The many faces of degeneracy in conic optimization

Henry Wolkowicz

Dept. Combinatorics and Optimization, University of Waterloo, Canada

Wed. Sept. 7, 2016, 16:00-16:30

COCA16 Continuous Optimization: Challenges and Applications
Celebrating Ronny Ben-Tal's 70th Birthday
Technion, Haifa, Israel

** Motivation: Loss of Slater CQ/Facial reduction

- Slater condition – existence of a strictly feasible solution – is at the heart of convex optimization.
- Without Slater: first-order optimality conditions may fail; dual problem may yield little information; small perturbations may result in infeasibility; many software packages can behave poorly.
- a pronounced phenomenon: though Slater holds generically, surprisingly many models arising from relaxations of hard nonconvex problems show loss of strict feasibility, e.g., Matrix completions/compressive sensing, sensor network localization, SNL, EDM, POP, Molecular Conformation, QAP, GP, strengthened Max-Cut
- We concentrate on appl. of Semidef. Progr., SDP.
  We look at various reasons and how to take advantage using
  two views of FACIAL REDUCTION, FR

Main Ref: (in progress)
"The many faces of degeneracy in conic optimization",
Drusvyatskiy, Wolkowicz '16 ;

** Facial Reduction/Preprocessing for LP

Primal-Dual Pair: A onto, $m \times n$, $\mathcal{P} = \{1, \dots, n\}$

$$\text{(LP-P)} \qquad \begin{array}{ll} \max & b^\top y \\ \text{s.t.} & A^\top y \le c \end{array} \qquad\qquad \text{(LP-D)} \qquad \begin{array}{ll} \min & c^\top x \\ \text{s.t.} & Ax = b, \\ & x \ge 0. \end{array}$$

Slater's CQ for (LP-D) / Theorem of alternative

Exactly One is True:

(I) $\exists \hat{x}$ s.t. $A\hat{x} = b, \hat{x} > 0$ $\quad$ ($\hat{x} \in \operatorname{ri} F$, feas. set)
Slater point

(II) $0 \ne z = A^\top y \ge 0, \ b^\top y = 0$ $\quad$ $(\langle z, F \rangle = 0)$
exposing vector

Linear Programming Example, $x \in \mathbb{R}^5$

$$\min \quad \begin{pmatrix} 2 & 6 & -1 & -2 & 7 \end{pmatrix} x$$
$$\text{s.t.} \quad \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & -1 & -1 & 0 & 1 \end{bmatrix} x = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$
$$x \geq 0$$

Sum the two constraints (multiply by: $y^T = (1\ 1)$):
get: $2x_1 + x_4 + x_5 = 0 \implies x_1 = x_4 = x_5 = 0$
i.e., equiv. simplified problem/smaller face/ fewer constr.

$\min 6x_2 - x_3$ s.t. $x_2 + x_3 = 1, x_2, x_3 \geq 0,$
$(x_1 = x_4 = x_5 = 0)$

Linear Programming, LP, $A^T y \leq c$

Slater's CQ for (LP-P) / Theorem of alternative

$\exists \hat{y}$ s.t. $c - A^\top \hat{y} > 0$, $\quad \left(\left(c - A^\top \hat{y}\right)_i > 0, \forall i \in \mathcal{P} =: \mathcal{P}^l \right)$

iff

$$Ad = 0,\ c^\top d = 0,\ d \geq 0 \implies d = 0 \qquad (*)$$

implicit equality constraints: $i \in \mathcal{P}^e$

Find $0 \neq d^*$ to $(*)$ with max number of non-zeros
(exposes minimal face containing feasible slacks)

$$d_i^* > 0 \quad \implies (c - A^\top y)_i = 0, \forall y \in \mathcal{F}^y \quad i \in \mathcal{P}^e)$$

(where $\mathcal{F}^y$ is primal feasible set)

$k = 1!$; we only need one step of FR for LP

d^* here exposes the minimal face (of slacks)

Make implicit-equalities explicit/ Regularizes LP

Facial Reduction: $A^\top y \leq_f c$; minimal face $f \trianglelefteq \mathbb{R}^n_+$

proper primal-dual pair; dual of dual is primal

$$\text{(LP}_{reg}\text{-P)} \qquad \begin{array}{ll} \max & b^\top y \\ \text{s.t.} & (A^I)^\top y \leq c^I \\ & (A^e)^\top y = c^e \end{array} \qquad \Bigg| \qquad \text{(LP}_{reg}\text{-D)} \qquad \begin{array}{ll} \min & (c^I)^\top x^I + (c^e)^\top x^e \\ \text{s.t.} & \begin{bmatrix} A^I & A^e \end{bmatrix} \begin{pmatrix} x^I \\ x^e \end{pmatrix} = b \\ & x^I \geq 0,\ x^e \text{ free} \end{array}$$

Generalized Slater CQ holds - And!

after deleting redundant equality constraints!

Mangasarian-Fromovitz CQ (MFCQ) holds

$$\left(\exists \hat{y} : \quad (A^I)^\top \hat{y} < c^I, \quad (A^e)^\top \hat{y} = c^e \right) \qquad (A^e)^\top \text{ is onto}$$

MFCQ holds iff dual optimal set is compact

Numerical difficulties if MFCQ fails; in particular for interior point methods! Modelling issue!

** Convex Programming

Ordinary convex programming, (OCP)

$$\text{(OCP)} \qquad \sup_y \, b^\top y \text{ subject to } g(y) \leq 0$$

$b \in \mathbb{R}^m$; $g(y) = (g_i(y)) \in \mathbb{R}^n$, $g_i : \mathbb{R}^m \to \mathbb{R}$ convex, $\forall i \in \mathbb{P}$

Slater's CQ; strict feasibility

$\exists \, \hat{y}$ s.t. $g_i(\hat{y}) < 0, \forall i$ (implies MFCQ)

Slater's CQ fails $\boxed{\iff}$ implicit equality constraints exist

$\mathcal{P}^e := \{i \in \mathcal{P} : g(y) \leq 0 \implies g_i(y) = 0\} \neq \emptyset$

Let $\mathcal{P}^l := \mathcal{P} \backslash \mathcal{P}^e$ and

$$g^l := (g_i)_{i \in \mathcal{P}^l}, \qquad g^e := (g_i)_{i \in \mathcal{P}^e}$$

implicit equalities to *equalities*/ Regularize OCP

Minimal face f

$$f = \{z \in \mathbb{R}^m_+ \ : \ z_i = 0, \forall i \in \mathcal{P}^e\} \trianglelefteq \mathbb{R}^m_+$$

(OCP) is equivalent to $g(y) \leq_f 0$

$$(\mathrm{OCP_{reg}}) \qquad \begin{array}{ll} \sup & b^\top y \\ \text{s.t.} & g^l(y) \leq 0 \\ & y \in \mathcal{F}^e \end{array}$$

where $\mathcal{F}^e := \{y : g^e(y) = 0\}$.

Then $\boxed{\mathcal{F}^e = \{y : g^e(y) \leq 0\}, \quad \text{so is a convex set!!}}$

Slater's CQ holds for (OCP_{reg}) $\quad \boxed{\exists \hat{y} \in \mathcal{F}^e : g^l(\hat{y}) < 0}$

(Ben-Israel, Ben-Tal, Zlobec: BBZ Conditions '76-80)

* (FR full generality) Abstract convex program

$$\text{(ACP)} \quad \inf_x f(x) \text{ s.t. } g(x) \preceq_K 0, x \in \Omega$$

where:

- $f : \mathbb{R}^n \to \mathbb{R}$ convex; $g : \mathbb{R}^n \to \mathbb{R}^m$ is K-convex
  - $K \subset \mathbb{R}^m$ closed convex cone; $\Omega \subseteq \mathbb{R}^n$ convex set
  - $a \preceq_K b \iff b - a \in K$, $a \prec_K b \iff b - a \in \operatorname{int} K$
  - $g(\alpha x + (1-\alpha y)) \preceq_K \alpha g(x) + (1-\alpha) g(y)$, $\forall x, y \in \mathbb{R}^n, \forall \alpha \in [0,1]$

Slater's CQ: $\boxed{\exists \hat{x} \in \Omega}$ s.t. $g(\hat{x}) \in -\operatorname{int} K$ $(g(x) \prec_K 0)$

- guarantees strong duality
  (zero duality gap AND dual attainmment)
- (near) loss of strict feasibility, nearness to infeasibility, correlates with number of iterations & loss of accuracy
- Recall that Slater (M-F) is equivalent to a nonempty bounded dual optimal set.

Faces of Convex Sets - Useful for Charact. of Opt.

Face of C, $\boxed{F \unlhd C}$

- $F \subseteq C$ is a face of C if F contains any line segment in C whose relative interior intersects F.
- A convex cone $F \subseteq K$ is a face of a convex cone K, $F \unlhd K$, if (simplified)

$$\boxed{x, y \in K \text{ and } x + y \in F \implies x, y \in F}$$

Polar (Dual) Cone/Conjugate Face

- polar cone $K^* := \{\phi : \langle \phi, k \rangle \geq 0, \ \forall k \in K\}$
- If $F \unlhd K$, the conjugate face of F is

$$\boxed{F^c := F^\perp \cap K^* \unlhd K^*}$$

Properties of Faces

General case

- A face of a face is a face
- intersection of a face with a face is a face.
- Let $C \subseteq K$, then $\text{face}(C)$ denotes the minimal face (intersection of faces) containing C.

$F \trianglelefteq K$ is an exposed face if there exists $\phi \in K^*$ with

$$F = K \cap \phi^\perp$$

F^c is always exposed by $x \in \text{ri}\, F$.

The SDP cone is facially exposed, all its faces are exposed. (In fact like $\mathbb{R}^n_+$: $\mathcal{S}^n_+$ is a proper closed convex cone, self-dual and facially exposed.)

Regularize abstract convex program (full generality)

in memorium: Jonathan Borwein 20 May 1951 - 2 Aug 2016,

jonborwein.org

$$\text{(ACP)} \quad \inf_x f(x) \text{ s.t. } g(x) \preceq_K 0, x \in \Omega$$

(Borwein-W.'78-79)

$$(\text{ACP}_R) \quad \inf_x f(x) \text{ s.t. } g(x) \preceq_{K^f} 0, x \in \Omega$$

where: K^f is the minimal face

Like LP, it is simple if we use the minimal face K^f.
We get a proper primal-dual pair!!

General Theorem of Alternative

Lemma (Facial Reduction (FR); find EXPOSING vector ϕ)

Suppose $\bar{x}$ is feasible. Then the LHS system

$$\left\{ \begin{array}{ll} (\Omega - \bar{x})^* \cap \partial\langle \phi, g(\bar{x})\rangle \neq \emptyset & \\ \phi \in K^*, & \langle \phi, g(\bar{x})\rangle = 0 \end{array} \right\} \quad \textit{implies} \quad K^f \subseteq \phi^\perp \cap K,$$

where: ∂ is subgradient; $\langle\cdot\rangle$ is inner-product.

Generally more than one step is needed to find K^f

Restrict to smaller face $\phi^\perp \cap K$;
repeat till Slater is obtained

* SDP Case/Replicating Cone/Faces

SDP case/Replicating cone

- Let $X \in \mathcal{S}^n_+$ with spectral decomposition,
$$X = [P\ Q] \begin{bmatrix} D_+ & 0 \\ 0 & 0 \end{bmatrix} [P\ Q]^T, \quad D_+ \in \mathbb{S}^r_{++} \quad (\text{rank}\, X = r)$$
- Then
$$\text{Range}(X) = \text{Range}(P), \quad \text{Null}(X) = \text{Range}(Q)$$
$$\text{face}(X) = P\mathbb{S}^r_+ P^T = (QQ^T)^\perp \cap \mathcal{S}^n_+.$$
($Z = QQ^T$ *exposing vector/matrix* for face.)
- $$\text{face}(X)^c = Q\mathbb{S}^{n-r}_+ Q^T$$

Range/Nullspace representations

$$\textit{face}(X) = \{Y \in \mathcal{S}^n_+ : \text{Range}(Y) \subseteq \text{Range}(X)\}$$
$$\textit{face}(X) = \{Y \in \mathcal{S}^n_+ : \text{Null}(Y) \supseteq \text{Null}(X)\}$$
$$\text{ri}\, \textit{face}(X) = \{Y \in \mathcal{S}^n_+ : \text{Range}(Y) = \text{Range}(X)\}$$

Semidefinite Programming, SDP, $\mathcal{S}^n_+$

$K = \mathcal{S}^n_+ = K^*$: nonpolyhedral, self-polar, facially exposed

$$\text{(SDP-P)} \quad v_P = \sup_{y \in \mathbb{R}^m} b^\top y \text{ s.t. } g(y) := \mathcal{A}^* y - c \preceq_{\mathcal{S}^n_+} 0$$

$$\text{(SDP-D)} \quad v_D = \inf_{x \in \mathcal{S}^n} \langle c, x \rangle \text{ s.t. } \mathcal{A}x = b,\ x \succeq_{\mathcal{S}^n_+} 0$$

where:

- PSD cone $\mathcal{S}^n_+ \subset \mathcal{S}^n$ symm. matrices
- $c \in \mathcal{S}^n$, $b \in \mathbb{R}^m$
- $\mathcal{A} : \mathcal{S}^n \to \mathbb{R}^m$ is an onto linear map, with adjoint $\mathcal{A}^*$
- $\mathcal{A}x = (\operatorname{trace} A_i x) = (\langle A_i, x \rangle) \in \mathbb{R}^m, \quad A_i \in \mathcal{S}^n$
  $\mathcal{A}^* y = \sum_{i=1}^m A_i y_i \in \mathcal{S}^n$

Regularization Using Minimal Face

Borwein-W.'78-79 , $f_P = \text{face}\, \mathcal{F}_P^s$; min. face of feasible slacks

(SDP-P) is equivalent to the regularized

$$(\text{SDP}_{reg}\text{-P}) \qquad v_{RP} := \sup_y \{\langle b, y\rangle \ : \ \mathcal{A}^* y \preceq_{f_P} c\}$$

f_p is minimal face of primal feasible slacks

$\{s \succeq 0 : s = c - \mathcal{A}^* y\} \subseteq f_p \trianglelefteq \mathcal{S}_+^n$

Lagrangian dual of regularized problem satisfies strong duality:

$$(\text{SDP}_{reg}\text{-D}) \quad v_{DRP} \quad := \inf_x \{\langle c, x\rangle \ : \ \mathcal{A} x = b, \ x \succeq_{f_P^*} 0\}$$

$v_P = v_{RP} = v_{DRP}$ and v_{DRP} is attained.

regularized PROPER primal-dual pair | dual of dual is primal

If we take the dual of $(\text{SDP}_{reg}\text{-D})$ we recover the primal regularized problem $(\text{SDP}_{reg}\text{-P})$.

Slater's CQ/Theorem of Alternative for Dual

Assume feasibility: $\exists \tilde{x}$ s.t. $\mathcal{A}\tilde{x} = b, \tilde{x} \succeq 0$.

Exactly one of the following alternatives holds/is consistent:

$$(I) \quad \exists \hat{x} \text{ s.t. } \mathcal{A}\hat{x} = b, \hat{x} \succ 0 \qquad (\textit{Slater})$$

<u>or</u>

$$(II) \quad 0 \neq z = \mathcal{A}^* y \succeq 0, \ \langle b, y \rangle = 0, \qquad (**)$$

(II) finds exposing vector: $0 \neq z \succeq 0$

z exposes a proper face containing all the dual feasible points

$$\mathcal{A}x = b, x \succeq 0 \implies zx = 0. \quad (\text{equiv. trace } zx = 0)$$

Regularization of Dual Using Minimal Face

Borwein-W.'78-79 , $f_D = \text{face}\, \mathcal{F}_D^x$; min. face of dual feasible set

(SDP-D) is equivalent to the regularized

$$(\text{SDP}_{reg}\text{-D}) \qquad v_{RD} := \inf_x \{\langle c, x\rangle \;:\; \mathcal{A}x = b, x \succeq_{f_D} 0\}$$

f_D is mininimal face of dual feasible set

$\{x \succeq 0 : \mathcal{A}x = b, x \succeq 0\} \subseteq f_D \trianglelefteq \mathcal{S}_+^n$

Lagrang. dual of regulariz. dual problem satisfies strong duality:

$$(\text{SDP}_{reg}\text{-DD}) \qquad v_{DRD} \;:= \sup_y \{\langle b, y\rangle \;:\; \mathcal{A}^* y \preceq_{f_D^*} c\}$$

$v_D = v_{RD} = v_{DRD}$ and v_{DRD} is attained.

regularized primal-dual pair

If we take the dual of $(\text{SDP}_{reg}\text{-DD})$ we recover the dual regularized problem $(\text{SDP}_{reg}\text{-P})$.

View One for FR in SDP

$(SDP_D) \quad \min\{\text{trace } CX \text{ s.t. } \mathcal{A}X = b, X \in \mathcal{S}^n_+\}$

Step 1: Let $0 \neq Z \succeq 0$ be an exposing vector.

add constraint trace $ZX = 0$. (Equivalently $ZX = 0$)

from spectral decomposition of Z, with Range $P = \text{Null } Z$:

$$\text{substitute:} \quad X = P\mathbb{S}^{t_1}_+ P^T, \qquad t_1 = \text{nullity}(Z)$$

We get the equivalent smaller problem

$$(SDP_{D1}) \quad \begin{array}{ll} \min & \text{trace}(P^T C P)R \\ \text{s.t.} & \text{trace}(P^T A_i P)R = b_i, i = 1, \ldots, m \\ & R \in \mathbb{S}^{t_1}_+ \end{array}$$

Remove/delete redundant linear constraints; repeat Step 1.
minimum number of steps is called the singularity degree
(ref. Sturm below)

View Two for FR in SDP

Lemma: Using exposing vectors

Let

$$Z_i \succeq 0, F_i = \mathcal{S}^n_+ \cap Z_i^\perp, i = 1, \ldots, m.$$

Then

$$\cap_{i=1}^m F_i = \mathcal{S}^n_+ \cap \left(\sum_{i=1}^m Z_i \right)^\perp$$

intersection of faces is exposed by sum of exposing vectors □

Singularity Degree d - Minimal Number of FR Steps

Sturm's error bounds Theorem for SDP, 2000

Given an affine subspace $\mathcal{V}$ of $\mathcal{S}^n$, the pair $(\mathcal{V}, \mathcal{S}^n_+)$ is $\frac{1}{2^d}$-Holder regular, $\gamma = \frac{1}{2^d}$, with displacement, where d is the singularity degree of $(\mathcal{V}, \mathcal{S}^n_+)$ with displacement.
(e.g., for intersecting sets, for all compact sets U there exists a constant $c > 0$ such that
$\operatorname{dist}(x, \mathcal{V} \cap \mathcal{S}^n_+) \leq c\left(\operatorname{dist}^\gamma(x, \mathcal{V}) + \operatorname{dist}^\gamma(x, \mathcal{S}^n_+)\right), \quad \forall x \in U$)

Cgnce rate alternating directions (MAP) for SDP

Theorem (Drusvyatskiy, Li, W. 2015) If the sequence X_k, Y_k converges, $d > 0$, then the rate is $\mathcal{O}\left(k^{-\frac{1}{2^{d+1}-2}}\right)$
(If Slater holds then cgnce is R-linear.)

(Paper includes Empirical Confirmation)

Applications?

- preprocessing is essential in commercial LP software.
- Can we do facial reduction in general?
- Is it efficient/worthwhile?
- important applications?
  - relation to feasibility questions, e.g., for matrix completion
  - iterative methods? convergence rates? (DR, MAP)

** FR - Motivation/Application; EDM, SNL

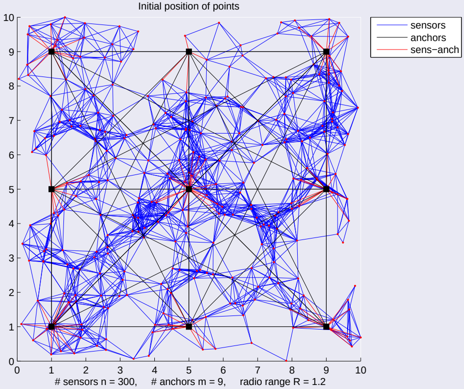
Highly (implicit) degenerate/low-rank problem

- high (implicit) degeneracy translates to low rank solutions
- take advantage of degeneracy; fast, high accuracy solutions

SNL - a Fundamental Problem of Distance Geometry; easy to describe - dates back to Grasssmann 1886

- r : embedding dimension
- n ad hoc wireless sensors $p_1, \ldots, p_n \in \mathbb{R}^r$ to locate in $\mathbb{R}^r$;
- m of the sensors $p_{n-m+1}, \ldots, p_n$ are anchors (positions known, using e.g. GPS)
- pairwise distances $D_{ij} = \|p_i - p_j\|^2, ij \in E$, are known within radio range $R > 0$
- $$P^\top = \begin{bmatrix} p_1 & \ldots & p_n \end{bmatrix} = \begin{bmatrix} X^\top & A^\top \end{bmatrix} \in \mathbb{R}^{r \times n}$$

Sensors ○ and Anchors ■



Popular Techniques; SDP Relax.; Highly Degen.

Nearest, Weighted, SDP Approx. (relax/discard $\operatorname{rank} B$)

- $\min_{B \succeq 0} \| H \circ (\mathcal{K}(B) - D) \|$

  $\operatorname{rank} B = r; \qquad H_{ij} = \begin{cases} 1/\sqrt{D_{ij}} & \text{if } ij \in E, \\ H_{ij} = 0 & \text{otherwise} \end{cases}$
- with rank constraint: a non-convex, NP-hard program
- SDP relaxation is convex
  BUT: expensive/low accuracy/implicitly highly degenerate

cliques restrict ranks of feasible B

Random Noisless Problems, Krislock W. '**2010**

- 2.16 GHz Intel Core 2 Duo, 2 GB of RAM
- Dimension $r = 2$
- Square region: $[0, 1] \times [0, 1]$
- $m = 9$ anchors
- Using only Rigid Clique Union and Rigid Node Absorption
- Error measure: Root Mean Square Deviation

$$\text{RMSD} = \left(\frac{1}{n} \sum_{i=1}^{n} \|p_i - p_i^{\text{true}}\|^2 \right)^{1/2}$$

Results - Large n (SDP size $O(n^2)$)

n # of Sensors Located

n # sensors \ R	0.07	0.06	0.05	0.04
2000	2000	2000	1956	1374
6000	6000	6000	6000	6000
10000	10000	10000	10000	10000

CPU Seconds

# sensors \ R	0.07	0.06	0.05	0.04
2000	1	1	1	3
6000	5	5	4	4
10000	10	10	9	8

RMSD (over located sensors)

n # sensors \ R	0.07	0.06	0.05	0.04
2000	$4e{-}16$	$5e{-}16$	$6e{-}16$	$3e{-}16$
6000	$4e{-}16$	$4e{-}16$	$3e{-}16$	$3e{-}16$
10000	$3e{-}16$	$5e{-}16$	$4e{-}16$	$4e{-}16$

Results - N Huge SDPs Solved

Large-Scale Problems (results from 2010)

# sensors	# anchors	radio range	RMSD	Time
20000	9	.025	$5e{-16}$	25s
40000	9	.02	$8e{-16}$	1m 23s
60000	9	.015	$5e{-16}$	3m 13s
100000	9	.01	$6e{-16}$	9m 8s

Size of SDPs Solved: $N = \binom{n}{2}$ (# vrbls)

$\mathcal{E}_n(\text{density of } \mathcal{G}) = \pi R^2$; $M = \mathcal{E}_n(|E|) = \pi R^2 N$ (# constraints)

Size of SDP Problems:

$M = [3,078,915 \quad 12,315,351 \quad 27,709,309 \quad 76,969,790]$

$N = 10^9 \, [0.2000 \quad 0.8000 \quad 1.8000 \quad 5.0000]$

View 2: Details with Exposing Vector/Numerics

Thm D.P.W. '15: $\mathcal{M} : \mathbb{E} \to \mathbb{Y}$, K proper convex cone

$\emptyset \neq F = \{X \in K : \mathcal{M}(X) = b\}$. Then a vector v exposes a proper face of $\mathcal{M}(K)$ containing b if, and only if, v satisfies the auxiliary system

$$0 \neq \mathcal{M}^* v \in \mathcal{K}^*, \quad \langle v, b \rangle = 0.$$

Let $N = \text{face}(b, \mathcal{M}(K))$ (smallest face containing b). Then:

- $K \cap \mathcal{M}^{-1}(N) = \text{face}(F, K)$
- v exposes N IFF $\mathcal{M}^*(v)$ exposes $\text{face}(F, K)$.

Corollary

If Slater's condition fails, then $d = 1$ IFF the minimal $\text{face}(b, \mathcal{M}(K))$ is exposed.

Noisy Case

Using exposing vectors

Successful numerics recently Drusvyatskiy/Krislock/Vronin/W. 2015.

* FR for Low-Rank Matrix Completion, LRMC, (Huang-W.'16)

Intractable (nonconvex) minimum rank completion

Given partial $m \times n$ real matrix $Z \in \mathbb{R}^{m\times n}$.

$$(LRMC) \qquad \begin{array}{ll} \min & \operatorname{rank}(M) \\ \text{s.t.} & \|M_{\hat{E}} - Z_{\hat{E}}\| \leq \delta, \end{array}$$

$\hat{E}$ sampled indices; $Z_{\hat{E}} \in \mathbb{R}^{\hat{E}}$; $\delta > 0$ tuning parameter

convex nuclear norm relaxation

$$\begin{array}{ll} \min & \|M\|_* \\ \text{s.t.} & \|M_{\hat{E}} - Z_{\hat{E}}\| \leq \delta, \end{array}$$

where $\|M\|_* = \sum_i \sigma_i(M)$.

SDP Equivalent to Nuclear Norm Minimization

Trace minimization

$$\begin{array}{ll} \min & \|Y\|_* = \operatorname{trace}(Y) \\ \text{s.t.} & \|Y_{\bar{E}} - Q_{\bar{E}}\| \leq \delta \\ & Y \in \mathbb{S}_+^{m+n}, \end{array}$$

$Q = \begin{bmatrix} 0 & Z \\ Z^T & 0 \end{bmatrix} \in \mathbb{S}_+^{m+n}$ and $\bar{E}$ indices in Y corresponding to $\hat{E}$

Noiseless case: strict feasibility trivially holds

$$Y_{\bar{E}} = Q_{\bar{E}}$$

choose diagonal of Y sufficiently large, positive.
(strict feas. holds for dual as well)

Why consider this here?

It has been shown recently by Huang-W. that one can exploit the structure at the optimum and efficiently apply FR.

Associated Undirected Weighted Graph $G = (V, E, W)$

node set $V = \{1, \dots, m, m+1, \dots, m+n\}$ Let:

$$E_{1,m} := \{ij \in V \times V : i < j \leq m\}$$
$$E_{m+1,m+n} := \{ij \in V \times V : m+1 \leq i < j \leq m+n\}$$

edge set

$$E := \bar{E} \cup E_{1,m} \cup E_{m+1,m+n}.$$

weights for all $ij \in E$

$$w_{ij} := \begin{cases} Z_{i(j-m)}, & \forall ij \in \bar{E} \\ 0, & \text{otherwise.} \end{cases}$$

Corresponding adjacency matrix A; cliques C

nontrivial cliques of interest (after row/col perms) corresp. to full (specified) submatrix X in Z; $C = \{i_1, \dots, i_k\}$ with cardinalities

$$|C \cap \{1, \dots, m\}| = p \neq 0, \quad |C \cap \{m+1, \dots, m+n\}| = q \neq 0.$$

Exposing Vector for Low-Rank Completions

Clique - X; generically rank r by lsc of rank

$$X \equiv \{Z_{i(j-m)} : ij \in C\}, \quad \text{specified } p \times q \text{ submatrix.}$$

let $\operatorname{rank} X = r_X$. Wlog

$$Z = \begin{bmatrix} Z_1 & Z_2 \\ X & Z_3 \end{bmatrix},$$

full rank factorization $X = \bar{P}\bar{Q}^T$ using SVD

$$X = \bar{P}\bar{Q}^T = U_X \Sigma_X V_X^T, \, \Sigma_X \in \mathbb{S}_{++}^{r_X}, \quad \bar{P} = U_X \Sigma_X^{1/2}, \, \bar{Q} = V_X \Sigma_X^{1/2}.$$

$$C_X = \{i, \ldots, m, m+1, \ldots, m+k\}, \qquad r < \max\{p, q\},$$

target rank r.

(From HW) rewrite optimality conditions SDP as

$$0 \preceq Y = \begin{bmatrix} U \\ P \\ Q \\ V \end{bmatrix} D \begin{bmatrix} U \\ P \\ Q \\ V \end{bmatrix}^T = \left[\begin{array}{c|cc|c} UDU^T & UDP^T & UDQ^T & UDV^T \\ \hline PDU^T & PDP^T & PDQ^T & PDV^T \\ QDU^T & QDP^T & QDQ^T & QDV^T \\ \hline VDU^T & VDP^T & VDQ^T & VDV^T \end{array}\right].$$

Using exposing vectors

Lemma (Basic FR)

Let $r < \min\{p,q\}$ and $X = PDQ^T = \bar{P}\bar{Q}^T$ as above. We find a pair of exposing vectors using

$$\mathrm{FR}(\bar{P},\bar{Q}):\ \bar{P}\bar{P}^T + \boxed{\bar{U}\bar{U}^T} \succ 0,\ \bar{P}^T\bar{U} = 0,$$

$$\bar{Q}\bar{Q}^T + \boxed{\bar{V}\bar{V}^T} \succ 0,\ \bar{Q}^T\bar{V} = 0.$$

Numerics LRMC/average over 5 instances

Table: noiseless: $r = 2$; $m \times n$ size $\uparrow$.

Specifications			Time (s)	Rank	Residual (%Z)
m	n	mean(p)			
700	2000	0.30	9.00	2.0	4.4605e-14
1000	5000	0.30	28.76	2.0	3.0297e-13
1400	9000	0.30	77.59	2.0	7.8674e-14
1900	14000	0.30	192.14	2.0	6.7292e-14
2500	20000	0.30	727.99	2.0	4.2753e-10

Table: noiseless: $r = 4$; $m \times n$ size $\uparrow$.

Specifications			Time (s)	Rank	Residual (%Z)
m	n	mean(p)			
700	2000	0.36	12.80	4.0	1.5217e-12
1000	5000	0.36	49.66	4.0	1.0910e-12
1400	9000	0.36	131.53	4.0	6.0304e-13
1900	14000	0.36	291.22	4.0	3.4847e-11
2500	20000	0.36	798.70	4.0	7.2256e-08

Numerics LRMC/average over 5 instances

Table: noiseless: $r = 3$; $m \times n$ size $\uparrow$; noise $\uparrow$; density $\downarrow$.

Specifications				Time (s)		Rank		Residual (%Z)	
m	n	% noise	p	initial	total	initial	refine	initial	refine
700	1000	0.00	0.40	2.22	1.82	2.40	2.40	3.961e-14	3.961e-14
700	1000	0.01	0.40	4.16	8.79	3.20	3.20	9.242e-01	9.360e-01
700	1000	0.15	0.40	3.64	6.32	2.40	2.40	9.416e-01	9.517e-01
700	1000	0.30	0.40	3.46	7.09	8.40	8.40	9.862e-01	9.862e-01
700	1000	0.45	0.40	3.45	4.26	3.80	3.80	9.539e-01	9.539e-01
1500	2000	10.00	0.40	14.07	19.13	2.40	2.40	9.281e-01	9.360e-01
1600	2100	10.00	0.35	13.85	18.03	2.40	2.40	9.535e-01	9.535e-01
1700	2200	10.00	0.30	10.48	30.81	11.00	11.00	8.000e-01	8.000e-01
1800	2300	10.00	0.25	4.22	15.22	4.60	4.60	4.000e-01	4.000e-01
1900	2500	10.00	0.40	21.39	29.03	2.20	2.20	9.506e-01	9.546e-01
2000	2600	10.00	0.35	18.58	50.70	10.20	10.20	9.894e-01	9.894e-01
2100	2700	10.00	0.30	22.75	40.97	6.40	6.40	9.759e-01	9.759e-01
2200	2800	10.00	0.25	6.61	26.14	5.20	5.20	4.000e-01	4.000e-01

** Conclusion

Preprocessing

- Though strict feasibility holds generically, failure appears in many applications. Loss of strict feasibility is directly related to ill-posedness and difficulty in numerical methods.
- Preprocessing based on structure can both *regularize* and simplify the problem. In many cases one gets an optimal solution without the need of any SDP solver.

Exploit structure at optimum

For low-rank matrix completion the structure at the optimum can be exploited to apply FR even though strict feasibility holds.

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Thanks for your attention!

The many faces of degeneracy in conic optimization

Henry Wolkowicz

Dept. Combinatorics and Optimization, University of Waterloo, Canada

Wed. Sept. 7, 2016, 16:00-16:30

COCA16 Continuous Optimization: Challenges and Applications
Celebrating Ronny Ben-Tal's 70th Birthday
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