Low-Rank Matrix Completion with Facial Reduction

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Outline

- Finding low rank matrix completions is a numerically hard nonconvex problem.
- A popular convex relaxation is the nuclear norm which is SDP-representable; and, both the SDP and its dual satisfy strict feasibility (Slater's constraint qualification).
- For inequality constrained optimization problems, perhaps the most important key is to identify the active constraints.
  We aim to do facial reduction for the optimal face of the SDP, i.e., identify the "'active" face.
- Thus we (try to) avoid a need for a SDP solver.

Low-Rank Matrix Completion

Example (Partial Matrix with Noise —— BUT Low Rank)

1.01	2	?
1	?	2.99

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1	2	3
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Problem Statement (non-convex & intractable)

Given a real partial matrix $z \in \mathbb{R}^{\hat{E}}$ with some level of noise,

$$\text{(LRMC)} \qquad \begin{array}{ll} \min & \operatorname{rank}(M) \\ s.t. & \|\mathcal{P}_{\hat{E}}(M) - z\| \le \delta, \quad M \in \mathbb{R}^{m \times n} \end{array}$$

- $\hat{E}$ indices for known entries (sampled data) in $Z \in \mathbb{R}^{m \times n}$;
  with coordinate projection/partial matrix $z = \mathcal{P}_{\hat{E}}(Z) \in \mathbb{R}^{\hat{E}}$
- $\delta > 0$ is a tuning parameter

Applications Include:

- data science
- model reduction
- collaborative filtering (Netflix problem)
- sensor network localization
- pattern recognition
- various machine learning scenarios

Low-Rank Matrix Completion

Minimizing rank is a hard nonconvex problem

Rank is a lower semi-continuous function.

Nuclear Norm Minimization (convex relaxation)

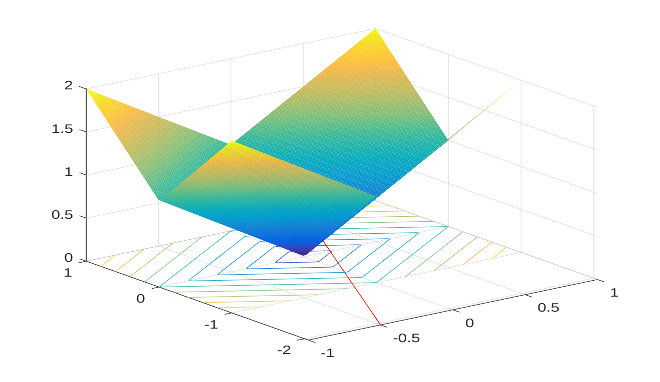
The problem (LRMC) can be approximated by

$$\text{(NN-LRMC)} \qquad \begin{array}{ll} \min & \|M\|_* \\ s.t. & \|\mathcal{P}_{\hat{E}}(M) - z\| \leq \delta \end{array}$$

- $\|M\|_* = \sum_i \sigma_i(M)$, sum of singular values, nuclear norm (Schatten 1-norm, Ky-Fan r-norm, trace norm)
- $\|UXV^T\|_* = \|X\|_*$ unitarily invariant

Generalization of Minimizing # Nonzeros (Sparsity)

smallest radius ℓ_1 ball to intersect line (constraint)



Nuclear Norm Minimization, Fazel-'02 thesis

Theorem (Fazel,Hindi,Boyd '01)

$\|X\|_*$ *is the convex envelope of* $\operatorname{rank} X$ *on* $\{X \in \mathbb{R}^{m \times n} : \|X\| \leq 1\}$.

Properties of nuclear norm:

- *"best"* convex lower approximation of rank function
- The nuclear ball is the convex hull of the intersection of rank-1 matrices with the unit ball:
  $\operatorname{conv}\{uv^T : u \in \mathbb{R}^n, v \in \mathbb{R}^m, \|u\| = 1, \|v\| = 1\}$
- SDP-representable
- Related references by: Candes,Fazel,Parrilo,Recht

SDP Representable

SDP Embedding Lemma

Let $M \in \mathbb{R}^{m\times n}$ and $t \in \mathbb{R}$. Then:

$\|M\|_* \leq t$

if, and only if,

there exist (symmetric) W_1 and W_2 such that

$$\begin{bmatrix} W_1 & M \\ M^T & W_2 \end{bmatrix} \succeq 0, \quad \text{trace}(W_1) + \text{trace}(W_2) \leq 2t.$$

- compact SVD: $M = U\Sigma V^T$, $\qquad \|M\|_* = \text{trace}\,\Sigma \leq t$
- $\begin{bmatrix} U\Sigma^{1/2} \\ V\Sigma^{1/2} \end{bmatrix} \begin{bmatrix} U\Sigma^{1/2} \\ V\Sigma^{1/2} \end{bmatrix}^T = \begin{bmatrix} U\Sigma U^T & U\Sigma V^T \\ V\Sigma U^T & V\Sigma V^T \end{bmatrix} \succeq 0$
- For necessity, set $W_1 = U\Sigma U^T$, $W_2 = V\Sigma V^T$; for sufficiency, exploit $\boxed{\text{range}\, M \subseteq \text{range}\, W_1, \text{range}\, M^T \subseteq \text{range}\, W_2}$

Nuclear Norm Low Rank Problem, (NN-LRMC)

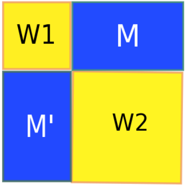
Semidefinite Embedding: Trace Minimization

Problem (NN-LRMC) can be formulated as:

$$\text{(SDP-LRMC)} \qquad \begin{array}{ll} \min & \frac{1}{2}\,\text{trace}(Y) \\ s.t. & \|\mathcal{P}_{\bar{E}}(Y) - z\| \leq \delta \\ & Y \succeq 0 \end{array}$$

where $Q = \begin{bmatrix} 0 & Z \\ Z^T & 0 \end{bmatrix}$, $z = \mathcal{P}_{\hat{E}}(Z) = \mathcal{P}_{\bar{E}}(Q)$;

$\bar{E}$ is set of indices in Q corresponding to known entries of Z.

Y = 

First, an Example of Facial Reduction, FR

Example (Facial Reduction in Linear Programming)

$$\begin{array}{ll} \min & \begin{pmatrix} 2 & 5 & -1 & 4 & 7 \end{pmatrix} x \\ \text{s.t.} & \begin{bmatrix} 1 & 1 & -1 & 3 & 1 \\ -1 & 1 & 2 & 2 & -1 \end{bmatrix} x = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\ & x \geq 0 \end{array}$$

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If we sum the two constraints we get a facial constraint

$$2x_2 + x_3 + 5x_4 = 0 \implies x \in \mathcal{F} = \left\{ x \in \mathbb{R}^5_+ : x_2 = x_3 = x_4 = 0 \right\}$$

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Thus, the problem can be reduced to (constraint 2 is redundant)

$$\begin{array}{ll} \min & \begin{pmatrix} 2 & 7 \end{pmatrix} x \\ \text{s.t.} & \begin{pmatrix} 1 & 1 \end{pmatrix} x = 1 \\ & x \geq 0 \end{array}$$

First Example of Facial Reduction, cont...

Example (Facial Reduction in Linear Programming)

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Example (Facial Reduction in Linear Programming)

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& x \geq 0
\end{array}
$$

Find y with $y^T b = 0, 0 \neq w = A^T y \geq 0$ to get:

$$y = (1\ 1)^T,\ 0 \neq w^T = (A^T y)^T = \begin{pmatrix} 0 & 2 & 1 & 5 & 0 \end{pmatrix} \geq 0.$$

Then w is an exposing vector of the feasible set:

First Example of Facial Reduction, cont...

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Then w is an exposing vector of the feasible set:

$$w^T x = 0, \forall \text{ feasible } x \implies x = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_5 \end{bmatrix}; x_2 = x_3 = x_4 = 0;$$

(simplified) FR problem is

$$\min \left\{ \begin{pmatrix} 2 & 7 \end{pmatrix} v \ : \ \begin{pmatrix} 1 & 1 \end{pmatrix} v = 1, \ v \geq 0 \right\}$$

Faces of a Closed Convex Cone, ccc

Face of a ccc $\mathcal{K}$, $\quad \mathcal{K}+\mathcal{K} \subseteq \mathcal{K}, \mathbb{R}\mathcal{K} \subseteq \mathcal{K}$

Let $\mathcal{K}$ be a ccc. A cone $F \subseteq \mathcal{K}$ is a face of $\mathcal{K}$, $F \trianglelefteq \mathcal{K}$, if

$$x, y \in \mathcal{K}, \quad x+y \in F \quad \Rightarrow \quad x, y \in F,$$

If $\emptyset \neq F \subsetneq \mathcal{K}$, then it is called a proper face.

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Faces of PSD Cone $\mathbb{S}^n_+$

Let $X\in\operatorname{relint}(F)$, $F\trianglelefteq\mathbb{S}^n_+$;

let $X=\begin{bmatrix}U & V\end{bmatrix}\begin{bmatrix}D & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}U & V\end{bmatrix}, D\in\mathbb{S}^k_{++}$

be the spectral decomposition.

two views are: $\qquad F=U\mathbb{S}^k_+U^T=\mathbb{S}^n_+\cap(VV^T)^\perp$

Exposed Vector View

Exposing Vector and Exposed Face

Let $\mathcal{K}$ be a ccc, $F \trianglelefteq \mathcal{K}$. Let $K^* = \{\phi : \langle \phi, k \rangle \geq 0, \forall k \in K\}$ denote the *dual cone*. If

$$F = \mathcal{K} \cap \phi^{\perp}, \quad \text{for some } \phi \in \mathcal{K}^*,$$

then the face F is called an exposed face and ϕ is the corresponding exposing vector.

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Example (Faces of PSD Cone)

In the PSD cone, we saw that a face can be expressed as

$$X \in \text{relint}\, F = \mathbb{S}^n_+ \cap \{VV^T\}^\perp, \qquad \text{null}(X) = \text{range}(V).$$

$VV^T \in \mathbb{S}^n_+$ is the exposing vector.

Properties of Faces

Some Useful Facts about Faces

- a face of a face is a face;
- an intersection of two faces is a face
- $F_i \trianglelefteq K, F_i = K \cap \phi_i^\perp$, $i = 1, \ldots, k$, implies

$$\cap_i F_i = K \cap (\sum_i \phi_i)^\perp$$

i.e., intersection exposed faces - exposed by sum of exposing vectors

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For PSD cone

- Self-replicating: a face of a PSD cone is *still* a PSD cone;
- Facially exposed: every face of PSD cone has exposing vector
- Self-dual: $\mathcal{K} = \mathcal{K}^* = \{x : \langle x, y \rangle \geq 0, \forall y \in \mathcal{K}\}$

Exposing Vector and Exposed Face

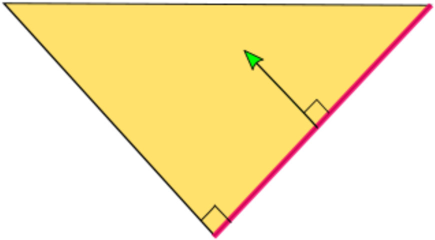

Figure: A Illustration of a Self-Dual Cone

Back to the Low-Rank Matrix Completion Problem

Recall (SDP-LRMC) Problem: Given $z \in \mathbb{R}^{\hat{E}}$ a partial matrix, find the matrix Z of minimum rank to complete z,
i.e., $\mathcal{P}_{\hat{E}}(Z) = \mathcal{P}_{\bar{E}}(Q) = z$,

Minimize nuclear norm using SDP

$$\text{(SDP-LRMC)} \qquad \begin{array}{rl} \min & \|Y\|_* = \frac{1}{2}\operatorname{trace}(Y) \\ \text{s.t.} & \mathcal{P}_{\bar{E}}(Y) = z \\ & Y \succeq 0, \end{array}$$

where $\bar{E}$ is the set of indices in Y that correspond to $\hat{E}$, the known entries of the upper right block of $\begin{bmatrix} 0 & Z \\ Z^T & 0 \end{bmatrix} \in \mathbb{S}_+^{m+n}$.

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• Since the diagonal is free, note that the Slater condition (strict feasibility) does hold for (SDP-LRMC). (And it holds for its dual.)

Facial Reduction of (SDP-LRMC) for Optimal Face

Bipartite Graph, $G_Z = (U_m, V_n, \hat{E})$

With Z and the sampled elements we get a bipartite graph G_Z.

Find Fully Known Submatrix X – a biclique α, $X \cong z[\alpha] \in \mathbb{R}^{p\times q}$

After permutation of row and columns, WLOG

$$Z = \begin{bmatrix} Z_1 & Z_2 \\ X & Z_3 \end{bmatrix}, \quad z = Z[\hat{E}], \quad \alpha \subseteq \hat{E}, \quad X \cong z[\alpha].$$

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Our algorithm is based on finding bicliques in G_Z; we do this by finding (nontrivial/nondiagonal-block) cliques within symmetric matrix Y.

$$Y = \begin{bmatrix} W_1 & Z \\ Z^T & W_2 \end{bmatrix}$$

Bipartite Graph and Biclique

Partial matrix

$$z \cong \begin{bmatrix} -5 & NA & 10 & -20 & NA & -6 \\ 4 & 0 & 4 & 4 & 6 & 6 \\ -3 & NA & NA & 32 & 27 & NA \\ 5 & NA & 0 & 10 & 12 & NA \\ NA & -30 & NA & NA & 27 & NA \\ 3 & -5 & -2 & 8 & NA & 4 \\ 5 & 5 & NA & 0 & 3 & NA \end{bmatrix}, \quad \hat{E} = \{11, 13, 14, 16, 21, \dots, 74, 75\}$$

biclique indices: $\bar{U}_m = \{6, 1, 2\}$, $\bar{V}_n = \{1, 4, 3, 6\}$, $\alpha = \{61, 64, 63, 66, 11, \dots, 26\}$

$$z[\alpha] \equiv X = \begin{bmatrix} 3 & 8 & -2 & 4 \\ -5 & -20 & 10 & -6 \\ 4 & 4 & 4 & 6 \end{bmatrix}.$$

$$Y[\alpha] = \begin{bmatrix} \text{FREE} & \begin{matrix} 3 & 8 & -2 & 4 \\ -5 & -20 & 10 & -6 \\ 4 & 4 & 4 & 6 \end{matrix} \\ \begin{matrix} 3 & -5 & 4 \\ 8 & -20 & 4 \\ -2 & 10 & 4 \\ 4 & -6 & 6 \end{matrix} & \text{FREE} \end{bmatrix}$$

Our view of facial reduction and exposed faces

Theorem (Drusvyatskiy, Pataki, W. '15)

Linear transformation $\mathcal{M} : \mathbb{S}^n \to \mathbb{R}^m$*, adjoint* $\mathcal{M}^*$*; feasible set* $\mathcal{F} := \{X \in \mathbb{S}^n_+ : \mathcal{M}(X) = b\} \neq \emptyset$*,* $b \in \mathbb{R}^m$*. Then a vector* v *exposes a proper face of* $\mathcal{M}(\mathbb{S}^n_+)$ *containing* b $\iff$ v *satisfies the auxiliary system*

$$0 \neq \mathcal{M}^* v \in \mathbb{S}^n_+ \quad \text{and} \quad \langle v, b \rangle = 0.$$

Let N *denote smallest face of* $\mathcal{M}(\mathbb{S}^n_+)$ *containing* b*. Then:*

1. $\mathbb{S}^n_+ \cap \mathcal{M}^{-1} N = \text{face}(\mathcal{F})$*, the smallest face containing* $\mathcal{F}$*.*
2. *For any vector* $v \in \mathbb{R}^m$ *the following equivalence holds:*

$$v \text{ exposes } N \iff \mathcal{M}^* v \text{ exposes } \text{face}(\mathcal{F})$$

Singularity Degree and Adjoint Representation

Adjoint of $\mathcal{P}_{\bar{E}[\alpha]}$

Recall: N denotes smallest face of $\mathcal{P}(\mathbb{S}^n_+)$ containing $z[\alpha]$. If v exposes N, then $V = \mathcal{P}^*(v)$ fills out v with zeros; V exposes $\text{face}(\mathcal{F})$.

Definition

The singularity degree, SD of: (LMI) $\mathcal{M}(X) = b, \quad X \succeq 0,$
is the minimal number of facial reduction steps needed to obtain strict feasibility.

Theorem

If Slater fails for (LMI), then the SD is one if, and only if, the minimal face of b in $\mathcal{M}(\mathbb{S}^n_+)$ is exposed.

Facial Reduction for (SDP-LRMC), r is target rank for Z

Biclique $\alpha \cong$ of G_Z, $z[\alpha] \equiv X \in \mathbb{R}^{p\times q}$

target rank $r \leq \min\{p,q\} < \max\{p,q\}$;

WLOG

$$Z = \begin{bmatrix} Z_1 & Z_2 \\ X & Z_3 \end{bmatrix},$$

SVD: $$X = \begin{bmatrix} U_1 & U_X \end{bmatrix} \begin{bmatrix} \Sigma \in \mathbb{S}^r_{++} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_1 & V_X \end{bmatrix}^T$$

We get full rank factorization

$$X = \bar{P}\bar{Q}^T = U_1 \Sigma V_1^T, \quad \bar{P} = U_1 \Sigma^{1/2}, \; \bar{Q} = V_1 \Sigma^{1/2}.$$

Since *rank* is lower semi-continuous: $\operatorname{rank} X = \operatorname{rank} Z$ generically.
In fact our tests form: $Z = \bar{P}\bar{Q}^T$
with $\bar{P}, \bar{Q}$ random, i.i.d. and full column rank r.

FR using Optimal Y

Rewrite Optimal Y

Assuming we have obtained the desired target rank $Y = r$

$$0 \preceq Y = \begin{bmatrix} U \\ P \\ Q \\ V \end{bmatrix} D \begin{bmatrix} U \\ P \\ Q \\ V \end{bmatrix}^T = \left[\begin{array}{c|cc|c} UDU^T & UDP^T & UDQ^T & UDV^T \\ \hline PDU^T & PDP^T & \boxed{PDQ^T} & PDV^T \\ QDU^T & QDP^T & QDQ^T & QDV^T \\ \hline VDU^T & VDP^T & VDQ^T & VDV^T \end{array}\right]$$

And assume rank $X = r$

$$X = PDQ^T = \bar{P}\bar{Q}^T.$$

implies the ranges satisfy
$U_1^T U_X = P^T U_X = 0, V_1^T V_X = Q^T V_X = 0$

$$\begin{array}{rcccccl} \text{range}(X) & = & \text{range}(P) & = & \text{range}(\bar{P}) & = & \text{range}(U_1), \\ \text{range}(X^T) & = & \text{range}(Q) & = & \text{range}(\bar{Q}) & = & \text{range}(V_1). \end{array}$$

Constructing Exposing Vectors

Key for facial reduction

We can use an exposing vector formed as $U_X U_X^T$ for block PDP^T as well as $V_X V_X^T$ for block QDQ^T and add appropriate blocks of zeros:

$$W_X = \left[\begin{array}{c|cc|c} 0 & 0 & 0 & 0 \\ \hline 0 & U_X U_X^T & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right] + \left[\begin{array}{c|cc|c} 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & V_X V_X^T & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right]$$
$$= \left[\begin{array}{c|cc|c} 0 & 0 & 0 & 0 \\ \hline 0 & U_X U_X^T & 0 & 0 \\ 0 & 0 & V_X V_X^T & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right].$$

All three matrices provide exposing vectors.

Facial reduction from exposing vector

$$F^* \trianglelefteq T\mathbb{S}_+^{((n+m)-(p+q-2r))}T^T, \quad \text{range } T = \text{null } W_X.$$

Exposing Vectors for (SDP-LRMC)

$X \in \mathbb{R}^{p\times q}$ known submatrix, $X = \bar{P}\bar{Q}$ full rank decomposition

Find A, B:

$$\bar{P}\bar{P}^T + AA^T \succ 0, \;\; \bar{Q}\bar{Q}^T + BB^T \succ 0, \qquad \bar{P}^T A = 0, \;\; \bar{Q}^T B = 0.$$

Then a pair of exposing vectors:

$$\mathcal{P}^*(AA^T), \quad \mathcal{P}^*(BB^T).$$

Measuring Noise of Biclique $\alpha \in \Theta$

Biclique: $\alpha \subseteq \hat{E}$, $\quad z[\alpha] \cong X \in \mathbb{R}^{p \times q}$, $\qquad$ target rank r

singular values of X: $\sigma_1 \geq \ldots \geq \sigma_{\min\{p,q\}}$

biclique noise: $$u_X^P := \frac{\sum_{i=r+1}^{\min\{p,q\}} \sigma_i^2}{0.5p(p-1)} \quad u_X^Q := \frac{\sum_{i=r+1}^{\min\{p,q\}} \sigma_i^2}{0.5q(q-1)}$$

Assign biclique weight

Total noise of all bicliques: $S = \sum_{X \in \Theta} (u_X^P + u_X^Q)$

for each $\alpha \in \Theta$: $$w_X^P = 1 - \frac{u_X^P}{S}, \quad w_X^Q = 1 - \frac{u_X^Q}{S}$$

Facial Reduction Process

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- Find set of bicliques Θ, of appropriate sizes

(Follows the framework in Drusvyatskiy/Krislock/Cheung-Voronin/W.)

Facial Reduction Process

- Find set of bicliques Θ, of appropriate sizes
- Find corresponding exposing vectors $\{Y_\alpha^{expo}\}_{\alpha\in\Theta}$
  calculate their weights $\{\omega_\alpha\}_{\alpha\in\Theta}$

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- Calculate the weighted sum of the exposing vectors

$$Y_{Final}^{expo} = \sum_{\alpha\in\Theta} \omega_\alpha Y_\alpha^{expo}$$

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$$Y_{Final}^{expo} = \sum_{\alpha\in\Theta} \omega_\alpha Y_\alpha^{expo}$$

- Find full column rank V such that range V = null Y_{Final}^{expo}.
- Solve equivalent smaller problem based on smaller dimensional matrix R, where

$$Y = VRV^T$$

(Follows the framework in Drusvyatskiy/Krislock/Cheung-Voronin/W.)

Exploit block structure

Y_{Final}^{expo} has block structure so V has a block structure too:

$$Y_{Final}^{expo} = \begin{bmatrix} \sum_{X\in\mathcal{C}} w_X^P W_X^P & 0 \\ 0 & \sum_{X\in\mathcal{C}} w_X^Q W_X^Q \end{bmatrix}, \quad V = \begin{bmatrix} V_P & 0 \\ 0 & V_Q \end{bmatrix}$$

Exploit block structure

Y_{Final}^{expo} has block structure so V has a block structure too:

$$Y_{Final}^{expo} = \begin{bmatrix} \sum_{X \in \mathcal{C}} w_X^P W_X^P & 0 \\ 0 & \sum_{X \in \mathcal{C}} w_X^Q W_X^Q \end{bmatrix}, \quad V = \begin{bmatrix} V_P & 0 \\ 0 & V_Q \end{bmatrix}$$

allows a computational speed up for eigenvalue subproblems.

Noiseless Case

FR dramatically reduces dimension of now overdetermined problem:

$$
\begin{array}{lll}
\min & \operatorname{trace}(R) & (= \operatorname{trace}(VRV^T)) \\
\text{s.t.} & \mathcal{P}_{\bar{E}}(V_P R_{pq} V_Q^T) = z & \\
& R = \begin{bmatrix} R_p & R_{pq} \\ R_{pq}^T & R_q \end{bmatrix} \succeq 0. &
\end{array}
$$

Noiseless Case

FR dramatically reduces dimension of now overdetermined problem:

$$\begin{array}{ll} \min & \operatorname{trace}(R) \qquad\qquad (= \operatorname{trace}(VRV^T)) \\ \text{s.t.} & \mathcal{P}_{\bar{E}}(V_P R_{pq} V_Q^T) = z \\ & R = \begin{bmatrix} R_p & R_{pq} \\ R_{pq}^T & R_q \end{bmatrix} \succeq 0. \end{array}$$

remove the redundant constraints

Use a compact QR to find well-conditioned full rank matrix representation. A simple semidefinite constrained least squares solution may be enough!

$$\min_{R \in \mathbb{S}_+^{r_v}} \|\mathcal{P}_{\tilde{E}}(V_P R_{pq} V_Q^T) - \tilde{z})\|.$$

(here $\tilde{E}, \tilde{z}$ denote the corresponding entries after removing redundant constraints.)

Noisy Case

Cannot simply remove redundant constraints;
use random sketch matrix A to reduce the number of constraints;
first solve:

$$\delta_0 = \min_{R \in \mathbb{S}_+^{r_v}} \left\| A \left(\mathcal{P}_{\hat{E}}(V_P R_{pq} V_Q^T) - z \right) \right\|.$$

and hopefully obtain the target rank!

Noisy Case

Cannot simply remove redundant constraints;
use random sketch matrix A to reduce the number of constraints;
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$$\delta_0 = \min_{R \in \mathbb{S}_+^{r_v}} \left\| A \left(\mathcal{P}_{\hat{E}}(V_P R_{pq} V_Q^T) - z \right) \right\| .$$

and hopefully obtain the target rank!
Otherwise, we use a refinement step.

Refinement Step in the Noisy Case

We would like to reduce the rank after the previous step using a parametric approach:

$$
\begin{array}{lrcl}
\min & \operatorname{trace}(R) & & \\
\text{s.t.} & \left\| A\left(\mathcal{P}_{\hat{E}}(V_P R_{pq} V_Q^T) - b\right) \right\| & \leq & \delta_0 \\
& R & \succeq & 0.
\end{array}
$$

Refinement Step in the Noisy Case

We would like to reduce the rank after the previous step using a parametric approach:

$$
\begin{array}{lrcl}
\min & \operatorname{trace}(R) & & \\
\text{s.t.} & \left\| A\left(\mathcal{P}_{\hat{E}}(V_P R_{pq} V_Q^T) - b\right)\right\| & \leq & \delta_0 \\
 & R & \succeq & 0.
\end{array}
$$

To ensure the rank can be reduce, we flip the problem:

$$
\begin{array}{rl}
\varphi(\tau) := \min & \left\| A\left(\mathcal{P}_{\hat{E}}(V_P R_{pq} V_Q^T) - b\right)\right\| + \gamma \|R\|_F \\
\text{s.t.} & \operatorname{trace}(R) \leq \tau \\
 & R \succeq 0.
\end{array}
$$

where γ is a regularization parameter, since the least squares problem can be underdetermined.

Sample Results

Table: noiseless: $r = 8$; $m \times n$ size; density p; mean 20 instances.

Specifications			r_v	Rcvrd (%Z)	Time (s)	Rank	Residual (%Z)
m	n	mean(p)					
1000	3000	0.53	16.10	96.39	37.29	8.0	1.1072e-10
1000	3000	0.50	17.65	88.99	36.50	8.0	4.6569e-10
1000	3000	0.48	32.15	71.66	72.14	8.5	2.0413e-07

Table: noisy: $r = 2$; $m \times n$ size; density p; mean 20 instances.

Specifications				Rcvd (%Z)	Time (s)		Rank		Residual (%Z)	
m	n	% noise	p		initial	refine	initial	refine	initial	refine
1100	3000	0.50	0.33	100.00	33.72	48.53	2.00	2.00	8.53e-03	8.53e-03
1100	3000	1.00	0.33	100.00	33.67	49.09	2.00	2.00	2.70e-02	2.70e-02
1100	3000	2.00	0.33	100.00	34.13	48.84	2.00	2.00	9.75e-02	9.75e-02
1100	3000	3.00	0.33	100.00	36.34	92.73	5.00	5.00	5.48e-01	1.40e-01
1100	3000	4.00	0.33	100.00	51.45	186.28	11.00	8.00	1.25e+00	1.28e-01

Conclusion

Preprocessing

- Though strict feasibility holds generically, failure appears in many applications. Loss of strict feasibility is directly related to ill-posedness and difficulty in numerical methods.
- Preprocessing based on structure can both *regularize* and simplify the problem. In many cases one gets an optimal solution without the need of any SDP solver.

Exploit structure at optimum

For low-rank matrix completion the structure at the optimum can be exploited to apply FR even though strict feasibility holds.

Thanks for your attention!

Low-Rank Matrix Completion with Facial Reduction

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