Geometric Rank Functions and Rational Points on Curves

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"Oh yes, I remember Clifford. I seem to always feel him near somehow."
– Jon Hendricks

The Chabauty-Coleman method

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Let C be a curve defined over $\mathbb{Q}$ with good reduction at a prime $p > 2g$. This means that viewed as a curve over $\mathbb{Q}_p$, it can be extended to $\mathbb{Z}_p$ such that the fiber over p is smooth. Let $\text{MWR} = \text{rank}(J(\mathbb{Q}))$ be the Mordell-Weil rank of C. Computing MWR is now an industry among number theorists.

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In the case $p \leq 2g$, there's a small error term.

Stoll's improvement

The Chabauty-Coleman method does give a bound on the number of rational points, but it doesn't tell you anything about their height. If the bound says that there are at most 5 points, and you've found 4, you don't know if there's an additional point. So you never know when to give up your search. It's important to get the bound as small as possible.

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The bound was lowered by Stoll in the case that MWR is even smaller than $g-1$:

Theorem: (Stoll) If MWR $< g$ then $\#C(\mathbb{Q}) \leq \#\mathcal{C}_0(\mathbb{F}_p) + 2\,\mathrm{MWR}$.

Idea of proof of Chabauty-Coleman:

First, work p-adically. If C has a rational point x_0 , use it for the base-point of the Abel-Jacobi map $C \to J$. If MWR $< g$ by an argument involving p-adic Lie groups, we can suppose that that $\overline{J(\mathbb{Q})}$ lies in an Abelian subvariety $A_{\mathbb{Q}_p} \subset J_{\mathbb{Q}_p}$ with $\dim(A_{\mathbb{Q}_p}) \leq \text{MWR} < g$.

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We might expect $C(\mathbb{Q}_p)$ to intersect $A_{\mathbb{Q}_p}$ in finitely many points. In fact, there is a 1-form ω on $J_{\mathbb{Q}_p}$ that vanishes on A, hence on the images of all points of $C(\mathbb{Q})$ under the Abel-Jacobi map. Pull back ω to $C_{\mathbb{Q}_p}$. By multiplying by a power of p, can suppose that ω does not vanish on the central fiber $\mathcal{C}_0$.

Idea of proof of Chabauty-Coleman (cont'd)

We should view a curve over $\mathbb{Z}_p$ as a family of curves over a disc with generic fiber being the curve over $\mathbb{Q}_p$ and the central fiber being its reduction over $\mathbb{F}_p$. Each rational point of $C(\mathbb{Q}_p)$ is a zero of ω. Think of zeroes of ω degenerating and slamming together as we approach the central fiber. Each residue class $\tilde{x} \in \mathcal{C}_0(\mathbb{F}_p)$ is the reduction of a tube $]\tilde{x}[$ of $\mathbb{Q}_p$-points. The vanishing behaviour of the restriction of ω near $\tilde{x}$ tells us about the zeroes of ω in $]\tilde{x}[$.

Outline of Coleman's proof (cont'd)

To make this insight precise, Coleman defines a function $\eta : C(\mathbb{Q}_p) \to \mathbb{Q}_p$ by a p-adic integral,

$$\eta(x) = \int_{x_0}^{x} \omega$$

that vanishes on points of $C(\mathbb{Q})$.
By a Newton polytope argument for any residue class $\tilde{x} \in \mathcal{C}_0(\mathbb{F}_p)$,

$$\#(\eta^{-1}(0) \cap [\tilde{x}[) \leq 1 + \mathrm{ord}_{\tilde{x}}(\omega|_{\mathcal{C}_0}).$$

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Summing over residue classes $\tilde{x} \in \mathcal{C}_0(\mathbb{F}_p)$, we get

$$\begin{aligned} \#C(\mathbb{Q}) \leq \#\eta^{-1}(0) &= \sum_{\tilde{x} \in \mathcal{C}_0(\mathbb{F}_p)} (1 + \mathrm{ord}_{\tilde{x}}(\omega|_{\mathcal{C}_0})) \\ &= \#\mathcal{C}_0(\mathbb{F}_p) + \deg(\omega) \\ &= \#\mathcal{C}_0(\mathbb{F}_p) + 2g - 2. \end{aligned}$$

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Let $\Lambda \subset \Gamma(J_{\mathbb{Q}_p}, \Omega^1)$ be the 1-forms vanishing on $\overline{J(\mathbb{Q})}$. For each residue class $\tilde{x} \in \mathcal{C}_0(\mathbb{F}_p)$, let

$$n(\tilde{x}) = \min\{\mathrm{ord}_{\tilde{x}}(\omega|_{\mathcal{C}_0}) | 0 \neq \omega \in \Lambda\}.$$

Let the Chabauty divisor on $\mathcal{C}_0$ be

$$D_0 = \sum_{\tilde{x}} n(\tilde{x})(\tilde{x}).$$

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Coleman integration works between points in the same tube, so by summing over residue classes, one gets

$$\#C(\mathbb{Q}) \leq \#\mathcal{C}_0(\mathbb{F}_p) + \deg(D_0).$$

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By a semi-continuity argument together with Clifford's theorem, one gets

$$\dim \Lambda \leq \dim H^0(\mathcal{C}_0, \Omega^1_{\mathcal{C}_0} - D_0) \leq g - \frac{\deg(D_0)}{2}.$$

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Therefore, we get

$$\#C(\mathbb{Q}) \leq \#\mathcal{C}_0(\mathbb{F}_p) + 2\,\mathrm{MWR}\,.$$

Bad reduction case

Now, the above argument breaks down in the bad reduction case because if $\mathcal{C}_0$ is reducible, even if replace $\Omega^1_{\mathcal{C}_0}$ by $K_{\mathcal{C}_0}$, $H^0(\mathcal{C}_0, K_{\mathcal{C}_0} - D_0)$ goes completely haywire with 1-forms vanishing on components. However,

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Theorem: (K-Zureick-Brown '12) Let $\mathcal{C}$ by a regular minimal model for C over $\mathbb{Z}_p$. Suppose MWR $< g$ then

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These are the Stoll bounds. The bad reduction case of Coleman's bound was proved independently by Lorenzini-Tucker and McCallum-Poonen. The bad reduction case of the Stoll bound was proved for hyperelliptic curves by Stoll and the general case was posed as a question in a paper of McCallum-Poonen.

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Since $\mathcal{C}$ is a regular minimal model, the total space is regular, but there are no conditions of the types of singularities on the central fiber. They can be worse than nodes.

A natural framework

If you adapt Stoll's proof and try to apply semi-continuity arguments, you end up in the following situation:
Let $\mathcal{C}$ be a regular minimal model of a curve C over a valuation field $\mathbb{K}$ with residue field $\mathbf{k}$. Let L be a line-bundle on C (think Ω^1_C). Let D_0 be a divisor on $C_0^{\text{sm}}(\mathbf{k})$. Let

$$|L|_{D_0} = \{D \in |L| \;\big|\; D_0 \subset \overline{D}\}$$

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Definition: We say the *rank* $r(L, -D_0)$ is greater than or equal to r if for any rank r effective divisor E supported on $C(\mathbb{K})$, $|L(-E)|_{D_0} \neq \emptyset$.

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One can prove by a specialization argument similar to Matt Baker's specialization lemma that if $\Lambda \subset H^0(C, L)$ is a linear subspace such that for every $s \in \Lambda$, $\overline{(s)} \supset D_0$, then $\dim \Lambda \leq r(L, -D_0) + 1$.

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Question: Can we prove a Clifford bound $r(\Omega^1_C, -D_0) \leq g - \frac{\deg(D_0)}{2} - 1$?

Bounding $r(L, -D_0)$

Problem: It is really hard to work with $|L|_{D_0}$ directly. It's a rigid analytic subspace of projective space and it's not even clear if its rank has nice properties. Working with it requires developing a missing theory of rigid analytic/algebraic compatibility.

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Reduction step: We can suppose that $\mathcal{C}$ is a semistable model. All rational points of C specialize to smooth points of $\mathcal{C}_0$ and they are not messed up too badly by the operations in semistable reduction. This does require a technical lemma.

Extension hierarchy for sections

We apply a certain extension hierarchy to this question. This is very closely related to tropical lifting. The steps have technical names which are inspired by the Néron model. The steps should be reminiscent of how one thinks about tropical lifting. Let D_0 be a divisor supported on smooth points of $\mathcal{C}_0(\mathbb{F}_p)$.

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We will concentrate on the first step.

The rank hierarchy

This hierarchy lets us define new rank functions following Baker-Norine. We say a pair (L, D_0) where L is a line-bundle on C and D_0 is a divisor on $\mathcal{C}_0^{\mathrm{sm}}$ has i-rank $\geq r$ if for any effective divisor E_0 on $\mathcal{C}_0^{\mathrm{sm}}(\mathbf{k})$ of degree r, steps $(1) - (i)$ are satisfied: for an extension $\mathcal{L}$ of L,

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The rank functions $r_{\mathrm{Ab}}, r_{\mathrm{tor}}$ are sensitive to the residue field $\mathbf{k}$ since bigger $\mathbf{k}$ allows for more divisors E. But they eventually stabilize.

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Theorem: We have the following inequalities:

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We have examples where the inequalities are strict.

Specialization lemma

These rank functions satisfy a specialization lemma:

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We have examples where the inequalities are strict.

So now, we have ways to bound $r(\Omega_C, -D_0)$.

Clifford Bounds

The appropriate bound would follow from an analogue of Clifford's theorem: let D_0 be an effective divisor supported on points of $\mathcal{C}_0^{\mathrm{sm}}(\mathbf{k})$; then we have

$$r(\Omega^1, -D_0) \leq g - \frac{\deg(D_0)}{2} - 1.$$

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Now, the multi-degree of its restriction to the central fiber is (considered as a divisor on the dual graph Γ),

$$\underline{\deg}(K_{\mathcal{C}_0}) = \sum_v (2g(C_v) - 2 + \deg(v))(v) = K_\Gamma + \sum_v 2g(C_v)(v)$$

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If all components are rational, then $\deg(K_\Gamma) = 2g - 2$ and the Baker-Norine's Clifford bounds for r_{num} are sufficient.

Clifford Bounds (cont'd)

In general, we have

Theorem: (Clifford-Brown-Amini-Baker-K) Let D_0 be a divisor supported on smooth **k**-points of $\mathcal{C}_0$ then

$$r_{\mathrm{Ab}}(K_{\mathcal{C}_0} - D_0) \leq g - \frac{\deg D_0}{2} - 1.$$

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Our proof uses the Baker-Norine version of Clifford's theorem, classical Clifford's theorem, and a general position argument. We cook up a divisor E_0 of degree at most $g - \frac{\deg D_0}{2}$ such that for any φ, there is some component C_v such that the line bundle $\mathcal{L}(\varphi)|_{C_v}(D_0 - E_0)$ on C_v has no non-zero sections.

Further Questions

1. Because Clifford's bounds are usually strict, in any given case, one can probably do better by bounding the Abelian rank by hand. Is there a general statement that incorporates the combinatorics of the dual graph?

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5. $r(L, -D_0)$?

Thanks!

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