WHAT IS TROPICAL GEOMETRY?

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This note was written to answer the question "What is tropical geometry?" That question can be interpreted in two ways: "Would you tell me something about this research area?"; and "What's with the name 'tropical geometry?'" To address the second question, tropical geometry is named in honor of Brazilian computer scientist Imre Simon. This naming is complicated by the fact that he lived in Sao Paolo and commuted across the Tropic of Capricorn. Whether or not his work is tropical depends on whether or not he preferred to do his research at home or in the office.

To address the first question, there's the rest of this note. This note is excerpted from an old research statement and is meant to be an advertisement for tropical geometry as it relates to my research. I make no claims to be complete or even-handed here.

The main theme of tropical geometry is transforming questions about algebraic varieties into questions about polyhedral complexes. One begins with an algebraic variety X, the common zero set of a system of polynomial equations in an algebraic torus $(\mathbb{K}^*)^n$ defined over a valued field $\mathbb{K}$. Here, an algebraic torus is the Cartesian product of finitely many copies of $\mathbb{K}^* = \mathbb{K} \setminus \{0\}$ and should be thought of as analogous to $(S^1)^n$. By the operation of *tropicalization*, one can define a *tropical variety*, $\mathrm{Trop}(X)$, which is a polyhedral complex, as a combinatorial shadow of X. The combinatorics of $\mathrm{Trop}(X)$ reflects the algebraic geometry of X.

Tropical geometry originally arose from a quite different point of view in which one considers algebraic geometry over the tropical semifield $(\mathbb{R} \cup \{\infty\}, \oplus, \otimes)$ with operations given by

$$a \oplus b = \min(a, b), \quad a \otimes b = a + b.$$

One can then find tropical analogues of classical mathematics and define tropical polynomials, tropical hypersurfaces, and tropical varieties. These objects do not look like their classical counterparts and instead are polyhedral complexes of differing combinatorial types. A number of researchers have developed tropical geometry by defining the appropriate analogues of notions from algebraic geometry and then proving the analogous theorems. Other results show that enumerative questions have the same answers tropically and classically. A spectacular early result of Mikhalkin [Mi03] established that the number of plane curves of degree d and genus g passing through $3d-1+g$ points in general position could be computed using tropical geometry. With collaborators, he found tropical analogues of theorems about algebraic curves and further developed the enumerative geometry of curves [Mi06, MZ08]. Gathmann, H. Markwig, and collaborators have transferred much of Gromov-Witten theory over to tropical geometry [Ga06, Mar07].

Another approach to tropical geometry, which originated in an idea of Kapranov, is to define a tropical variety as a shadow of an algebraic variety [EKL06]. Let $\mathbb{K} = \mathbb{C}\{\{t\}\}$ be the field of formal Puiseux series, that is, the field of Laurent series whose exponents may be fractions but with bounded denominator (one may use other algebraically closed non-Archimedean fields instead). This field has a valuation $v : \mathbb{K}^* \to \mathbb{Q} \subset \mathbb{R}$. One considers a subvariety of an algebraic torus, $X \subset (\mathbb{K}^*)^n$, and defines its tropical variety as $\mathrm{Trop}(X) = \overline{v(X)}$, the closure of the image of X under the product of valuation maps $v : (\mathbb{K}^*)^n \to \mathbb{R}^n$. This process of constructing $\mathrm{Trop}(X)$ from X is called *tropicalization*. It is known from the work of Bergman and Bieri-Groves that $\mathrm{Trop}(X)$ is a polyhedral complex of dimension equal to that of X. The affine span of each polyhedral cell is a rational affine subspace. It was shown by Speyer [Sp05] that the resulting complex has natural weights and satisfies a certain balancing condition. If X is defined over $\mathbb{C}$, then $\mathrm{Trop}(X)$ (defined by base-changing to $\mathbb{K}$) is a fan. We use the term *tropical varieties* to refer to balanced weighted rational polyhedral complexes. Those that arise from algebraic varieties by the operation of tropicalization are called *tropicalizations*. This approach to tropical geometry is tied to the theory of Gröbner bases in combinatorial algebraic geometry.

$\mathrm{Trop}(X)$ is very closely related to the dual complex of a degeneration of X over a DVR when X is defined over a valued field $\mathbb{K}$. When X is defined over $\mathbb{C}$, by a result of Tevelev [Te07], $\mathrm{Trop}(X)$ is instead related to the dual complex of a compactification of X. Therefore, $\mathrm{Trop}(X)$ captures the combinatorics of stratifications (in the $\mathbb{C}$ case) and of degenerations (in the $\mathbb{K}$ case). In the complex case, tropical geometry is a combinatorial theory of subvarieties of a toric variety with stratification induced by the toric strata. In that sense, tropical geometry is an enlargement of the theory of toric varieties. In the valued field case, understanding $\mathrm{Trop}(X)$ comes down to understanding the combinatorics of a degeneration of X. There is a certain tension between algebraic geometry and combinatorics in tropical geometry: if the combinatorics of $\mathrm{Trop}(X)$ are simple, the algebraic geometry of the components is likely to be complicated and rich; if the components of the degeneration are simple, the combinatorics of $\mathrm{Trop}(X)$ are rich and capture the geometry of X.

The tropicalization of X reflects many of the properties of X. $\mathrm{Trop}(X)$ contains a lot of information about the intersection theory of X which is why enumerative tropical geometry has been successful. Moreover, $\mathrm{Trop}(X)$ captures properties of the monodromy of X, considered as a family over a punctured disc. It is a subtle question to determine whether a polyhedral complex is the tropicalization of an algebraic variety.

Tropical varieties are novel because they encode degenerations with much more complicated combinatorics than previously studied. For example, Cools, Draisma, Payne, and Robeva [CFPR] were able to give a new proof of the Brill-Noether theorem by degenerating a high genus curve into a union of rational curves with complicated dual graph and then bounding the dimension of a linear system on the smooth curve using the specialization lemma of Baker [Ba08]. This is orthogonal to the approach using limit linear series where the dual graphs are all trees [EH86]. Moreover, tropical geometry gives an explicit way of approaching Berkovich spaces which are a certain type of analytic space: it is a theorem of Payne [P09] that the Berkovich analytification X^{an} of an affine variety X is homeomorphic to a certain inverse limit of tropicalizations $\varprojlim \mathrm{Trop}(X, \iota)$. In a certain sense, any given tropicalization can be viewed as an approximation of the Berkovich space.

There are many applications of tropical geometry. The work of Hacking, Keel, and Tevelev [HKT09] constructs compactifications of moduli spaces of del Pezzo surfaces using tropical geometry. Tillmann [Ti05] has used tropical geometry to study ideal points in the space of hyperbolic structures on 3-manifolds. Recent work by Gubler [Gu07a, Gu07b] and Rabinoff [R09] have applied tropical techniques to answering questions about the arithmetic of abelian varieties. Tropical intersection theory [AR10] generalizes Newton polytope techniques in number theory [R10] and is likely to have many applications there in the future. Tropical geometry is used in the approaches to mirror symmetry taken by Kontsevich-Soibelman

[KS06] and Gross-Siebert [GS06, GS07], and it is related to numerical homotopy methods [HS95] in scientific computing. It has also been applied to phylogenetics in mathematical biology [PS08] and to integrable systems [IT08].

Since this note was originally written, there have been a number of exciting developments. I will only mention a few, as I am limited by space as well as by my research interests and expertise. The topology of non-Archimedean spaces has grown into an area of study in its own right [P15]. There have been significant contributions to the theory of degenerations of linear systems [BJ15]. In particular, Jensen and Payne have resolved the maximal rank conjecture for quadrics [JP15] and given a new proof of the Gieseker-Petri theorem [JP14]. Cartwright [C13] has introduced abstract tropical complexes which, in the two-dimensional case [C13], have a rich theory mirroring that of algebraic surfaces. Jeffrey Giansiracusa and Noah Giansiracusa [GG13] have developed a scheme-theoretic approach for doing algebraic geometry over semifields. Babaee and Huh [BH15] have used ideas from tropical geometry and several complex variables to disprove Demailly's strongly positive Hodge conjecture. Abramovich, Chen and collaborators [ACMUW15] have made significant connections between log structures and fans giving new applications and providing the technical groundwork for degeneration formulas [ACGS] for Gross and Siebert's logarithmic Gromov-Witten invariants which are expected to encompass and extend tropical curve counting. Rabinoff, Zureick-Brown, and the author [KRZB15] have used Berkovich curves and the theory of linear systems on metric graphs to give uniform bounds for the number of rational points on curves of small Mordell-Weil rank, extending a result of Stoll [St13] on hyperelliptic curves. Adiprasito, Huh, and the author [AHK15] have used tropical ideas to put a Hodge structure on matroidal Stanely-Reisner rings and in doing so have proved a long-standing conjecture of Rota on the log-concavity of the characteristic polynomial of matroids.

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