An Eigenvalue Majorization Inequality for Positive Semidefinite Block Matrices: In Memory of Ky Fan

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Abstract

Let $H = \begin{bmatrix} M & K \\ K^* & N \end{bmatrix}$ be a Hermitian matrix. It is known that the vector of diagonal elements of H, $\operatorname{diag}(H)$, is majorized by the vector of the eigenvalues of H, $\lambda(H)$, and that this majorization can be extended to the eigenvalues of diagonal blocks of H. Reverse majorization results for the eigenvalues are our goal. Under the additional assumptions that H is positive semidefinite and the block K is Hermitian, the main result of this paper provides a reverse majorization inequality for the eigenvalues. This results in the following majorization inequalitites when combined with known majorization inequalities on the left:

$$\operatorname{diag}(H) \prec \lambda(M \oplus N) \prec \lambda(H) \prec \lambda((M+N) \oplus 0).$$

1 Introduction

An early result concerning eigenvalue majorization is the fundamental result due to I. Schur (see e.g., [1, 5, 6]), which states that the diagonal entries of a Hermitian matrix A are majorized by its

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eigenvalues. i.e., $\mathrm{diag}(A) \prec \lambda(A)$. This result can be extended to block Hermitian matrices. For example, if $H = \begin{bmatrix} M & K \\ K^* & N \end{bmatrix}$ is Hermitian, then

$$\mathrm{diag}(H) \prec \lambda(M \oplus N) \prec \lambda(H)\ ^1.$$

Reverse majorization results are our goal. Here and throughout, K^* denotes the conjugate transpose of K; $M \oplus N$ denotes the direct sum of M and N, i.e., the block diagonal matrix $\begin{bmatrix} M & 0 \\ 0 & N \end{bmatrix}$; and 0 is a zero block matrix of compatible size.

Majorization inequalities are useful and important; see e.g., [6]. The main result of this paper is the following reverse majorization inequality for a Hermitian positive semidefinite 2×2 block matrix. (We delay the proof until Section 2.)

Theorem 1.1. *Let $H = \begin{bmatrix} M & K \\ K^* & N \end{bmatrix}$ be a Hermitian positive semidefinite matrix. If, in addition, the block K is Hermitian, then the following majorization inequality holds:*

$$\lambda(H) \prec \lambda((M+N) \oplus 0). \tag{1}$$

1.1 Preliminary Results

Let $\mathbb{M}^{m\times n}(\mathbb{C})$ be the space of all complex matrices of size $m \times n$ with $\mathbb{M}^n(\mathbb{C}) = \mathbb{M}^{n\times n}(\mathbb{C})$. For $A \in \mathbb{M}^n(\mathbb{C})$, the vector of eigenvalues of A is denoted by $\lambda(A) = (\lambda_1(A), \lambda_2(A), \ldots, \lambda_n(A))$. If A is Hermitian, we arrange the eigenvalues of A in nonincreasing order: $\lambda_1(A) \geq \lambda_2(A) \geq \cdots \geq \lambda_n(A)$.

For two sequences of real numbers arranged in nonincreasing order,

$$x = (x_1, x_2, \ldots, x_n), \qquad y = (y_1, y_2, \ldots, y_n),$$

we say that x is majorized by y, denoted by $x \prec y$ (or $y \succ x$), if

$$\sum_{j=1}^{k} x_j \leq \sum_{j=1}^{k} y_j \quad (k = 1, \ldots, n-1), \quad \text{and} \quad \sum_{j=1}^{n} x_j = \sum_{j=1}^{n} y_j.$$

We make use of the following lemmas in our proof of Theorem 1.1.

Lemma 1.2. *If $A, B \in \mathbb{M}^n(\mathbb{C})$ are Hermitian, then*

$$2\lambda(A) \prec \lambda(A+B) + \lambda(A-B). \tag{2}$$

Proof. The lemma is equivalent to Ky Fan's eigenvalue inequality, i.e., $\lambda(A+B) \prec \lambda(A) + \lambda(B)$, [2]. A proof can be found in [4, Theorem 4.3.27] and [7, Theorem 7.15]. □

Lemma 1.3. *Let $A \in \mathbb{M}^{m\times n}(\mathbb{C})$ with $m \geq n$, then we have*

$$\lambda(AA^*) = \lambda(A^*A \oplus 0). \tag{3}$$

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[1] To see the second inequality, let $M = U^*D_1U$, $N = V^*D_2V$, where D_1, D_2 are diagonal matrices, be the spectral decomposition of M, N, respectively. Then $\lambda(H) = \lambda\left(\begin{bmatrix} D_1 & UKV^* \\ VK^*U^* & D_2 \end{bmatrix}\right) \succ \lambda(D_1 \oplus D_2) = \lambda(M \oplus N)$.

2 Proof of Main Result; Corollaries

Before we prove Theorem 1.1, we show by an example that the requirement K being Hermitian is necessary.

Example 2.1. *Let* $M = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$, $N = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ *and* $K = \begin{bmatrix} 1 & 0 \\ 2 & 2 \end{bmatrix}$. *Then*

$$\begin{aligned} \lambda((M+N)\oplus 0) &= (4+\sqrt{2}, 4-\sqrt{2}, 0, 0), \\ \lambda\left(\begin{bmatrix} M & K \\ K^* & N \end{bmatrix}\right) &= (4+\sqrt{5}, 4-\sqrt{5}, 0, 0). \end{aligned}$$

Therefore $\lambda\left(\begin{bmatrix} M & K \\ K^* & N \end{bmatrix}\right) \not\prec \lambda(M+N)\oplus 0)$.

Proof of Theorem 1.1. Since $H := \begin{bmatrix} M & K \\ K & N \end{bmatrix}$ is positive semidefinite, we may write $H = P^*P$, where $P = \begin{bmatrix} X & Y \end{bmatrix}$, for some $X, Y \in \mathbb{M}^{2n\times n}(\mathbb{C})$. Therefore, we have $M = X^*X$, $N = Y^*Y$ and $K = X^*Y = Y^*X$. Note that by Lemma 1.3, we have $\lambda\left(\begin{bmatrix} M & K \\ K & N \end{bmatrix}\right) = \lambda(PP^*)$. The conclusion (1) is then equivalent to showing

$$\{X^*Y = Y^*X\} \implies \{\lambda\left((X^*X+Y^*Y)\oplus 0\right) \succ \lambda(XX^*+YY^*)\}. \tag{4}$$

First, note that

$$\begin{aligned} (X+iY)^*(X+iY) &= X^*X+Y^*Y+i(X^*Y-Y^*X) \\ &= X^*X+Y^*Y \\ (X-iY)^*(X-iY) &= X^*X+Y^*Y-i(X^*Y-Y^*X) \\ &= X^*X+Y^*Y \\ (X+iY)(X+iY)^* &= XX^*+YY^*-i(XY^*-YX^*) \\ (X-iY)(X-iY)^* &= XX^*+YY^*+i(XY^*-YX^*). \end{aligned}$$

Therefore we see that

$$\begin{aligned} \lambda\left((X^*X+Y^*Y)\oplus 0\right) &= \tfrac{1}{2}\left\{\lambda\left((X+iY)^*(X+iY)\oplus 0\right) + \lambda\left((X-iY)^*(X-iY)\oplus 0\right)\right\} \\ &= \tfrac{1}{2}\left\{\left(\lambda\left((X+iY)(X+iY)^*\right) + \lambda\left((X-iY)(X-iY)^*\right)\right)\right\} \\ &\succ \lambda(XX^*+YY^*), \end{aligned}$$

where the second equality is by Lemma 1.3 and the majorization follows from applying Lemma 1.2 with $A = (XX^*+YY^*)$, $B = i(XY^*-YX^*)$. □

As we can see from the above proof, a special case of Theorem 1.1 can be stated as follows.

Corollary 2.2. *Let* $X, Y \in \mathbb{M}^n(\mathbb{C})$ *with* X^*Y *Hermitian. Then we have*

$$\lambda(XX^*+YY^*) \prec \lambda(X^*X+Y^*Y). \tag{5}$$

Corollary 2.3. *Let* $k \geq 1$ *be an integer. If* $A, B \in \mathbb{M}^n(\mathbb{C})$ *are Hermitian, then we have*

$$\lambda(A^2+(AB)^k(BA)^k) \succ \lambda(A^2+(BA)^k(AB)^k). \tag{6}$$

Proof. Let $X = A$ and $Y = (BA)^k$. Then $XY = A(BA)^k$ is Hermitian. The result now follows from Corollary 2.2. $\square$

Corollary 2.4. *Let $k \geq 1$ be an integer, $p \in [0, \infty)$; and let $A, B \in \mathbb{M}^n(\mathbb{C})$ be Hermitian. Then we have*

1. $\operatorname{trace}[(A^2 + (AB)^k(BA)^k)^p] \geq \operatorname{trace}[(A^2 + (BA)^k(AB)^k)^p]$, *for $p \geq 1$;*
2. $\operatorname{trace}[(A^2 + (AB)^k(BA)^k)^p] \leq \operatorname{trace}[(A^2 + (BA)^k(AB)^k)^p]$, *for $0 \leq p \leq 1$.*

Proof. Since $f(x) = x^p$ is a convex function for $p \geq 1$ and concave for $0 \leq p \leq 1$, Corollary 2.4 follows from Corollary 2.3 and a general property of majorization (See [5, p. 56]). $\square$

Remark 2.5. *The case where $k = 1$ in Corollary 2.4 was proved in [3].*

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