) \leq r_{\mathrm{num}}(D).$$

We have examples where the inequalities are strict.

Proof of Specialization lemma

The proof is essentially the same as Baker's specialization lemma.

Proof of Specialization lemma

The proof is essentially the same as Baker's specialization lemma.

First by definition, we have

$$r_{\text{tor}}(D) \leq r_{\text{Ab}}(D) \leq r_{\text{num}}(D),$$

so it suffices to show $r_C(D) \leq r_{\text{tor}}(D)$.

Proof of Specialization lemma

The proof is essentially the same as Baker's specialization lemma.

First by definition, we have

$$r_{\mathrm{tor}}(D) \leq r_{\mathrm{Ab}}(D) \leq r_{\mathrm{num}}(D),$$

so it suffices to show $r_C(D) \leq r_{\mathrm{tor}}(D)$.

One can characterize $r_C(D)$ by saying $r_C(D) \geq r$ if and only if for any effective divisor E of degree r supported on $C(\mathbb{K})$ that

$$H^0(C, \mathcal{O}(D-E)) \neq \{0\}.$$

Proof of Specialization lemma

The proof is essentially the same as Baker's specialization lemma.

First by definition, we have

$$r_{\mathrm{tor}}(D) \leq r_{\mathrm{Ab}}(D) \leq r_{\mathrm{num}}(D),$$

so it suffices to show $r_C(D) \leq r_{\mathrm{tor}}(D)$.

One can characterize $r_C(D)$ by saying $r_C(D) \geq r$ if and only if for any effective divisor E of degree r supported on $C(\mathbb{K})$ that

$$H^0(C, \mathcal{O}(D - E)) \neq \{0\}.$$

Consequently, there's a section s of $\mathcal{O}(D - E)$. The section can be extended to a rational section of $\mathcal{O}(\mathcal{D} - \mathcal{E})$ on $\mathcal{C}$. The associated divisor can be decomposed as

$$(s) = H - V$$

where H is the closure of a divisor in C and V is supported on $\mathcal{C}_0$.

Proof of Specialization lemma (cont'd)

Consequently, we can write

$$\varphi \equiv V = \sum_v a_v C_v.$$

Proof of Specialization lemma (cont'd)

Consequently, we can write

$$\varphi \equiv V = \sum_v a_v C_v.$$

Now, s can be viewed as a regular section of $\mathcal{O}(\mathcal{D} - \mathcal{E})(\varphi)$. Set $s_v = s|_{C_v}$. These are the desired sections on components.

Proof of Specialization lemma (cont'd)

Consequently, we can write

$$\varphi \equiv V = \sum_v a_v C_v.$$

Now, s can be viewed as a regular section of $\mathcal{O}(\mathcal{D} - \mathcal{E})(\varphi)$. Set $s_v = s|_{C_v}$. These are the desired sections on components.

It follows that $r_{\text{tor}}(D) \geq r$.

Clifford's theorem for r_{Ab}

Let $K_{\mathcal{C}_0}$ be the relative dualizing sheaf of the closed fiber. This is characterized by being the natural extension of the canonical bundle on C to $\mathcal{C}$, restricted to the closed fiber. Note $\underline{\deg}(K_{\mathcal{C}_0}) = \sum_v (2g(C_v) - 2 + \deg(v))(v) = K_\Gamma + \sum_v 2g(C_v)(v)$.

Clifford's theorem for r_{Ab}

Let $K_{\mathcal{C}_0}$ be the relative dualizing sheaf of the closed fiber. This is characterized by being the natural extension of the canonical bundle on C to $\mathcal{C}$, restricted to the closed fiber. Note $\underline{\deg}(K_{\mathcal{C}_0}) = \sum_v (2g(C_v) - 2 + \deg(v))(v) = K_\Gamma + \sum_v 2g(C_v)(v)$.

(No longer as much of a) Question: Is Riemann-Roch true for r_{Ab} and r_{tor}?

$$r_i(D_0) - r_i(K_{\mathcal{C}_0} - D_0) = 1 - g + \deg(D_0)?$$

Yes for r_{Ab}! By Amini-Baker.

Clifford's theorem for r_{Ab}

Let $K_{\mathcal{C}_0}$ be the relative dualizing sheaf of the closed fiber. This is characterized by being the natural extension of the canonical bundle on C to $\mathcal{C}$, restricted to the closed fiber. Note
$\underline{\deg}(K_{\mathcal{C}_0}) = \sum_v (2g(C_v) - 2 + \deg(v))(v) = K_\Gamma + \sum_v 2g(C_v)(v)$.

(No longer as much of a) Question: Is Riemann-Roch true for r_{Ab} and r_{tor}?

$$r_i(D_0) - r_i(K_{\mathcal{C}_0} - D_0) = 1 - g + \deg(D_0)?$$

Yes for r_{Ab}! By Amini-Baker.

Theorem: (Clifford-K-Zureick-Brown) Let D_0 be a divisor supported on smooth $\mathbf{k}$-points of $\mathcal{C}_0$ then

$$r_{\mathrm{Ab}}(K_{\mathcal{C}_0} - D_0) \leq g - \frac{\deg D_0}{2} - 1.$$

Clifford's theorem for r_{Ab}

Let $K_{\mathcal{C}_0}$ be the relative dualizing sheaf of the closed fiber. This is characterized by being the natural extension of the canonical bundle on C to $\mathcal{C}$, restricted to the closed fiber. Note $\underline{\deg}(K_{\mathcal{C}_0}) = \sum_v (2g(C_v) - 2 + \deg(v))(v) = K_\Gamma + \sum_v 2g(C_v)(v)$.

(No longer as much of a) Question: Is Riemann-Roch true for r_{Ab} and r_{tor}?

$$r_i(D_0) - r_i(K_{\mathcal{C}_0} - D_0) = 1 - g + \deg(D_0)?$$

Yes for r_{Ab}! By Amini-Baker.

Theorem: (Clifford-K-Zureick-Brown) Let D_0 be a divisor supported on smooth $\mathbf{k}$-points of $\mathcal{C}_0$ then

$$r_{\mathrm{Ab}}(K_{\mathcal{C}_0} - D_0) \leq g - \frac{\deg D_0}{2} - 1.$$

Note that Clifford Brown a.k.a. "Brownie" does not appear to have had a middle name. If he did, it certainly wasn't "K-Zureick."

Outline of proof of Clifford's theorem

The theorem follows by Amini-Baker's Riemann-Roch theorem which uses a version of reduced divisors, but we gave another proof.

Outline of proof of Clifford's theorem

The theorem follows by Amini-Baker's Riemann-Roch theorem which uses a version of reduced divisors, but we gave another proof.

To prove Clifford's theorem, given D_0 supported on $\mathcal{C}_0^{\mathrm{sm}}(\mathbf{k})$, we must cook up a divisor E_0 of degree at most $g - \frac{\deg D_0}{2}$ such that for any φ, there is some component C_v such that the line bundle

$$\mathcal{O}(D_0 - E_0)(\varphi)|_{C_v}$$

on C_v has no non-zero sections.

Outline of proof of Clifford's theorem

The theorem follows by Amini-Baker's Riemann-Roch theorem which uses a version of reduced divisors, but we gave another proof.

To prove Clifford's theorem, given D_0 supported on $\mathcal{C}_0^{\mathrm{sm}}(\mathbf{k})$, we must cook up a divisor E_0 of degree at most $g - \frac{\deg D_0}{2}$ such that for any φ, there is some component C_v such that the line bundle

$$\mathcal{O}(D_0 - E_0)(\varphi)|_{C_v}$$

on C_v has no non-zero sections.

The idea is to choose E_0 to vandalize any possible section on any component as efficiently as possible. It's a piece of combinatorics that uses the classical Clifford's theorem, Clifford's theorem for linear systems on graphs, and a general position argument.

Further Questions

1. What can we say about the number of rational points specializing to different components of the closed fiber? This probably involves more global data, not just expanding in residue discs. Our more recent work is a first step in that direction.

Further Questions

1. What can we say about the number of rational points specializing to different components of the closed fiber? This probably involves more global data, not just expanding in residue discs. Our more recent work is a first step in that direction.
2. What about r_{tor}? Does that help us improve the bounds?

Further Questions

1. What can we say about the number of rational points specializing to different components of the closed fiber? This probably involves more global data, not just expanding in residue discs. Our more recent work is a first step in that direction.
2. What about r_{tor}? Does that help us improve the bounds?
3. What about passing from the special fiber to the generic fiber? This should give even better bounds. We can use deformation-theoretic obstructions from tropical lifting here.

Further Questions

1. What can we say about the number of rational points specializing to different components of the closed fiber? This probably involves more global data, not just expanding in residue discs. Our more recent work is a first step in that direction.
2. What about r_{tor}? Does that help us improve the bounds?
3. What about passing from the special fiber to the generic fiber? This should give even better bounds. We can use deformation-theoretic obstructions from tropical lifting here.
4. $r(D, F_0)$?

Thanks!

O. Amini and M. Baker. *Linear series on metrized complexes of algebraic curves.*

M. Baker. *Specialization of linear systems from curves to graphs.*

M. Baker and S. Norine. *Riemann-Roch and Abel-Jacobi theory on a finite graph.*

E. Katz and D. Zureick-Brown. *The Chabauty-Coleman bound at a prime of bad reduction and Clifford bounds for geometric rank functions.*

D. Lorenzini and T. Tucker. *Thue equations and the method of Chabauty-Coleman.* Invent. Math.

W. McCallum and B. Poonen. *The method of Chabauty and Coleman.*

M. Stoll. *Independence of rational points on twists of a given curve.*