On the Kronecker Product

Kathrin Schäcke

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Abstract

In this paper, we review basic properties of the Kronecker product, and give an overview of its history and applications. We then move on to introducing the symmetric Kronecker product, and we derive several of its properties. Furthermore, we show its application in finding search directions in semidefinite programming.

Contents

1 Introduction

The Kronecker product of two matrices, denoted by $A \otimes B$, has been researched since the nineteenth century. Many properties about its trace, determinant, eigenvalues, and other decompositions have been discovered during this time, and are now part of classical linear algebra literature. The Kronecker product has many classical applications in solving matrix equations, such as the Sylvester equation: $AX+XB = C$, the Lyapunov equation: $XA + A^*X = H$, the commutativity equation: $AX = XA$, and others. In all cases, we want to know which matrices X satisfy these equations. This can easily be established using the theory of Kronecker products.

A similar product, the symmetric Kronecker product, denoted by $A \otimes_s B$, has been the topic of recent research in the field of semidefinite programming. Interest in the symmetric Kronecker product was stimulated by its appearance in the equations needed for the computation of search directions for semidefinite programming primal–dual interior–point algorithms. One type of search direction is the AHO direction, named after Alizadeh, Haeberly, and Overton. A generalization of this search direction is the Monteiro–Zhang family of directions. We will introduce those search directions and show where the symmetric Kronecker product appears in the derivation. Using properties of the symmetric Kronecker product, we can derive conditions for when search directions of the Monteiro–Zhang family are uniquely defined.

We now give a short overview of this paper. In Section 2, we discuss the ordinary Kronecker product, giving an overview of its history in Section 1.2. We then list many of its properties without proof in Section 2.1, and conclude with some of its applications in Section 2.2. In Section 3, we introduce the symmetric Kronecker product. We prove a number of its properties in Section 3.1, and show its application in semidefinite programming in Section 3.2. We now continue this section with some background and notation.

1.1 Background and Notation

Let $\mathbb{M}^{m,n}$ denote the space of $m \times n$ real (or complex) matrices and $\mathbb{M}^n$ the square analog. If needed, we will specify the field of the real numbers by $\mathbb{R}$, and of the complex numbers by $\mathbb{C}$. Real or complex matrices are denoted by $\mathbb{M}^{m,n}(\mathbb{R})$ or $\mathbb{M}^{m,n}(\mathbb{C})$. We skip the field if the matrix can be either real or complex without changing the result. Let $\mathbb{S}^n$ denote the space of $n \times n$ real symmetric matrices, and let $\mathbb{R}^n$ denote the space of n–dimensional real vectors. The (i, j)th entry of a matrix $A \in \mathbb{M}^{m,n}$ is referred to by $(A)_{ij}$, or by a_{ij}, and the ith entry of a vector $v \in \mathbb{R}^n$ is referred to by v_i. Upper case letters are used to denote matrices, while lower case letters are used for vectors. Scalars are usually denoted by Greek letters.

The following symbols are being used in this paper:

$$
\begin{array}{ll}
\otimes & \text{for the Kronecker product,} \\
\oplus & \text{for the Kronecker sum,} \\
\otimes_s & \text{for the symmetric Kronecker product.}
\end{array}
$$

Let A be a matrix. Then we note by A^T its transpose, by A^* its conjugate transpose, by A^{-1} its inverse (if existent, i.e. A nonsingular), by $A^{\frac{1}{2}}$ its positive semidefinite square root (if existent, i.e. A positive semidefinite), and by $\det(A)$ or $|A|$ its determinant.

Furthermore, we introduce the following special vectors and matrices:

I_n is the identity matrix of dimension n. The dimension is omitted if it is clear from the context. The ith unit vector is denoted by e_i. E_{ij} is the (i, j)th elementary matrix, consisting of all zeros except for a one in row i and column j.

We work with the **standard inner product** in a vector space

$$\langle u, v\rangle = u^T v, \qquad u, v \in \mathbb{R}^n\,,$$

and with the **trace inner product** in a matrix space

$$\langle M, N\rangle = \operatorname{trace} M^T N, \qquad M, N \in \mathbb{M}^n\,(\mathbb{R}), \text{or}$$

$$\langle M, N\rangle = \operatorname{trace} M^* N, \qquad M, N \in \mathbb{M}^n\,(\mathbb{C}),$$

where

$$\operatorname{trace} M = \sum_{i=1}^{n} m_{ii}.$$

This definition holds in $\mathbb{M}^n$ as well as in $\mathbb{S}^n$. The corresponding norm is the **Frobenius norm**, defined by $\|M\|_F = \sqrt{\operatorname{trace} M^T M}, \quad M \in \mathbb{M}^n\,(\mathbb{R})$ (or $\sqrt{\operatorname{trace} M^* M}, \quad M \in \mathbb{M}^n\,(\mathbb{C})$).

The trace of a product of matrices has the following property:

$$\operatorname{trace} AB = \operatorname{trace} BA, \quad \forall \text{ compatible } A, B,$$

i.e. the factors can be commuted.

A symmetric matrix $S \in \mathbb{S}^n$ is called **positive semidefinite**, denoted $S \succeq 0$, if

$$p^T S p \geq 0, \qquad \forall \quad p \in \mathbb{R}^n .$$

It is called **positive definite** if the inequality is strict for all nonzero $p \in \mathbb{R}^n$.

The following factorizations of a matrix will be mentioned later:

The **LU factorization with partial pivoting** of a matrix $A \in \mathbb{M}^n(\mathbb{R})$ is defined as

$$PA = LU,$$

where P is a permutation matrix, L is a lower triangular square matrix and U is an upper triangular square matrix.

The **Cholesky factorization** of a matrix $A \in \mathbb{M}^n(\mathbb{R})$ is defined as

$$A = LL^T,$$

where L is a lower triangular square matrix. It exists if A is positive semidefinite.

The **QR factorization** of a matrix $A \in \mathbb{M}^{m,n}(\mathbb{R})$ is defined as

$$A = QR,$$

where $Q \in \mathbb{M}^n(\mathbb{R})$ is orthogonal and $R \in \mathbb{M}^{m,n}(\mathbb{R})$ is upper triangular.

The **Schur factorization** of a matrix $A \in \mathbb{M}^{m,n}$ is defined as

$$U^* A U = D + N =: T,$$

where $U \in \mathbb{M}^n$ is unitary, $N \in \mathbb{M}^n$ is strictly upper triangular, and D is diagonal, containing all eigenvalues of A.

A **linear operator** $\mathcal{A} : \mathbb{S}^n \longrightarrow \mathbb{R}^m$ is a mapping from the space of symmetric $n \times n$ real matrices to the space of m–dimensional real vectors, which has the following two properties known as linearity:

$$\mathcal{A}(M + N) = \mathcal{A}(M) + \mathcal{A}(N), \qquad \forall\ M, N \in \mathbb{S}^n ,$$

and

$$\mathcal{A}(\lambda M) = \lambda \mathcal{A}(M), \qquad \forall\ M \in \mathbb{S}^n , \lambda \in \mathbb{R}.$$

The **adjoint operator** of $\mathcal{A}$ is another linear operator $\mathcal{A}^* : \mathbb{R}^m \longrightarrow \mathbb{S}^n$, which has the following property:

$$\langle u, \mathcal{A}(S)\rangle = \langle \mathcal{A}^*(u), S\rangle \qquad \forall \quad u \in \mathbb{R}^m, S \in \mathbb{S}^n .$$

1.2 History of the Kronecker product

The following information is interpreted from the paper "On the History of the Kronecker Product" by Henderson, Pukelsheim, and Searle [10].

Apparently, the first documented work on Kronecker products was written by Johann Georg Zehfuss between 1858 and 1868. It was he, who established the determinant result

$$|A \otimes B| = |A|^b |B|^a, \tag{1}$$

where A and B are square matrices of dimension a and b, respectively.

Zehfuss was acknowledged by Muir (1881) and his followers, who called the determinant $|A \otimes B|$ the Zehfuss determinant of A and B.

However, in the 1880's, Kronecker gave a series of lectures in Berlin, where he introduced the result (1) to his students. One of these students, Hensel, acknowledged in some of his papers that Kronecker presented (1) in his lectures.

Later, in the 1890's, Hurwitz and Stéphanos developed the same determinant equality and other results involving Kronecker products such as:

$$\begin{aligned}
I_m \otimes I_n &= I_{mn}, \\
(A \otimes B)(C \otimes D) &= (AC) \otimes (BD), \\
(A \otimes B)^{-1} &= A^{-1} \otimes B^{-1}, \\
(A \otimes B)^T &= A^T \otimes B^T.
\end{aligned}$$

Hurwitz used the symbol $\times$ to denote the operation. Furthermore, Stéphanos derives the result that the eigenvalues of $A \otimes B$ are the products of all eigenvalues of A with all eigenvalues of B.

There were other writers such as Rados in the late 1800's who also discovered property (1) independently. Rados even thought that he wrote the original paper on property (1) and claims it for himself in his paper published in 1900, questioning Hensel's contributing it to Kronecker.

Despite Rados' claim, the determinant result (1) continued to be associated with Kronecker. Later on, in the 1930's, even the definition of the matrix operation $A \otimes B$ was associated with Kronecker's name.

Therefore today, we know the Kronecker product as "Kronecker" product and not as "Zehfuss", "Hurwitz", "Stéphanos", or "Rados" product.

2 The Kronecker Product

The **Kronecker product** is defined for two matrices of arbitrary size over any ring. However in the succeeding sections we consider only the fields of the real and complex numbers, denoted by $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

Definition 2.1 *The* **Kronecker product** *of the matrix $A \in \mathbb{M}^{p,q}$ with the matrix $B \in \mathbb{M}^{r,s}$ is defined as*

$$A \otimes B = \begin{bmatrix} a_{11}B & \dots & a_{1q}B \\ \vdots & & \vdots \\ a_{p1}B & \dots & a_{pq}B \end{bmatrix}. \tag{2}$$

Other names for the Kronecker product include **tensor product**, **direct product** (Section 4.2 in [9]) or **left direct product** (e.g. in [8]).

In order to explore the variety of applications of the Kronecker product we introduce the notation of the vec**−operator**.

Definition 2.2 *For any matrix $A \in \mathbb{M}^{m,n}$ the* vec**−operator** *is defined as*

$$\mathrm{vec}\,(A) = (a_{11}, \dots, a_{m1}, a_{12}, \dots, a_{m2}, \dots, a_{1n}, \dots, a_{mn})^T, \tag{3}$$

i.e. the entries of A are stacked columnwise forming a vector of length mn.

Note that the inner products for $\mathbb{R}^{n^2}$ and $\mathbb{M}^n$ are compatible:

$$\mathrm{trace}\,(A^T B) = \mathrm{vec}\,(A)^T \mathrm{vec}\,(B), \qquad \forall\ A, B \in \mathbb{M}^n\,.$$

2.1 Properties of the Kronecker Product

The Kronecker product has a lot of interesting properties, many of them are stated and proven in the basic literature about matrix analysis (e.g. [9, Chapter 4]).

2.1.1 Basic Properties

KRON 1 (4.2.3 in [9]) *It does not matter where we place multiplication with a scalar, i.e.*

$$(\alpha A) \otimes B = A \otimes (\alpha B) = \alpha(A \otimes B) \quad \forall \alpha \in \mathbb{K}, A \in \mathbb{M}^{p,q}, B \in \mathbb{M}^{r,s}.$$

KRON 2 (4.2.4 in [9]) *Taking the transpose before carrying out the Kronecker product yields the same result as doing so afterwards, i.e.*

$$(A \otimes B)^T = A^T \otimes B^T \quad \forall A \in \mathbb{M}^{p,q}, B \in \mathbb{M}^{r,s}.$$

KRON 3 (4.2.5 in [9]) *Taking the complex conjugate before carrying out the Kronecker product yields the same result as doing so afterwards, i.e.*

$$(A \otimes B)^* = A^* \otimes B^* \quad \forall A \in \mathbb{M}^{p,q}(\mathbb{C}), B \in \mathbb{M}^{r,s}(\mathbb{C}).$$

KRON 4 (4.2.6 in [9]) *The Kronecker product is associative, i.e.*

$$(A \otimes B) \otimes C = A \otimes (B \otimes C) \quad \forall A \in \mathbb{M}^{m,n}, B \in \mathbb{M}^{p,q}, C \in \mathbb{M}^{r,s}.$$

KRON 5 (4.2.7 in [9]) *The Kronecker product is right–distributive, i.e.*

$$(A + B) \otimes C = A \otimes C + B \otimes C \quad \forall A, B \in \mathbb{M}^{p,q}, C \in \mathbb{M}^{r,s}.$$

KRON 6 (4.2.8 in [9]) *The Kronecker product is left–distributive, i.e.*

$$A \otimes (B + C) = A \otimes B + A \otimes C \quad \forall A \in \mathbb{M}^{p,q}, B, C \in \mathbb{M}^{r,s}.$$

KRON 7 (Lemma 4.2.10 in [9]) *The product of two Kronecker products yields another Kronecker product:*

$$\begin{aligned}(A \otimes B)(C \otimes D) = AC \otimes BD \qquad & \forall A \in \mathbb{M}^{p,q}, B \in \mathbb{M}^{r,s}, \\ & C \in \mathbb{M}^{q,k}, D \in \mathbb{M}^{s,l}.\end{aligned}$$

KRON 8 (Exercise 4.2.12 in [9]) *The trace of the Kronecker product of two matrices is the product of the traces of the matrices, i.e.*

$$\begin{aligned}\operatorname{trace}(A \otimes B) &= \operatorname{trace}(B \otimes A) \\ &= \operatorname{trace}(A)\operatorname{trace}(B) \quad \forall A \in \mathbb{M}^m, B \in \mathbb{M}^n.\end{aligned}$$

KRON 9 (Exercise 4.2.1 in [9]) *The famous determinant result (1) in our notation reads:*

$$\begin{aligned}\det(A \otimes B) &= \det(B \otimes A) \\ &= (\det(A))^n (\det(B))^m \quad \forall A \in \mathbb{M}^m, B \in \mathbb{M}^n.\end{aligned}$$

This implies that $A \otimes B$ is nonsingular if and only if both A and B are nonsingular.

KRON 10 (Corollary 4.2.11 in [9]) *If $A \in \mathbb{M}^m$ and $B \in \mathbb{M}^n$ are non-singular then*

$$(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}.$$

This property follows directly from the **mixed product property** ***KRON 7***.

The Kronecker product does not commute. Since the entries of $A \otimes B$ contain all possible products of entries in A with entries in B one can derive the following relation:

KRON 11 (Section 3 in [11]) *For $A \in \mathbb{M}^{p,q}$ and $B \in \mathbb{M}^{r,s}$,*

$$B \otimes A = S_{p,r}(A \otimes B)S_{q,s}^T,$$

where

$$S_{m,n} = \sum_{i=1}^{m}(e_i^T \otimes I_n \otimes e_i) = \sum_{j=1}^{n}(e_j \otimes I_m \otimes e_j^T)$$

is the **perfect shuffle** *permutation matrix. It is described in full detail in [6].*

2.1.2 Factorizations, Eigenvalues and Singular Values

First, let us observe that the Kronecker product of two upper (lower) triangular matrices is again upper (lower) triangular. This fact in addition to the nonsingularity property **KRON 9** and the mixed product property **KRON 7** allows us to derive several results on factors of Kronecker products.

KRON 12 (Section 1 in [14]) *Let $A \in \mathbb{M}^m, B \in \mathbb{M}^n$ be invertible, and let $P_A, L_A, U_A, P_B, L_B, U_B$ be the matrices corresponding to their LU factorizations with partial pivoting. Then we can easily derive the LU factorization with partial pivoting of their Kronecker product:*

$$A \otimes B = (P_A^T L_A U_A) \otimes (P_B^T L_B U_B) = (P_A \otimes P_B)^T (L_A \otimes L_B)(U_A \otimes U_B).$$

KRON 13 (Section 1 in [14]) *Let $A \in \mathbb{M}^m, B \in \mathbb{M}^n$ be positive (semi)-definite, and let L_A, L_B be the matrices corresponding to their Cholesky factorizations. Then we can easily derive the Cholesky factorization of their Kronecker product:*

$$A \otimes B = (L_A L_A^T) \otimes (L_B L_B^T) = (L_A \otimes L_B)(L_A \otimes L_B)^T.$$

The fact that $A \otimes B$ is positive (semi)definite follows from the eigenvalue theorem established below.

KRON 14 (Section 1 in [14]) *Let $A \in \mathbb{M}^{q,r}, B \in \mathbb{M}^{s,t}$, $1 \leq r \leq q,\ 1 \leq t \leq s$, be of full rank, and let Q_A, R_A, Q_B, R_B be the matrices corresponding to their QR factorizations. Then we can easily derive the QR factorization of their Kronecker product:*

$$A \otimes B = (Q_A R_A) \otimes (Q_B R_B) = (Q_A \otimes Q_B)(R_A \otimes R_B).$$

KRON 15 (in proof of Theorem 4.2.12 in [9]) *Let $A \in \mathbb{M}^m, B \in \mathbb{M}^n$, and let U_A, T_A, U_B, T_B be the matrices corresponding to their Schur factorizations. Then we can easily derive the Schur factorization of their Kronecker product:*

$$A \otimes B = (U_A T_A U_A^*) \otimes (U_B T_B U_B^*) = (U_A \otimes U_B)(T_A \otimes T_B)(U_A \otimes U_B)^*.$$

A consequence of this property is the following result about eigenvalues.

Recall that the **eigenvalues** of a square matrix $A \in \mathbb{M}^n$ are the factors λ that satisfy $Ax = \lambda x$ for some $x \in \mathbb{C}^n$. This vector x is then called the **eigenvector** corresponding to λ. The **spectrum**, which is the set of all eigenvalues, is denoted by $\sigma(A)$.

Theorem 2.3 (Theorem 4.2.12 in [9]) *Let $A \in \mathbb{M}^m$ and $B \in \mathbb{M}^n$. Furthermore, let $\lambda \in \sigma(A)$ with corresponding eigenvector x, and let $\mu \in \sigma(B)$ with corresponding eigenvector y. Then $\lambda\mu$ is an eigenvalue of $A \otimes B$ with corresponding eigenvector $x \otimes y$. Any eigenvalue of $A \otimes B$ arises as such a product of eigenvalues of A and B.*

Corollary 2.4 *It follows directly that if $A \in \mathbb{M}^m, B \in \mathbb{M}^n$ are positive (semi)definite matrices, then $A \otimes B$ is also positive (semi)definite.*

Recall that the **singular values** of a matrix $A \in \mathbb{M}^{m,n}$ are the square roots of the $\min(m, n)$ (counting multiplicities) largest eigenvalues of A^*A. The **singular value decomposition** of A is $A = V\Sigma W^*$, where $V \in \mathbb{M}^m, W \in \mathbb{M}^n$ are unitary and Σ is a diagonal matrix containing the singular values ordered by size on the diagonal. It follows that the rank of A is the number of its nonzero singular values.

KRON 16 (Theorem 4.2.15 in [9]) *Let $A \in \mathbb{M}^{q,r}, B \in \mathbb{M}^{s,t}$, have rank r_A, r_B, and let $V_A, W_A, \Sigma_A, V_B, W_B, \Sigma_B$ be the matrices corresponding to their singular value decompositions. Then we can easily derive the singular value decomposition of their Kronecker product:*

$$A \otimes B = (V_A \Sigma_A W_A^*) \otimes (V_B \Sigma_B W_B^*) = (V_A \otimes V_B)(\Sigma_A \otimes \Sigma_B)(W_A \otimes W_B)^*.$$

It follows directly that the singular values of $A \otimes B$ are the $r_A r_B$ possible positive products of singular values of A and B (counting multiplicities), and therefore $\operatorname{rank}(A \otimes B) = \operatorname{rank}(B \otimes A) = r_A r_B$.

For more information on these factorizations and decompositions see e.g. [7].

2.1.3 The Kronecker Sum

The **Kronecker sum** of two square matrices $A \in \mathbb{M}^m, B \in \mathbb{M}^n$ is defined as

$$A \oplus B = (I_n \otimes A) + (B \otimes I_m).$$

Choosing the first identity matrix of dimension n and the second of dimension m ensures that both terms are of dimension mn and can thus be added.

Note that the definition of the Kronecker sum varies in the literature. Horn and Johnson ([9]) use the above definition, whereas Amoia et al ([2]) as well as Graham ([6]) use $A \oplus B = (A \otimes I_n) + (I_m \otimes B)$. In this paper we will work with Horn and Johnson's version of the Kronecker sum.

As for the Kronecker product, one can derive a result on the eigenvalues of the Kronecker sum.

Theorem 2.5 (Theorem 4.4.5 in [9]) *Let $A \in \mathbb{M}^m$ and $B \in \mathbb{M}^n$. Furthermore, let $\lambda \in \sigma(A)$ with corresponding eigenvector x, and let $\mu \in \sigma(B)$ with corresponding eigenvector y. Then $\lambda + \mu$ is an eigenvalue of $A \oplus B$ with corresponding eigenvector $y \otimes x$. Any eigenvalue of $A \oplus B$ arises as such a sum of eigenvalues of A and B.*

Note that the distributive property does not hold in general for the Kronecker product and the Kronecker sum:

$$(A \oplus B) \otimes C \neq (A \otimes C) \oplus (B \otimes C),$$

and

$$A \otimes (B \oplus C) \neq (A \otimes B) \oplus (A \otimes C).$$

The first claim can be illustrated by the following example: Let $A = 1, B = 2, C = I_2$.

$$(A \oplus B) \otimes C = (1 * 1 + 2 * 1) \otimes I_2 = 3I_2,$$

whereas the right hand side works out to

$$(A \otimes C) \oplus (B \otimes C) = (1I_2) \oplus (2I_2) = I_2 \otimes 1I_2 + 2I_2 \otimes I_2 = I_4 + 2I_4 = 3I_4.$$

A similar example can be used to validate the second part.

2.1.4 Matrix Equations and the Kronecker Product

The Kronecker product can be used to present linear equations in which the unknowns are matrices. Examples for such equations are:

$$AX = B, \tag{4}$$

$$AX + XB = C, \tag{5}$$

$$AXB = C, \tag{6}$$

$$AX + YB = C. \tag{7}$$

These equations are equivalent to the following systems of equations:

$$(I \otimes A)\operatorname{vec} X = \operatorname{vec} B \quad \text{corresponds to (4)}, \tag{8}$$

$$\left[(I \otimes A) + (B^T \otimes I)\right] \operatorname{vec} X = \operatorname{vec} C \quad \text{corresponds to (5)}, \tag{9}$$

$$(B^T \otimes A)\operatorname{vec} X = \operatorname{vec} C \quad \text{corresponds to (6)}, \tag{10}$$

$$(I \otimes A)\operatorname{vec} X + (B^T \otimes I)\operatorname{vec} Y = \operatorname{vec} C \quad \text{corresponds to (7)}. \tag{11}$$

Note that with the notation of the Kronecker sum, equation (9) can be written as

$$(A \oplus B^T)\operatorname{vec} X = \operatorname{vec} C.$$

2.2 Applications of the Kronecker Product

The above properties of the Kronecker product have some very nice applications.

Equation (5) is known to numerical linear algebraists as the **Sylvester equation**. For given $A \in \mathbb{M}^m, B \in \mathbb{M}^n, C \in \mathbb{M}^{m,n}$, one wants to find all

$X \in \mathbb{M}^{m,n}$ which satisfy the equation. This system of linear equations plays a central role in control theory, Poisson equation solving, or invariant subspace computation to name just a few applications. In the case of all matrices being square and of the same dimension, equation (5) appears frequently in system theory (see e.g. [3]).

The question is often, whether there is a solution to this equation or not. In other words one wants to know if the Kronecker sum $A \oplus B^T$ is nonsingular. From our knowledge about eigenvalues of the Kronecker sum, we can immediately conclude that this matrix is nonsingular if and only if the spectrum of A has no eigenvalue in common with the negative spectrum of B:

$$\sigma(A) \cap (-\sigma(B)) = \emptyset.$$

An important special case of the Sylvester equation is the **Lyapunov equation**:

$$XA + A^*X = H,$$

where $A, H \in \mathbb{M}^n$ are given and H is Hermitian. This special type of matrix equation arises in the study of matrix stability. A solution of this equation can be found by transforming it into the equivalent system of equations:

$$\left[(A^T \otimes I) + (I \otimes A^*)\right] \operatorname{vec}(X) = \operatorname{vec}(H),$$

which is equivalent to

$$\left[A^* \oplus A^T\right] \operatorname{vec}(X) = \operatorname{vec}(H).$$

It has a unique solution X if and only if A^* and $-A^T$ have no eigenvalues in common. For example, consider the computation of the Nesterov–Todd search direction (see e.g. [5]). The following equation needs to be solved:

$$\frac{1}{2}(D_V V + V D_V) = \mu I - V^2,$$

where V is a real symmetric positive definite matrix and the right hand side is real and symmetric, therefore Hermitian. Now, we can conclude that this equation has a unique symmetric solution since V is positive definite, and therefore V and $-V^T$ have no eigenvalues in common.

Another application of the Kronecker product is the commutativity equation. Given a matrix $A \in \mathbb{M}^n$, we want to know all matrices $X \in \mathbb{M}^n$ that commute with A, i.e. $\{X : AX = XA\}$. This can be rewritten as $AX - XA = 0$, and hence as

$$\left[(I \otimes A) - (A^T \otimes I)\right] \text{vec}\,(X) = 0.$$

Now we have transformed the commutativity problem into a null space problem which can be solved easily.

Graham ([6]) mentions another interesting application of the Kronecker product. Given $A \in \mathbb{M}^n$ and $\mu \in \mathbb{K}$, we want to know when the equation

$$AX - XA = \mu X \tag{12}$$

has a nontrivial solution. By transforming the equation into

$$\left[(I \otimes A) - (A^T \otimes I)\right] \text{vec}\,(X) = \mu \text{vec}\,(X),$$

which is equivalent to

$$\left[A \oplus (-A^T)\right] \text{vec}\,(X) = \mu \text{vec}\,(X),$$

we find that μ has to be an eigenvalue of $\left[A \oplus (-A^T)\right]$, and that all X satisfying (12) are eigenvectors of $\left[A \oplus (-A^T)\right]$ (after applying vec to X). From our results on the eigenvalues and eigenvectors of the Kronecker sum, we know that those X are therefore Kronecker products of eigenvectors of A^T with the eigenvectors of A.

This also ties in with our result on the commutativity equation. For $\mu = 0$, we get that 0 has to be an eigenvalue of $\left[A \oplus (-A^T)\right]$ in order for a nontrivial commutating X to exist.

There are many other applications of the Kronecker product in e.g. signal processing, image processing, quantum computing and semidefinite programming. The latter will be discussed in the following sections on the symmetric Kronecker product.

3 The Symmetric Kronecker Product

The **symmetric Kronecker product** has many applications in semidefinite programming software and theory. Much of the following can be found in De Klerk's book [5], or in Todd, Toh, and Tütüncü's paper [12].

Definition 3.1 *For any symmetric matrix $S \in \mathbb{S}^n$ we define the vector* $\operatorname{svec}(S) \in \mathbb{R}^{\frac{1}{2}n(n+1)}$ *as*

$$\operatorname{svec}(S) = (s_{11}, \sqrt{2}s_{21}, \ldots, \sqrt{2}s_{n1}, s_{22}, \sqrt{2}s_{32}, \ldots, \sqrt{2}s_{n2}, \ldots, s_{nn})^T. \tag{13}$$

Note that this definition yields another inner product equivalence:

$$\operatorname{trace}(ST) = \operatorname{svec}(S)^T \operatorname{svec}(T), \qquad \forall\ S, T \in \mathbb{S}^n.$$

Definition 3.2 *The* **symmetric Kronecker product** *can be defined for any two (not necessarily symmetric) matrices $G, H \in \mathbb{M}^n$ as a mapping on a vector* $\operatorname{svec}(S)$*, where $S \in \mathbb{S}^n$:*

$$(G \otimes_s H)\operatorname{svec}(S) = \frac{1}{2}\operatorname{svec}(HSG^T + GSH^T).$$

This is an implicit definition of the symmetric Kronecker product. We can also give a direct definition if we first introduce the orthonormal matrix $Q \in \mathbb{M}^{\frac{1}{2}n(n+1) \times n^2}$, which has the following property:

$$Q\operatorname{vec}(S) = \operatorname{svec}(S) \quad \text{and} \quad Q^T \operatorname{svec}(S) = \operatorname{vec}(S) \qquad \forall\ S \in \mathbb{S}^n. \tag{14}$$

Orthonormal is used in the sense of Q having orthonormal rows, i.e. $QQ^T = I_{\frac{1}{2}n(n+1)}$. For every dimension n, there is only one such matrix Q. It can be characterized as follows (compare to [12]).

Consider the entries of the symmetric matrix $S \in \mathbb{S}^n$:

$$S = \begin{bmatrix} s_{11} & s_{12} & \ldots & s_{1n} \\ s_{12} & s_{22} & \ldots & s_{2n} \\ \vdots & \ldots & \ddots & \vdots \\ s_{1n} & s_{2n} & \ldots & s_{nn} \end{bmatrix},$$

then

$$\operatorname{vec}(S) = \begin{pmatrix} s_{11} \\ s_{21} \\ \vdots \\ s_{n1} \\ s_{12} \\ \vdots \\ s_{n2} \\ \vdots \\ s_{1n} \\ \vdots \\ s_{nn} \end{pmatrix} \quad \text{and} \quad \operatorname{svec}(S) = \begin{pmatrix} s_{11} \\ \sqrt{2}s_{21} \\ \vdots \\ \sqrt{2}s_{n1} \\ s_{22} \\ \sqrt{2}s_{32} \\ \vdots \\ \sqrt{2}s_{n2} \\ \vdots \\ s_{nn} \end{pmatrix}.$$

We can now characterize the entries of Q using the equation

$$Q\operatorname{vec}(S) = \operatorname{svec}(S).$$

Let $q_{ij,kl}$ be the entry in the row defining element s_{ij} in $\operatorname{svec}(S)$, and in the column that is multiplied with the element s_{kl} in $\operatorname{vec}(S)$. Then

$$q_{ij,kl} = \begin{cases} 1 & \text{if } i = j = k = l, \\ \frac{1}{\sqrt{2}} & \text{if } i = k \neq j = l, \text{or } i = l \neq j = k, \\ 0 & \text{otherwise.} \end{cases}$$

We will work out the details for dimension $n = 2$:

$$Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

we can check equations (14):

$$Q\operatorname{vec}(S) = \begin{pmatrix} s_{11} \\ \frac{1}{\sqrt{2}}s_{21} + \frac{1}{\sqrt{2}}s_{12} \\ s_{22} \end{pmatrix} = \begin{pmatrix} s_{11} \\ \frac{2}{\sqrt{2}}s_{21} \\ s_{22} \end{pmatrix} = \operatorname{svec}(S),$$

and

$$Q^T\operatorname{svec}(S) = \begin{pmatrix} s_{11} \\ s_{21} \\ s_{21} \\ s_{22} \end{pmatrix} = \begin{pmatrix} s_{11} \\ s_{21} \\ s_{12} \\ s_{22} \end{pmatrix} = \operatorname{vec}(S).$$

Note that equations (14) imply that

$$Q^T Q \text{vec}\, S = Q^T \text{svec}\, S = \text{vec}\,(S), \qquad \forall\ S \in \mathbb{S}^n\,.$$

Furthermore, these equations show that $Q^T Q$ is the orthogonal projection matrix onto the space of symmetric matrices in vector form, i.e. onto vec (S), where S is a symmetric matrix.

Let us now define the symmetric Kronecker product using the matrix introduced above.

Definition 3.3 *Let Q be the unique $\frac{1}{2}n(n+1) \times n$ matrix which satisfies (14). Then the symmetric Kronecker product can be defined as follows:*

$$G \otimes_s H = \frac{1}{2} Q(G \otimes H + H \otimes G)Q^T, \qquad \forall\ G, H \in \mathbb{M}^n\,.$$

The two definitions for the symmetric Kronecker product are equivalent. Let $G, H \in \mathbb{M}^n$, $U \in \mathbb{S}^n$, and Q as before.

$$\begin{aligned}
(G \otimes_s H)\text{svec}\,(U) &= \frac{1}{2} Q(G \otimes H + H \otimes G)Q^T \text{svec}\,(U) \\
&= \frac{1}{2} Q(G \otimes H + H \otimes G)\text{vec}\,(U) \quad \text{by (14)} \\
&= \frac{1}{2} Q((G \otimes H)\text{vec}\,(U) + (H \otimes G)\text{vec}\,(U)) \\
&= \frac{1}{2} Q(\text{vec}\,(HUG^T) + \text{vec}\,(GUH^T)) \quad \text{by (10)} \\
&= \frac{1}{2} Q\text{vec}\,(HUG^T + GUH^T) \\
&= \frac{1}{2} \text{svec}\,(HUG^T + GUH^T),
\end{aligned}$$

where the last equality follows since $HUG^T + GUH^T = HUG^T + (HUG^T)^T$, and therefore symmetric, and by applying equation (14).

3.1 Properties of the symmetric Kronecker product

The symmetric Kronecker product has many interesting properties. Some follow directly from the properties of the Kronecker product, others hold for the symmetric but not for the ordinary Kronecker product, and vice versa.

The symmetric Kronecker product is commutative:

$$A \otimes_s B = B \otimes_s A,$$

for any $A, B \in \mathbb{M}^n$. This follows directly from the definition.

Furthermore, we can prove properties according to **KRON 1 -KRON 8** with the exception of **KRON 4** for which we provide a counter example.

SKRON 1

$$(\alpha A) \otimes_s B = A \otimes_s (\alpha B) = \alpha(A \otimes_s B) \quad \forall \alpha \in \mathbb{R}, A, B \in \mathbb{M}^n$$

Proof.

$$\begin{aligned}
(\alpha A) \otimes_s B &= \frac{1}{2}Q((\alpha A) \otimes B + B \otimes (\alpha A))Q^T \\
&= \frac{1}{2}Q(A \otimes (\alpha B) + (\alpha B) \otimes A)Q^T \\
&= A \otimes_s (\alpha B) = \alpha(A \otimes_s B).
\end{aligned}$$

■

SKRON 2

$$(A \otimes_s B)^T = A^T \otimes_s B^T \quad \forall A, B \in \mathbb{M}^n$$

Proof.

$$\begin{aligned}
(A \otimes_s B)^T &= (\frac{1}{2}Q(A \otimes B + B \otimes A)Q^T)^T \\
&= \frac{1}{2}Q((A \otimes B)^T + (B \otimes A)^T)Q^T \\
&= \frac{1}{2}Q(A^T \otimes B^T + B^T \otimes A^T)Q^T \\
&= A^T \otimes_s B^T.
\end{aligned}$$

■

Corollary 3.4 *An immediate consequence of this property is that $A \otimes_s I$ is symmetric if and only if A is symmetric.*

SKRON 3

$$(A \otimes_s B)^* = A^* \otimes_s B^* \quad \forall A, B \in \mathbb{M}^n(\mathbb{C})$$

Proof.

$$\begin{aligned}
(A \otimes_s B)^* &= (\frac{1}{2}Q(A \otimes B + B \otimes A)Q^T)^* \\
&= \frac{1}{2}(Q^T)^*((A \otimes B)^* + (B \otimes A)^*)Q^* \\
&=^{1} \frac{1}{2}Q(A^* \otimes B^* + B^* \otimes A^*)Q^T \\
&= A^* \otimes_s B^*.
\end{aligned}$$

■

SKRON 4

$$(A \otimes_s B) \otimes_s C = A \otimes_s (B \otimes_s C) \quad \textit{does not hold in general}$$

Proof. Consider the left hand side, i.e. $(A \otimes_s B) \otimes_s C$. The symmetric Kronecker product is defined for any two square matrices of equal dimension, say $A, B \in \mathbb{M}^n$. The resulting matrix is a square matrix of dimension $\frac{1}{2}n(n+1)$. In order for the outer symmetric Kronecker product to be defined, we require C to be in $\mathbb{M}^{\frac{1}{2}n(n+1)}$.

Now, consider the right hand side, i.e. $A \otimes_s (B \otimes_s C)$. Here, the inner symmetric Kronecker product is defined if and only if the matrices B and C are of equal dimensions. This holds if and only if $n = \frac{1}{2}n(n+1)$, which holds if and only if $n = 0$ or $n = 1$. In both cases the result holds trivially. However, for any bigger dimension, the left hand side and right hand side are never simultaneously well defined. ■

SKRON 5

$$(A + B) \otimes_s C = A \otimes_s C + B \otimes_s C \quad \forall A, B, C \in \mathbb{M}^n$$

[4]since Q is a real matrix

Proof.

$$\begin{aligned}
(A+B)\otimes_s C &= \frac{1}{2}Q((A+B)\otimes C + C\otimes(A+B))Q^T \\
&= \frac{1}{2}Q(A\otimes C + B\otimes C + C\otimes A + C\otimes B)Q^T \\
&= \frac{1}{2}Q(A\otimes C + C\otimes A)Q^T + \frac{1}{2}Q(B\otimes C + C\otimes B)Q^T \\
&= A\otimes_s C + B\otimes_s C.
\end{aligned}$$

■

SKRON 6

$$A\otimes_s (B+C) = A\otimes_s B + A\otimes_s C \quad \forall A,B,C\in\mathbb{M}^n$$

Proof.

$$\begin{aligned}
A\otimes_s (B+C) &= \frac{1}{2}Q(A\otimes(B+C) + (B+C)\otimes A)Q^T \\
&= \frac{1}{2}Q(A\otimes B + A\otimes C + B\otimes A + C\otimes A)Q^T \\
&= \frac{1}{2}Q(A\otimes B + B\otimes A)Q^T + \frac{1}{2}Q(A\otimes C + C\otimes A)Q^T \\
&= A\otimes_s B + A\otimes_s C.
\end{aligned}$$

■

SKRON 7 (see e.g. Lemma E.1.2 in [5])

$$(A\otimes_s B)(C\otimes_s D) = \frac{1}{2}(AC\otimes_s BD + AD\otimes_s BC) \quad \forall A,B,C,D\in\mathbb{M}^n$$

Furthermore,

$$(A\otimes_s B)(C\otimes_s C) = AC\otimes_s BC, \quad \textit{and} \quad (A\otimes_s A)(B\otimes_s C) = AB\otimes_s AC.$$

Proof. This proof is directly taken from [5]. Let S be a symmetric matrix, then

$$\begin{aligned}
& (A \otimes_s B)(C \otimes_s D)\text{svec}\,(S) \\
= \; & \frac{1}{2}(A \otimes_s B)\text{svec}\,(CSD^T + DSC^T) \\
= \; & \frac{1}{4}\text{svec}\,(ACSD^T B^T + BCSD^T A^T + ADSC^T B^T + BDSC^T A^T) \\
= \; & \frac{1}{4}\text{svec}\,((AC)S(BD)^T + (BC)S(AD)^T + (AD)S(BC)^T + (BD)S(AC)^T) \\
= \; & \frac{1}{2}(AC \otimes_s BD + AD \otimes_s BC)\text{svec}\,(S).
\end{aligned}$$

■

SKRON 8

$$\begin{aligned}
\text{trace}\,(A \otimes_s B) = \; & \text{trace}\,(AB) + \; \frac{1}{2} \sum_{1 \le i < j \le n} (a_{ii}b_{jj} + a_{jj}b_{ii} \\
& \qquad -(a_{ij}b_{ji} + a_{ji}b_{ij})) \quad \forall\; A, B \in \mathbb{M}^n
\end{aligned}$$

Proof. Note that

$$e_k = \text{svec}\,(E_{jj}), \quad \text{if } k = (j-1)n + 1 - \frac{(j-2)(j-1)}{2} \quad \forall\; j = 1, \ldots, n,$$

and

$$e_k = \frac{1}{\sqrt{2}}\text{svec}\,(E_{ij} + E_{ji}), \quad \text{if } k = (j-1)n + i - \frac{j(j-1)}{2} \quad \forall\; 1 \le j < i \le n.$$

Now, the proof follows straight forward:

$$\begin{aligned}
\text{trace}\,(A \otimes_s B) \; &= \sum_{k=1}^{\frac{n(n+1)}{2}} (A \otimes_s B)_{kk} = \sum_{k=1}^{\frac{n(n+1)}{2}} e_k^T (A \otimes_s B) e_k \\
&= \sum_{k=1}^{n} \text{svec}\,(E_{kk})^T (A \otimes_s B)\text{svec}\,(E_{kk}) \\
&\quad + \frac{1}{2} \sum_{1 \le i < j \le n} \text{svec}\,(\frac{1}{2}(E_{ij} + E_{ji}))(A \otimes_s B)\text{svec}\,(\frac{1}{2}(E_{ij} + E_{ji})).
\end{aligned}$$

Now, for any $k = 1, \dots, n$, consider

$$\begin{aligned}
\text{svec}\,(E_{kk})^T (A \otimes_s B) \text{svec}\,(E_{kk}) &= \frac{1}{2} \text{svec}\,(E_{kk})^T \text{svec}\,(B E_{kk} A^T + A E_{kk} B^T) \\
&= \frac{1}{2} \text{trace}\,(E_{kk} B E_{kk} A^T + E_{kk} A E_{kk} B^T) \\
&= \text{trace}\,(e_k e_k^T B e_k e_k^T A^T) \\
&= \text{trace}\,(e_k^T B e_k)(e_k^T A^T e_k) = a_{kk} b_{kk},
\end{aligned}$$

and for any $1 \leq i < j \leq n$, we have

$$\begin{aligned}
& \text{svec}\,(E_{ij} + E_{ji})^T (A \otimes_s B) \text{svec}\,(E_{ij} + E_{ji}) \\
= \; & \frac{1}{2} \text{svec}\,(E_{ij} + E_{ji})^T \text{svec}\,(B(E_{ij} + E_{ji}) A^T + A(E_{ij} + E_{ji}) B^T) \\
= \; & \frac{1}{2} \text{trace}\,(E_{ij} B E_{ij} A^T + E_{ij} B E_{ji} A^T + E_{ij} A E_{ij} B^T + E_{ij} A E_{ji} B^T \\
& + E_{ji} B E_{ij} A^T + E_{ji} B E_{ji} A^T + E_{ji} A E_{ij} B^T + E_{ji} A E_{ji} B^T) \\
= \; & \text{trace}\,(E_{ij} B E_{ij} A^T + E_{ij} A E_{ij} B^T + E_{ij} B E_{ji} A^T + E_{ij} A E_{ji} B^T) \\
= \; & (e_j^T B e_i)(e_j^T A^T e_i) + (e_j^T A e_i)(e_j^T B^T e_i) \\
& + (e_j^T B e_j)(e_i^T A^T e_i) + (e_j^T A e_j)(e_i^T B^T e_i) \\
= \; & b_{ji} a_{ij} + a_{ji} b_{ij} + b_{jj} a_{ii} + a_{jj} b_{ii}.
\end{aligned}$$

Putting the pieces together, we get

$$\begin{aligned}
\text{trace}\,(A \otimes_s B) &= \sum_{k=1}^{n} a_{kk} b_{kk} + \frac{1}{2} \sum_{1 \leq i < j \leq n} ((a_{ij} b_{ji} + a_{ji} b_{ij} \\
& \quad + a_{ii} b_{jj} + a_{jj} b_{ii}) \\
&= \text{trace}\,(AB) + \frac{1}{2} \sum_{1 \leq i < j \leq n} (a_{ii} b_{jj} + a_{jj} b_{ii} \\
& \quad - (a_{ij} b_{ji} + a_{ji} b_{ij})).
\end{aligned}$$

■

So far, no one has discovered properties corresponding to **KRON 9**, but we can say something about the next property.

SKRON 10

$$(A \otimes_s A)^{-1} = (A^{-1}) \otimes_s (A^{-1}) \quad \forall \; nonsingular \; A \in \mathbb{M}^n \,,$$

but

$$(A \otimes_s B)^{-1} \neq (A^{-1}) \otimes_s (B^{-1}) \quad \forall \; nonsingular \; A, B \in \mathbb{M}^n \,,$$

in general.

Proof. Try to find matrices B and C such that

$$(A \otimes_s A)(B \otimes_s C) = I.$$

From **SKRON 7** it follows that

$$(A \otimes_s A)(B \otimes_s C) = AB \otimes_s AC,$$

and

$$AB \otimes_s AC = I,$$

if and only if $B = A^{-1}$ and $C = A^{-1}$.

For the second part of the claim consider

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Both matrices are invertible. However, the sum

$$A \otimes B + B \otimes A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

is singular with rank two. $A \otimes_s B$ has dimension three. Using this and the fact that multiplication with Q does not increase the rank of this matrix, we conclude that the inverse

$$(A \otimes_s B)^{-1} = \left(\frac{1}{2} Q (A \otimes B + B \otimes A) Q^T \right)^{-1}$$

does not exist. ∎

As for the Kronecker product and the Kronecker sum, we can also establish a result on expressing the eigenvalues and eigenvectors of the symmetric Kronecker product of two matrices A and B in terms of the eigenvalues and eigenvectors of A and of B. Let us first prove a preliminary lemma.

Lemma 3.5 (adapted from Lemma 7.1 in [1]) *Let V be defined as the matrix which contains the orthonormal eigenvectors v_i, $i = 1, \ldots, n$, of the simultaneously diagonalizable matrices $A, B \in \mathbb{M}^n$ columnwise. Then, the (i,j)th column vector, $1 \le j \le i \le n$, of the matrix $V \otimes_s V$ can be written in terms of the ith and jth eigenvectors of A and B as follows:*

$$\begin{array}{rl} \operatorname{svec}(v_i v_j^T) & \text{if } i = j, \\ \frac{1}{\sqrt{2}}\operatorname{svec}(v_i v_j^T + v_j v_i^T) & \text{if } i > j. \end{array}$$

Furthermore, the matrix $V \otimes_s V$ is orthonormal.

Proof. The (i,j)th column of $V \otimes_s V$, $1 \le j \le i \le n$, can be written as

$$(V \otimes_s V)e_{ij},$$

where e_{ij} denotes the corresponding unit vector. Recall that E_{ij} denotes the matrix containing all zeros except for a one in position (i,j). Now, observe that for $i \neq j$,

$$\begin{aligned} (V \otimes_s V)e_{ij} &= (V \otimes_s V)\frac{1}{\sqrt{2}}\operatorname{svec}(E_{ij} + E_{ji}) \\ &= \frac{1}{2}\frac{1}{\sqrt{2}}\operatorname{svec}(V(E_{ij} + E_{ji})V^T + V(E_{ij} + E_{ji})V^T) \\ &= \frac{1}{\sqrt{2}}\operatorname{svec}(VE_{ij}V^T + VE_{ji}V^T) \\ &= \frac{1}{\sqrt{2}}\operatorname{svec}(v_i v_j^T + v_j v_i^T). \end{aligned}$$

To prove orthogonality of $V \otimes_s V$, consider columns $(i,j) \neq (k,l)$, i.e. $i \neq k$ or $j \neq l$.

$$\begin{aligned} & \frac{1}{\sqrt{2}}\operatorname{svec}(v_i v_j^T + v_j v_i^T)^T \frac{1}{\sqrt{2}}\operatorname{svec}(v_k v_l^T + v_l v_k^T) \\ = \; & \frac{1}{2}\operatorname{trace}(v_i v_j^T + v_j v_i^T)(v_k v_l^T + v_l v_k^T) \\ = \; & \frac{1}{2}\operatorname{trace}(v_i v_j^T v_k v_l^T + v_i v_j^T v_l v_k^T + v_j v_i^T v_k v_l^T + v_j v_i^T v_l v_k^T) \\ = \; & \frac{1}{2}(v_i v_j^T v_k v_l^T + 0 + 0 + v_j v_i^T v_l v_k^T) \\ = \; & 0. \end{aligned}$$

The last equality holds since in order for one of the terms, say $v_i v_j^T v_k v_l^T$, to be one, we need $j = k$ and $i = l$. Recall that $i \geq j$ and $k \geq l$. This yields $i \geq j = k \geq l = i$, forcing all indices to be equal. But we excluded that option.

Furthermore, for all indices $i \neq j$, the following proves normality:

$$\begin{aligned}
& \frac{1}{\sqrt{2}}\text{svec}\,(v_i v_j^T + v_j v_i^T)^T \frac{1}{\sqrt{2}}\text{svec}\,(v_i v_j^T + v_j v_i^T) \\
= \; & \frac{1}{2}\text{trace}\,(v_i v_j^T + v_j v_i^T)(v_i v_j^T + v_j v_i^T) \\
= \; & \frac{1}{2}\text{trace}\,(v_i v_j^T v_i v_j^T + v_i v_j^T v_j v_i^T + v_j v_i^T v_i v_j^T + v_j v_i^T v_j v_i^T) \\
= \; & \frac{1}{2}(0 + 1 + 1 + 0) \\
= \; & 1.
\end{aligned}$$

On the other hand, for $i = j$, we yield the above claims in a similar fashion by writing the unit vector e_{ii} as svec (E_{ii}). ∎

Having proven this result, we can now establish the following theorem on eigenvalues and eigenvectors of the symmetric Kronecker product.

Theorem 3.6 (Lemma 7.2 in [1]) *Let $A, B \in \mathbb{M}^n$ be simultaneously diagonalizable matrices. Furthermore, let $\lambda_1, \ldots, \lambda_n$ and $\mu_1, \ldots, \mu_n$ be their eigenvalues, and $v_1, \ldots, v_n$ a common basis of orthonormal eigenvectors. Then, the eigenvalues of $A \otimes_s B$ are given by*

$$\frac{1}{2}(\lambda_i \mu_j + \lambda_j \mu_i), \quad 1 \leq i \leq j \leq n,$$

and their corresponding eigenvectors can be written as

$$\begin{aligned}
\text{svec}\,(v_i v_j^T) \qquad & \text{if } i = j, \\
\frac{1}{\sqrt{2}}\text{svec}\,(v_i v_j^T + v_j v_i^T) \qquad & \text{if } i < j.
\end{aligned}$$

Proof.

$$(A \otimes_s B)\frac{1}{\sqrt{2}}\text{svec}\,(v_i v_j^T + v_j v_i^T)$$

$$
\begin{aligned}
&= \frac{1}{\sqrt{2}}\frac{1}{2}\mathrm{svec}\,(B(v_iv_j^T + v_jv_i^T)A^T + A(v_iv_j^T + v_jv_i^T)B^T) \\
&= \frac{1}{\sqrt{2}}\frac{1}{2}\mathrm{svec}\,(Bv_iv_j^TA^T + Bv_jv_i^TA^T + Av_iv_j^TB^T + Av_jv_i^TB^T) \\
&= \frac{1}{\sqrt{2}}\frac{1}{2}\mathrm{svec}\,(\mu_iv_i\lambda_jv_j^T + \mu_jv_j\lambda_iv_i^T + \lambda_iv_i\mu_jv_j^T + \lambda_jv_j\mu_iv_i^T) \\
&= \frac{1}{\sqrt{2}}\frac{1}{2}\mathrm{svec}\,((\mu_i\lambda_j + \mu_j\lambda_i)(v_iv_j^T + v_jv_i^T)) \\
&= \frac{1}{2}(\mu_i\lambda_j + \mu_j\lambda_i)\frac{1}{\sqrt{2}}\mathrm{svec}\,(v_iv_j^T + v_jv_i^T).
\end{aligned}
$$

Note that these eigenvectors are exactly the columns of $V \otimes_s V$ in the lemma above. We proved that these column vectors are orthogonal. Since all these vectors are eigenvectors, and since there are $\frac{n(n+1)}{2}$ of them, we have shown that they span the complete eigenspace of $A \otimes_s B$. ■

We have seen in Section 2.1 that the Kronecker product of two positive (semi)definite matrices is positive (semi)definite as well. A similar property holds for the symmetric Kronecker product.

Theorem 3.7 (see e.g. Lemma E.1.4 in [5]) *If $A, B \in \mathbb{S}^n$ are positive (semi)definite, then $A \otimes_s B$ is positive (semi)definite (not necessarily symmetric).*

Proof. We need to show the following for any $s \in \mathbb{R}^{\frac{1}{2}n(n+1)},\ s \neq 0$:

$$s^T(A \otimes_s B)s > 0,$$

(or ≥ 0 in the case of positive semidefinite). By denoting s as $\mathrm{svec}\,(S)$, we can show that

$$
\begin{aligned}
s^T(A \otimes_s B)s &= \mathrm{svec}\,(S)^T(A \otimes_s B)\mathrm{svec}\,(S) \\
&= \frac{1}{2}\mathrm{svec}\,(S)^T\mathrm{svec}\,(BSA + ASB) \\
&= \frac{1}{2}\mathrm{trace}\,(SBSA + SASB) \\
&= \mathrm{trace}\,SASB = \mathrm{trace}\,B^{\frac{1}{2}}SA^{\frac{1}{2}}A^{\frac{1}{2}}SB^{\frac{1}{2}} \\
&= \mathrm{trace}\,(A^{\frac{1}{2}}SB^{\frac{1}{2}})^T(A^{\frac{1}{2}}SB^{\frac{1}{2}}) = \|A^{\frac{1}{2}}SB^{\frac{1}{2}}\|_F \\
&> 0,
\end{aligned}
$$

(or ≥ 0 in case of positive semidefinite). The strict inequality for the positive definite case holds since S is nonzero and both $A^{\frac{1}{2}}$ and $B^{\frac{1}{2}}$ are positive definite. ■

The following property relates positive (semi)definiteness of the ordinary Kronecker product to positive (semi)definiteness of the symmetric Kronecker product.

Theorem 3.8 (Theorem 2.10 in [13]) *Let A and B be in $\mathbb{S}^n$. Then, $A \otimes B$ is positive (semi)definite if and only if $A \otimes_s B$ is positive (semi)definite.*

Proof. Assume that $A \otimes B$ is positive (semi)definite. Let U be a symmetric matrix. We need to show that $\mathrm{svec}\,(U)^T(A \otimes_s B)\mathrm{svec}\,(U) > 0$ (≥ 0.)

$$\begin{aligned}
\mathrm{svec}\,(U)^T(A \otimes_s B)\mathrm{svec}\,(U) &= \frac{1}{2}\mathrm{svec}\,(U)^T\mathrm{svec}\,(BUA + AUB) \\
&= \frac{1}{2}\mathrm{trace}\,(UBUA + UAUB) \\
&= \mathrm{trace}\,(UBUA) \\
&= \mathrm{vec}\,(U)^T\mathrm{vec}\,(BUA) \\
&= \mathrm{vec}\,(U)^T(A \otimes B)\mathrm{vec}\,(U) > 0,
\end{aligned}$$

(or ≥ 0 respectively).

To prove the other direction, assume first that $A \otimes_s B$ is positive definite. Further, assume for contradiction that $A \otimes B$ is not positive definite, i.e. that there exists an eigenvalue $\lambda \leq 0$. Since any eigenvalue of $A \otimes B$ is a product $\mu\eta$, where μ is an eigenvalue of A and η is an eigenvalue of B, we must assume that one of the matrices A and B, say A, has a nonpositive eigenvalue μ and the other one, say B, has a nonnegative eigenvalue η. We denote the corresponding eigenvectors by a and b.

Let $U = ab^T + ba^T$. Then,

$$\begin{aligned}
& \mathrm{svec}\,(U)^T(A \otimes_s B)\mathrm{svec}\,(U) \\
= \; & \mathrm{svec}\,(U)^T\frac{1}{2}\mathrm{svec}\,(BUA + AUB) \\
= \; & \frac{1}{2}\mathrm{trace}\,(UBUA + UAUB) = \mathrm{trace}\,(UBUA) \\
= \; & \mathrm{trace}\,(ab^T + ba^T)B(ab^T + ba^T)A
\end{aligned}$$

$$
\begin{aligned}
&= \operatorname{trace}(ab^TBab^TA + ab^TBba^TA + ba^TBab^TA + ba^TBba^TA) \\
&= \operatorname{trace}((Bb)^Tab^T(Aa) + (b^TBb)(a^TAa) + (a^TBa)(b^TAb) \\
&\quad +a^T(Bb)(Aa)^Tb) \\
&= \eta\mu(b^Ta)^2 + (b^TBb)(a^TAa) + (a^TBa)(b^TAb) + \eta\mu(a^Tb)^2 \\
&=: P1 + P2 + P3 + P4.
\end{aligned}
$$

Parts one and four $(P1, P4)$ are nonpositive since $\eta\mu \leq 0$. Part two is nonpositive since

$$b^TBb = \eta b^Tb \geq 0, \tag{15}$$

and

$$a^TAa = \mu a^Ta \leq 0. \tag{16}$$

To prove that part three is nonpositive, consider an arbitrary rank one matrix vv^T. Now,

$$
\begin{aligned}
\operatorname{svec}(vv^T)^T(A \otimes_s B)\operatorname{svec}(vv^T) &= \operatorname{svec}(vv^T)^T\frac{1}{2}\operatorname{svec}(Bvv^TA + Avv^TB) \\
&= \frac{1}{2}\operatorname{trace}(vv^TBvv^TA + vv^TAvv^TB) \\
&= \operatorname{trace}(v^TBvv^TAv) = (v^TBv)(v^TAv) \\
&> 0.
\end{aligned}
$$

This implies that $\lambda < 0$ since for $v = b$, we can say that $b^TBb \neq 0$, and for $v = a$, it follows that $a^TAa \neq 0$. Furthermore, $b^TBb > 0$ implies $b^TAb > 0$, and $a^TAa < 0$ implies $a^TBa < 0$. From this and equations (15) and (16), we conclude that $b^TAb > 0$ and $a^TBa < 0$, and therefore, $P3 < 0$.

This yields that

$$\operatorname{svec}(U)^T(A \otimes_s B)\operatorname{svec}(U) = P1 + P2 + P3 + P4 < 0,$$

contradicting the positive definiteness of $A \otimes_s B$.

For $A \otimes_s B$ positive semidefinite, the result follows analogously. ∎

With this theorem, **Theorem 3.7** can be established as a corollary of **Corollary 2.4** and **Theorem 3.8**.

3.2 Applications of the symmetric Kronecker product

A major application of the symmetric Kronecker product comes up in defining the search direction for primal–dual interior–point methods in semidefinite programming. Here, one tries to solve the following system of equations (see Section 3.1 in [12]):

$$\begin{aligned} \mathcal{A}X &= b, \\ \mathcal{A}^*y + S &= C, \\ XS &= \nu I, \end{aligned} \tag{17}$$

where $\mathcal{A}$ is a linear operator from $\mathbb{S}^n$ to $\mathbb{R}^m$ with full row rank, $\mathcal{A}^*$ is its adjoint, b is a vector in $\mathbb{R}^m$, C is a matrix in $\mathbb{S}^n$, I is the n dimensional identity matrix, and ν is a scalar.

The solutions to this system for different $\nu > 0$, $(X(\nu), y(\nu), S(\nu))$, represent the central path. We try to find approximate solutions by taking Newton steps. The search direction for a single Newton step is the solution $(\Delta X, \Delta y, \Delta S)$ of the following system of equations:

$$\begin{aligned} \mathcal{A}\Delta X &= b - \mathcal{A}X, \\ \mathcal{A}^*\Delta y + \Delta S &= C - S - \mathcal{A}^*y, \\ \Delta X S + X\Delta S &= \nu I - XS. \end{aligned} \tag{18}$$

In order to yield useful results, system (18) needs to be changed to produce symmetric solutions ΔX and ΔS.

One approach was used by Alizadeh, Haeberly, and Overton [1]. They symmetrized the third equation of (17) by writing it as:

$$\frac{1}{2}(XS + (XS)^T) = \nu I.$$

Now the last row of system (18) reads

$$\frac{1}{2}(\Delta X S + X\Delta S + S\Delta X + \Delta S X) = \nu I - \frac{1}{2}(XS + SX).$$

Let A be the matrix representation of $\mathcal{A}$ and let A^T be the matrix representation of $\mathcal{A}^*$. Note that if ΔX is a solution of (18) then so is $\frac{1}{2}(\Delta X + \Delta X^T)$, so we can use svec and $\otimes_s$ in this context. The same holds for ΔS. The modified system of equations can now be written in block form:

$$\begin{bmatrix} 0 & A & 0 \\ A^T & 0 & I \\ 0 & E_{AHO} & F_{AHO} \end{bmatrix} \begin{pmatrix} \Delta y \\ \mathrm{svec}\,(\Delta X) \\ \mathrm{svec}\,(\Delta S) \end{pmatrix} = \begin{pmatrix} b - \mathcal{A}X \\ \mathrm{svec}\,(C - S - \mathcal{A}^* y) \\ \mathrm{svec}\,(\nu I - \frac{1}{2}(XS + SX)) \end{pmatrix},$$

where I is the identity matrix of dimension $\frac{1}{2}n(n+1)$, and E_{AHO} and F_{AHO} are defined using the symmetric Kronecker product.

$$E_{AHO} := I \otimes_s S, \quad F_{AHO} := X \otimes_s I.$$

The solution to this system of equations is called the AHO direction. This search direction is a special case of the more general Monteiro–Zhang family of search directions. For this family of search directions the product XS is being symmetrized via the following linear transformation

$$H_P(XS) = \frac{1}{2}(P(XS)P^{-1} + P^{-T}(XS)^T P^T), \tag{19}$$

where P is an invertible matrix. Note, that for $P = I$, we get

$$H_I(XS) = \frac{1}{2}(XS + SX),$$

which yields the AHO direction.

Using the Monteiro–Zhang symmetrization (see e.g. [12]), we get the following system of equations:

$$\begin{bmatrix} 0 & A & 0 \\ A^T & 0 & I \\ 0 & E & F \end{bmatrix} \begin{pmatrix} \Delta y \\ \mathrm{svec}\,(\Delta X) \\ \mathrm{svec}\,(\Delta S) \end{pmatrix} = \begin{pmatrix} b - \mathcal{A}X \\ \mathrm{svec}\,(C - S - \mathcal{A}^* y) \\ \mathrm{svec}\,(\nu I - H_P(XS)) \end{pmatrix}, \tag{20}$$

where the more general matrices E and F are defined as

$$E := P \otimes_s P^{-T} S, \quad F := PX \otimes_s P^{-T}.$$

Note that, using property **SKRON 7**, these matrices can be written as

$$E = (I \otimes_s P^{-T} S P^{-1})(P \otimes_s P),$$

and

$$F = (PXP^T \otimes_s I)(P^{-T} \otimes_s P^{-T}),$$

which yields the following lemma.

Lemma 3.9 *If X and S are positive definite, then E and F are nonsingular.*

Proof. This can be shown by proving the nonsingularity of each factor. We observe that the matrix $P^{-T}SP^{-1}$ is positive definite. Denote its eigenvalues by $\mu_i, i = 1, \ldots, n$, and note that I and $P^{-T}SP^{-1}$ are simultaneously diagonalizable. Then, the eigenvalues of $I \otimes_s P^{-T}SP^{-1}$ are $\frac{1}{2}(\mu_j + \mu_i), 1 \leq i \leq j \leq n$, which is positive. Therefore, the matrix $I \otimes_s P^{-T}SP^{-1}$ is invertible. Also, $P \otimes_s P$ is invertible because of property **SKRON 10**. The result for F can be obtained similarly. ■

Having established nonsingularity of E and F, we can now state the following theorem. It provides a sufficient condition for the uniqueness of the solution $(\Delta X, \Delta y, \Delta S)$ of system (20).

Theorem 3.10 (Theorem 3.1 in [12]) *Let X, S and $E^{-1}F$ be positive definite ($E^{-1}F$ does not need to be symmetric). Then system (20) has a unique solution.*

Proof. We want to show that

$$\begin{bmatrix} 0 & A & 0 \\ A^T & 0 & I \\ 0 & E & F \end{bmatrix} \begin{pmatrix} \Delta y \\ \mathrm{svec}\,(\Delta X) \\ \mathrm{svec}\,(\Delta S) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \tag{21}$$

has only the trivial solution. Consider the equations

$$A\mathrm{svec}\,(\Delta X) = 0, \tag{22}$$

$$A^T\Delta y + \mathrm{svec}\,(\Delta S) = 0, \tag{23}$$

and

$$E\mathrm{svec}\,(\Delta X) + F\mathrm{svec}\,(\Delta S) = 0. \tag{24}$$

Solving equation (23) for $\mathrm{svec}\,(\Delta S)$, and plugging the result into equation (24) yields

$$E\mathrm{svec}\,(\Delta X) - FA^T\Delta y = 0. \tag{25}$$

When multiplying this by E^{-1} from the left and then by A, we get

$$A\mathrm{svec}\,(\Delta X) - AE^{-1}FA^T\Delta y = 0,$$

which is

$$AE^{-1}FA^T\Delta y = 0,$$

because of equation (22). Since A has full row rank, and since $E^{-1}F$ is positive definite, it follows that $AE^{-1}FA^T$ is positive definite, and therefore $\Delta y = 0$.

Plugging this back into equations (23) and (25) establishes the desired result. ■

Now, we want to know conditions for which $E^{-1}F$ is positive definite. The following results establish several such conditions.

Lemma 3.11 (part of Theorem 3.1 in [12]) *$E^{-1}F$ is positive definite if X and S are positive definite and $H_P(XS)$ is positive semidefinite.*

Proof. Let $u \in \mathbb{R}^{\frac{n(n+1)}{2}}$ be a nonzero vector. Denote by k the product $E^{-T}u$, and define K by $k = \mathrm{svec}\,(K)$. Then we have

$$\begin{aligned}
u^TE^{-1}Fu &= k^TFE^Tk = k^T(PX \otimes_s P^{-T})(P^T \otimes_s SP^{-1})k \\
&= \frac{1}{2}k^T(PXP^T \otimes_s P^{-T}SP^{-1} + PXSP^{-1} \otimes_s P^{-T}P^T)k \\
&= \frac{1}{2}k^T(PXP^T \otimes_s P^{-T}SP^{-1})k + \frac{1}{2}k^T(PXSP^{-1} \otimes_s I)k \\
&> \frac{1}{2}k^T(PXSP^{-1} \otimes_s I)k \\
&= \frac{1}{2}\mathrm{svec}\,(K)^T(PXSP^{-1} \otimes_s I)\mathrm{svec}\,(K) \\
&= \frac{1}{4}\mathrm{svec}\,(K)^T\mathrm{svec}\,(KP^{-T}SXP^T + PXSP^{-1}K) \\
&= \frac{1}{4}\mathrm{trace}\,(KKP^{-T}SXP^T + KPXSP^{-1}K) \\
&= \frac{1}{4}\mathrm{trace}\,K(P^{-T}SXP^T + PXSP^{-1})K \\
&= \frac{1}{2}\mathrm{trace}\,KH_P(XS)K \geq 0,
\end{aligned}$$

where the second equality follows from **SKRON 7**, and the strict inequality holds since $PXP^T \succ 0$ and $P^{-T}SP^{-1} \succ 0$ and from **Theorem 3.7**, it follows that $PXP^T \otimes_s P^{-T}SP^{-1} \succ 0$. ■

Lemma 3.12 (Theorem 3.2 in [12]) *Let X and S be positive definite. Then the following are equivalent:*

1. *PXP^T and $P^{-T}SP^{-1}$ commute,*

2. *$PXSP^{-1}$ is symmetric,*

3. *FE^T is symmetric, and*

4. *$E^{-1}F$ is symmetric.*

Proof. The first two statements are equivalent since

$$\begin{aligned}(PXSP^{-1})^T &= P^{-T}SXP^T = (P^{-T}SP^{-1})(PXP^T) \\ &= (PXP^T)(P^{-T}SP^{-1}) = PXSP^{-1},\end{aligned}$$

if and only if the first statement holds.

Note that

$$FE^T = (PX \otimes_s P^{-T})(P^T \otimes_s SP^{-1}) = \frac{1}{2}(PXP^T \otimes_s P^{-T}SP^{-1} + PXSP^{-1} \otimes_s I).$$

The last equality follows from property **SKRON 7**. We know that PXP^T and $P^{-T}SP^{-1}$ are symmetric. Therefore, FE^T is symmetric if and only if $PXSP^{-1} \otimes_s I$ is symmetric. From **Corollary 3.4**, it follows that $PXSP^{-1} \otimes_s I$ is symmetric if and only if $PXSP^{-1}$ is symmetric. This establishes the equivalence between the second and the third statement.

The equivalence between the last two statements follows from the equation

$$E^{-1}F = E^{-1}(FE^T)E^{-T} = E^{-1}(EF^T)E^{-T} = F^TE^{-T} = (E^{-1}F)^T,$$

if and only if the third statement holds. ■

Theorem 3.13 (part of Theorem 3.2 in [12]) *Let X and S be positive definite. Then any of the conditions in* **Lemma 3.12** *imply that system (20) has a unique solution.*

Proof. We want to show that one of the conditions in **Lemma 3.12** implies that $H_P(XS)$ is positive semidefinite. Assume that the second (and therefore also the first) statement in **Lemma 3.12** is true. Let u be a nonzero vector in $\mathbb{R}^n$. Then

$$\begin{aligned} u^T H_P(XS)u &= \frac{1}{2}u^T(P^{-T}SXP^T + PXSP^{-1})u \\ &= u^T PXSP^{-1}u = u^T(PXP^T)(P^{-T}SP^{-1})u. \end{aligned}$$

Since PXP^T and $P^{-T}SP^{-1}$ commute and are symmetric positive definite, we can denote their eigenvalue decompositions by $\bar{Q}^T D_{PXP^T}\bar{Q}$ and $\bar{Q}^T D_{P^{-T}SP^{-1}}\bar{Q}$ respectively, where $\bar{Q}$ is an orthogonal matrix containing their eigenvectors rowwise. Now we continue the above equation

$$\begin{aligned} u^T(PXP^T)(P^{-T}SP^{-1})u &= u^T\bar{Q}^T D_{PXP^T}\bar{Q}\bar{Q}^T D_{P^{-T}SP^{-1}}\bar{Q}u \\ &= u^T(\bar{Q}^T D_{PXP^T} D_{P^{-T}SP^{-1}}\bar{Q})u. \end{aligned}$$

Since $\bar{Q}^T D_{PXP^T} D_{P^{-T}SP^{-1}}\bar{Q}$ is again a positive definite matrix, it follows that $u^T H_P(XS)u > 0$ for all nonzero u. We now conclude from **Lemma 3.11** that $E^{-1}F$ is positive definite. Applying this information to **Theorem 3.10**, we get the desired result. ■

4 Conclusion

We have shown that the symmetric Kronecker product has several properties according to the properties of the ordinary Kronecker product. However, factorizations of the symmetric Kronecker product cannot easily be derived unless we consider special cases (e.g. A and B simultaneously diagonalizable).

When trying to find search directions of the Monteiro–Zhang family, the properties of the symmetric Kronecker product lead to some nice conditions for when the search direction is unique.

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