Geometric Rank Functions and Rational Points on Curves

Eric Katz (University of Waterloo)
joint with David Zureick-Brown (Emory University)

April 4, 2012

"Oh yes, I remember Clifford. I seem to always feel him near somehow."
– Jon Hendricks

Linear systems on curves and graphs

Let $\mathbb{K}$ be a discretely valued field with valuation ring $\mathcal{O}$ and residue field $\mathbf{k}$. Let C be a curve with semistable reduction over $\mathbb{K}$. In other words, C can be completed to a family of curves $\mathcal{C}$ over $\mathcal{O}$ such that the total space is regular and that the central fiber $\mathcal{C}_0$ has ordinary double-points as singularities. Think: extending a family of curves over a punctured disc across the puncture while allowing mild singularities.

Let D be a divisor on C, supported on $C(\mathbb{K})$. Would like to bound the dimension of $H^0(C, \mathcal{O}(D))$ by using the central fiber.

Baker-Norine linear systems on graphs

The Baker-Norine theory of linear systems on graphs gives such bounds. Let the multi-degree $\underline{\deg}$ of a divisor D to be the formal sum

$$\underline{\deg}(D) = \sum_v \deg(\mathcal{O}(D)|_{C_v})(v)$$

where C_v are the components of $\mathcal{C}_0$.

Baker-Norine define a rank $r(\underline{\deg}(D))$ in terms of the combinatorics of the dual graph Γ of $\mathcal{C}_0$.

The bound obeys the specialization lemma:

$$\dim(H^0(C, \mathcal{O}(D))) - 1 \leq r(\underline{\deg}(D)).$$

These bounds are particularly nice in the case where all components of $\mathcal{C}_0$ are rational (the maximally degenerate case).

Non-maximal degeneration case

The Baker-Norine theory is not ideal for the non-maximally degenerate case for the following reasons:

1. The bound is not very sharp,
2. The canonical divisor of the dual graph Γ does not have much to do with the canonical bundle K_C of C; unclear what Riemann-Roch says in this case.

In fact, we have the following examples of things going haywire:

1. If C has good reduction, Γ is just a vertex and so $r(\underline{\deg}(D)) = \deg(D)$. Lots of other pathological cases.
2. $\underline{\deg}(K_C) = K_\Gamma + \sum_v (2g(C_v) - 2)(v)$.

Amini-Caporaso approach

Amini-Caporaso have a combinatorial approach to handle this case by inserting loops at vertices corresponding to higher genus components. Their approach obeys the specialization lemma and the appropriate Riemann-Roch theorem.

Their bound is sharper than the Baker-Norine bound and in their theory, one has

$$\underline{\deg}(K_C) = K_\Gamma$$

where K_Γ is the canonical divisor of the *weighted* dual graph Γ.

Today, I'll give an approach that incorporates the geometry of the components. The approach I'll explain was developed independently by Amini-Baker.

Our approach: extending linear equivalence

Our definition of rank is inspired by the following question:

Let D_1, D_2 be divisors on C supported on $C(\mathbb{K})$. Let $\mathcal{D}_1, \mathcal{D}_2$ be their closures on $\mathcal{C}$,

Question: Are the generic fibers D_1, D_2 linearly equivalent?

Try to construct a section s with $(s) = D_1 - D_2$.

Extension hierarchy for linear equivalence problem

We apply a certain extension hierarchy to this question. The steps have technical names which are inspired by the Néron model. The steps should be reminiscent of how one thinks about tropical lifting.

1. Try to construct s_0 on the central fiber such that $(s_0) = (\mathcal{D}_1)_0 - (\mathcal{D}_2)_0$.
   1. numerical: Is there an extension $\mathcal{L}$ of $\mathcal{O}(D_1 - D_2)$ to $\mathcal{C}$ that has degree 0 on every component of the central fiber?
   2. Abelian: For each component C_v of the central fiber, is there a section s_v on C_v of $\mathcal{L}|_{C_v}$ with $(s_v) = ((\mathcal{D}_1)_0 - (\mathcal{D}_2))|_{C_v}$?
   3. toric: Can the sections s_v be chosen to agree on nodes?
2. Use deformation theory to extend the glued together section s_0 to $\mathcal{C}$.

We will concentrate on the first step.

The rank hierarchy

This hierarchy lets us define new rank functions following Baker-Norine. We say a divisor D on C has i-rank $\geq r$ if for any effective divisor E in $C(\mathbb{K})$ of degree r, steps $(1) - (i)$ are satisfied for $\mathcal{D} = \overline{D}, \mathcal{E} = \overline{E}$:

1. numerical: there is a divisor $\varphi = \sum_v a_v C_v$ supported on the central fiber such that
$$\deg(\mathcal{O}(\mathcal{D} - \mathcal{E})(\varphi)|_{C_v}) \geq 0$$
for all v.
2. Abelian: For each component C_v of the central fiber, there is a non-vanishing section s_v on C_v of $\mathcal{O}(\mathcal{D} - \mathcal{E})(\varphi)|_{C_v}$.
3. toric: The sections s_v be chosen to agree across nodes.

New rank functions

So we have rank functions $r_{\text{num}}, r_{\text{Ab}}, r_{\text{tor}}$.

1. $r_{\text{num}}(D)$ depends only on the multi-degree of D, that is $\deg(D|_{C_v})$ for all v
2. $r_{\text{Ab}}, r_{\text{tor}}$ depend only on $\mathcal{D}_0$.

The rank functions $r_{\text{Ab}}, r_{\text{tor}}$ are sensitive to the residue field $\mathbf{k}$ since bigger $\mathbf{k}$ allows for more divisors E. But they eventually stabilize.

Specialization map

To show that r_{Ab} and r_{tor} only depend on D_0, we need to introduce the specialization (a.k.a. reduction) map

$$\begin{array}{rcl} \rho : C(\mathbb{K}) & \rightarrow & \mathcal{C}_0^{\mathrm{sm}}(\mathbf{k}) \\ x & \mapsto & \overline{\{x\}} \cap \mathcal{C}_0(\mathbf{k}). \end{array}$$

Note that $\mathbb{K}$-points always specialize to smooth points of the central fiber. The specialization map is surjective so any divisor E_0 of $\mathcal{C}_0$ supported on $\mathcal{C}_0^{\mathrm{sm}}(\mathbf{k})$ extends to a divisor E supported on $C(\mathbb{K})$ with

$$\rho(E) = E_0.$$

Therefore, we need only check effective divisors E_0 supported on $\mathcal{C}_0^{\mathrm{sm}}(\mathbf{k})$.

A natural question inspired by number theory

Our approach was designed to give an approximate answer to the following natural question motivated by number theory. Let D be a divisor on C supported on $C(\mathbb{K})$. Let F_0 be a divisor on $C_0^{\mathrm{sm}}(\mathbf{k})$. Let

$$|D|_{F_0} = \{D' \in |D| \; \big| \; F_0 \subset \overline{D'}\}.$$

Definition: We say the *rank* $r(D, F_0)$ is greater than or equal to r if for any rank r effective divisor E supported on $C(\mathbb{K})$, $|D - E|_{F_0} \neq \emptyset$.

Question: Can we bound $r(D, F_0)$ in terms of $\mathcal{C}_0$, $\deg(D)$ and F_0?

It's unclear what kind of object $|D|_{F_0}$ is. It's a rigid analytic subspace of projective space and it's not even quite clear if its rank has nice properties. Working with it requires developing a missing theory of rigid analytic/algebraic compatibility. But it is very natural to consider as we shall see.

Numerical rank and Baker-Norine rank

But $r_{\text{num}}(D)$ is not new. In fact, it is the Baker-Norine rank of $\underline{\deg}(D)$. What is called here a *multi-degree* is what Baker and Norine call a divisor on a graph.

One observes that for $\varphi = \sum_v a_v C_v$, treated as a function on $V(\Gamma)$, we have

$$\underline{\deg}(\varphi) = \Delta(\varphi)$$

where Δ is the graph Laplacian.

Also after possible unramified field extension of $\mathbb{K}$ for any multi-degree, $\underline{E} = \sum a_v(v)$, there is a divisor E on C with $\underline{\deg}(E) = \underline{E}$.

Consequently, unpacking the definition of r_{num}, we see that it says $r_{\text{num}}(D) \geq r$ if and only if for any multi-degree $\underline{E} \geq 0$ with $\deg(\underline{E}) = r$, there is a $\varphi : V(\Gamma) \to \mathbb{Z}$ with

$$\underline{D} - \underline{E} + \Delta(\varphi) \geq 0.$$

Specialization lemma

These rank functions satisfy a specialization lemma. For D, a divisor supported on $C(\mathbb{K})$, set

$$r_C(D) = \dim H^0(C, \mathcal{O}(D)) - 1.$$

Then

$$r_C(D) \leq r_{\mathrm{tor}}(D) \leq r_{\mathrm{Ab}}(D) \leq r_{\mathrm{num}}(D).$$

We have examples where the inequalities are strict.

Proof of Specialization lemma

The proof is essentially the same as Baker's specialization lemma.

First by definition, we have

$$r_{\mathrm{tor}}(D) \leq r_{\mathrm{Ab}}(D) \leq r_{\mathrm{num}}(D),$$

so it suffices to show $r_C(D) \leq r_{\mathrm{tor}}(D)$.

One can characterize $r_C(D)$ by saying $r_C(D) \geq r$ if and only if for any effective divisor E of degree r supported on $C(\mathbb{K})$ that

$$H^0(C, \mathcal{O}(D - E)) \neq \{0\}.$$

Consequently, there's a section s of $\mathcal{O}(D - E)$. The section can be extended to a rational section of $\mathcal{O}(\mathcal{D} - \mathcal{E})$ on $\mathcal{C}$. The associated divisor can be decomposed as

$$(s) = H - V$$

where H is the closure of a divisor in C and V is supported on $\mathcal{C}_0$.

Proof of Specialization lemma (cont'd)

Consequently, we can write

$$\varphi \equiv V = \sum_{v} a_v C_v.$$

Now, s can be viewed as a regular section of $\mathcal{O}(\mathcal{D} - \mathcal{E})(\varphi)$. Set $s_v = s|_{C_v}$. These are the desired sections on components.

It follows that $r_{\text{tor}}(D) \geq r$.

Clifford's theorem for r_{Ab}

Let $K_{\mathcal{C}_0}$ be the relative dualizing sheaf of the central fiber. This is characterized by being the natural extension of the canonical bundle on C to $\mathcal{C}$, restricted to the central fiber. Note $\underline{\deg}(K_{\mathcal{C}_0}) = \sum_v (2g(C_v) - 2 + \deg(v))(v) = K_\Gamma + \sum_v 2g(C_v)(v)$.

(No longer as much of a) Question: Is Riemann-Roch true for r_{Ab} and r_{tor}?

$$r_i(D_0) - r_i(K_{\mathcal{C}_0} - D_0) = 1 - g + \deg(D_0)?$$

Yes for r_{Ab}! By Amini-Baker.

Theorem: (Clifford-Brown-K) Let D_0 be a divisor supported on smooth $\mathbf{k}$-points of $\mathcal{C}_0$ then

$$r_{\mathrm{Ab}}(K_{\mathcal{C}_0} - D_0) \leq g - \frac{\deg D_0}{2} - 1.$$

Proof uses the Baker-Norine version of Clifford's theorem, classical Clifford's theorem, and a general position argument.

Proof of Clifford's theorem

The theorem follows by Amini-Baker's Riemann-Roch theorem which uses a version of reduced divisors, but we give another proof...

To prove Clifford's theorem, given D_0 supported on $\mathcal{C}_0^{\mathrm{sm}}(\mathbf{k})$, we must cook up a divisor E_0 of degree at most $g - \frac{\deg D_0}{2}$ such that for any φ, there is some component C_v such that the line bundle

$$\mathcal{O}(D_0 - E_0)(\varphi)|_{C_v}$$

on C_v has no non-zero sections.

The idea is to choose E_0 to vandalize any possible section on any component as efficiently as possible. Now, we need only look at φ such that

$$\deg(\mathcal{O}(D_0 - E_0)(\varphi)|_{C_v}) \geq 0$$

for all C_v. Up to addition of a multiple of the central fiber, there are finitely many such φ.

Proof of Clifford's theorem (cont'd)

To vandalize efficiently, we need the following general position principle: We make an unramified field extension of $\mathbb{K}$ to ensure that $\mathbf{k}$ is infinite. Now we can choose an effective degree n divisor P_0 on $C_v^{\mathrm{sm}}(\mathbf{k})$ such that for any φ,

$$h^0(C_v, \mathcal{O}(D_0 - E_0 - P_0)(\varphi)|_{C_v}) = \max(0, h^0(C_v, \mathcal{O}(D_0 - E_0)(\varphi)|_{C_v}) - n).$$

Now if C_v has $deg(\mathcal{O}(D_0 - E_0)(\varphi)|_{C_v}) \leq 2g - 1$, by ordinary Clifford's theorem,

$$h^0(C_v, \mathcal{O}(D_0 - E_0)(\varphi)|_{C_v}) \leq \frac{d}{2} + 1.$$

Such components can be vandalized with fewer points of E_0 than expected.

One keeps track of these components and vandalizes their sections. If necessary, one also uses Baker-Norine's version of Clifford's theorem to add points to E_0 to ensure that there are always such components C_v. The numbers work out correctly.

Application: Chabauty-Coleman method

The Chabauty-Coleman method is an effective method for bounding the number of rational points on a curve of genus $g \geq 2$. It does not work for all higher genus curves unlike Faltings' theorem, but it gives bounds that can be helpful for explicitly determining the number of points.

Let C be a curve defined over $\mathbb{Q}$ with good reduction at a prime $p > 2g$. This means that viewed as a curve over $\mathbb{Q}_p$, it can be extended to $\mathbb{Z}_p$ such that the fiber over p is smooth. Let $\mathrm{MWR} = \mathrm{rank}(J(\mathbb{Q}))$ be the Mordell-Weil rank of C. Computing MWR is now an industry among number theorists.

Theorem: (Coleman) If $\mathrm{MWR} < g$ then $\#C(\mathbb{Q}) \leq \#\mathcal{C}_0(\mathbb{F}_p) + 2g - 2$.

In the case $p \leq 2g$, there's a small error term.

Theorem: (Stoll) If $\mathrm{MWR} < g$ then $\#C(\mathbb{Q}) \leq \#\mathcal{C}_0(\mathbb{F}_p) + 2\,\mathrm{MWR}$.

This improvement is important! A sharper bound means less searching for a rational point that may not exist.

Outline of Coleman's proof

First, work p-adically. If C has a rational point x_0 , use it for the base-point of the Abel-Jacobi map $C \to J$. Applying Chabauty's argument involving p-adic Lie groups, can assume that that $\overline{J(\mathbb{Q})}$ lies in an Abelian subvariety $A_{\mathbb{Q}_p} \subset J_{\mathbb{Q}_p}$ with $\dim(A_{\mathbb{Q}_p}) \leq \text{MWR}$. Then there is a 1-form ω on $J_{\mathbb{Q}_p}$ that vanishes on A, hence on the images of all points of $C(\mathbb{Q})$ under the Abel-Jacobi map. Pull back ω to $C_{\mathbb{Q}_p}$. By multiplying by a power of p, can suppose that ω does not vanish on the central fiber $\mathcal{C}_0$.

Coleman defines a function $\eta : C(\mathbb{Q}_p) \to \mathbb{Q}_p$ by a p-adic integral,

$$\eta(x) = \int_{x_0}^{x} \omega$$

that vanishes on points of $C(\mathbb{Q})$.

Outline of Coleman's proof (cont'd)

Let $\rho : C(\mathbb{Q}_p) \to \mathcal{C}_0(\mathbb{F}_p)$ be the specialization map

$$\rho(x) = \overline{\{x\}} \cap \mathcal{C}_0(\mathbb{F}_p),$$

By a Newton polytope argument for any residue class $\tilde{x} \in \mathcal{C}_0(\mathbb{F}_p)$,

$$\#(\eta^{-1}(0) \cap \rho^{-1}(\tilde{x})) \leq 1 + \mathrm{ord}_{\tilde{x}}(\omega|_{\mathcal{C}_0}).$$

Summing over residue classes $\tilde{x} \in \mathcal{C}_0(\mathbb{F}_p)$, we get

$$\begin{aligned} \#C(\mathbb{Q}) \leq \#\eta^{-1}(0) &= \sum_{\tilde{x} \in \mathcal{C}_0(\mathbb{F}_p)} (1 + \mathrm{ord}_{\tilde{x}}(\omega|_{\mathcal{C}_0})) \\ &= \#\mathcal{C}_0(\mathbb{F}_p) + \deg(\omega) \\ &= \#\mathcal{C}_0(\mathbb{F}_p) + 2g - 2. \end{aligned}$$

Proof of Stoll's improvement

Stoll improved the bound by picking a good choice of ω for each residue class.

Let $\Lambda \subset \Gamma(J_{\mathbb{Q}_p}, \Omega^1)$ be the 1-forms vanishing on $\overline{J(\mathbb{Q})}$. For each residue class $\tilde{x} \in \mathcal{C}_0(\mathbb{F}_p)$, let

$$n(\tilde{x}) = \min\{\mathrm{ord}_{\tilde{x}}(\omega|_{\mathcal{C}_0}) | 0 \neq \omega \in \Lambda\}.$$

Let the Chabauty divisor on $\mathcal{C}_0$ be

$$D_0 = \sum_{\tilde{x}} n(\tilde{x})(\tilde{x}).$$

Note that by Coleman's argument,

$$\#(\eta^{-1}(0) \cap \rho^{-1}(\tilde{x})) \leq 1 + n(\tilde{x}).$$

By summing over residue classes, one gets

$$\#C(\mathbb{Q}) \leq \#\mathcal{C}_0(\mathbb{F}_p) + \deg(D_0).$$

Proof of Stoll's improvement (cont'd)

Now, we just need to bound D_0. Every $\omega \in \Lambda$ extends (up to a multiple by a power of p) to a regular 1-form vanishing on D_0.

By a semi-continuity argument, one gets

$$\dim \Lambda \leq \dim H^0(\mathcal{C}_0, K_{\mathcal{C}_0} - D_0) \leq g - \frac{\deg(D_0)}{2}.$$

Since $\dim \Lambda = g - \mathrm{MWR}$, $\deg(D_0) \leq 2\,\mathrm{MWR}$.

Therefore, we get

$$\#C(\mathbb{Q}) \leq \#\mathcal{C}_0(\mathbb{F}_p) + 2\,\mathrm{MWR}.$$

Bad reduction case

The bad reduction case of Coleman's bound was proved independently by Lorenzini-Tucker and McCallum-Poonen. The bad reduction case of the Stoll bound was proved for hyperelliptic curves by Stroll and the general case was posed as a question in a paper of McCallum-Poonen.

The set-up for the bad reduction case is where $\mathcal{C}$ is a regular minimal model over $\mathbb{Z}_p$. This means that the total space is regular, but there are no conditions of the types of singularities on the central fiber. They can be worse than nodes.

Theorem:(Lorenzini-Tucker,McCallum-Poonen) Suppose MWR $< g$ then

$$C(\mathbb{Q}) \leq \#\mathcal{C}_0^{\mathrm{sm}}(\mathbb{F}_p) + 2g - 2.$$

The reason why we only need to look at the smooth points is that any rational point of C specializes to a smooth point of $\mathcal{C}_0$. Therefore, we need only consider the residue classes in $\mathcal{C}_0^{\mathrm{sm}}(\mathbb{F}_p)$.

Stoll bounds in the bad reduction case

Theorem: (Brown-K '12) Suppose MWR $< g$ then

$$C(\mathbb{Q}) \leq \#\mathcal{C}_0^{\mathrm{sm}}(\mathbb{F}_p) + 2\,\mathrm{MWR}$$

Now, we outline the proof which is formally similar to Stoll's.

The first step is to go from a regular minimal model to a semistable model. We can make finite ramified field extension $\mathbb{Q}_p \subset \mathbb{K}$ such that $C' = C \times_{\mathbb{Q}_p} \mathbb{K}$ has a semistable model $\mathcal{C}'$. There is a map

$$\mathcal{C}' \to \mathcal{C} \times_{\mathbb{Z}_p} \mathcal{O}.$$

Now, $\mathcal{C}'^{\mathrm{sm}}_0(\mathbf{k})$ may have many more points than $\mathcal{C}_0(\mathbb{F}_p)$. Fortunately, we only need to consider points lying over $\mathcal{C}_0^{\mathrm{sm}}(\mathbb{F}_p)$. But over points of $\mathcal{C}_0^{\mathrm{sm}}$, $\mathcal{C}'_0 \to \mathcal{C}_0$ is an isomorphism. We only need to look at ω near those points.

Proof of Stoll bounds in bad reduction case (cont'd)

Produce the Chabauty divisor nearly as before: for $\tilde{x} \in \mathcal{C}_0^{\mathrm{sm}}(\mathbb{F}_p)$, set

$$n(\tilde{x}) = \min\{\mathrm{ord}_{\tilde{x}}(\omega|_{\mathcal{C}_0}) \big| 0 \neq \omega \in \Lambda\}.$$

where each ω is normalized so that it does not vanish identically on the component C_v containing $\tilde{x}$.

Let the Chabauty divisor supported on $\mathcal{C}_0'(\mathbf{k}')$ be

$$D_0 = \sum_{\tilde{x} \in \mathcal{C}_0^{\mathrm{sm}}(\mathbf{k})} n(\tilde{x})(\tilde{x}).$$

Nearly all the Coleman machinery works in the bad reduction case. The Coleman integral is now multivalued, but it is well-defined as long as one integrates between points in the same residue class. Consequently,

$$\#C(\mathbb{Q}) \leq \#\mathcal{C}_0(\mathbb{F}_p) + \deg(D_0).$$

Proof of Stoll bounds in the bad reduction case (cont'd)

Since every ω in Λ vanishes on D_0, we can use the proof of the specialization lemma to show that

$$\dim \Lambda \leq r_{\mathrm{Ab}}(K_{\mathcal{C}_0} - D_0) + 1.$$

Then apply Clifford's theorem for r_{Ab} to conclude

$$\deg(D_0) \leq 2\,\mathrm{MWR}\,.$$

And that's it!

Further Questions

1. Because Clifford's bounds are usually strict, in any given case, one can probably do better by bounding the Abelian rank by hand. Is there a general statement that incorporates the combinatorics of the dual graph?
2. What can we say about the number of rational points specializing to different components of the central fiber?
3. What about r_{tor}? Does that help us improve the bounds?
4. What about passing from the special fiber to the generic fiber? This should give even better bounds. We can use deformation-theoretic obstructions from tropical lifting here. Probably really need to understand the bad reduction analogue of the Coleman integral which is the Berkovich integral.
5. $r(D, F_0)$?

Thanks!

M. Baker. *Specialization of linear systems from curves to graphs.* Algebra Number Theory, 2:613–653, 2008.

M. Baker and S. Norine. *Riemann-Roch and Abel-Jacobi theory on a finite graph.* Adv. Math., 215:766–788, 2007.

E. Katz and D. Zureick-Brown. *The Chabauty-Coleman bound at a prime of bad reduction and Clifford bounds for geometric rank functions.*, next week.

D. Lorenzini and T. Tucker. *Thue equations and the method of Chabauty-Coleman.* Invent. Math., 148:47–77, 2002.

W. McCallum and B. Poonen. *The method of Chabauty and Coleman.* Panoramas et Synthèses, to appear.

M. Stoll. *Independence of rational points on twists of a given curve.* Compos. Math., 142:1201–1214, 2006.