Linear Programming:

Part (i): Strict Feasibility and Degeneracy
Part (ii): Best Approximation and Exterior Point Path Following

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Monday 11:15AM, April 10, 2023, in M103

at: School of Mathematical and Statistical Sciences
Clemson® University

LP Part (i): Strict Feasibility and Degeneracy

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joint work with: Jiyoung Im, Univ. of Waterloo

Motivation/Main Results

Background

- Currently: simplex and interior point methods are most popular algorithms for solving linear programs, LPs.
- Unlike general conic programs, (finite) LPs do not require strict feasibility for strong duality. Hence strict feasibility (no variable fixed at zero; one type of degeneracy) is often less emphasized.

History Degeneracy

- techniques for resolving degeneracy:
  • (symbolic) perturbation Charnes '52 [10];
  • lexicographic Dantzig-Orden-Wolfe '55 [14];
  • modified lexicographic Wolfe '63 [38] (more efficient Ryan-Osborne '88 [32]);
  • Bland finite pivoting rule 77 [5] (simple/less efficient)
- Megiddo '86 [26]: "exiting degenerate vertex as hard as solving general LP"

Motivation cont...

We show that lack of strict feasibility:

1. causes numerical difficulties in both simplex and interior point methods.
2. and $\Longrightarrow$ all basic feasible solutions, BFS, are degenerate

We introduce:

1. the notion of implicit singularity when strict feasibility fails;
2. an extension of Phase-I of simplex method for the two part preprocessing for strict feasibility

Background and Notation

Feasible LPs; standard form (with FINITE opt. value)

$$(\mathcal{P}) \quad \text{(finite) } p^* = \begin{array}{ll} \min_x & c^T x \\ \text{s.t.} & Ax = b \in \mathbb{R}^m \\ & x \in \mathbb{R}^n_+ \end{array}$$

assume wlog $\operatorname{rank}(A) = m$;

with feasible set: $\mathcal{F} = \{x \in \mathbb{R}^n : Ax = b, \ x \geq 0\}$

Dual LP

$$(\mathcal{D}) \quad p^* = d^* = \begin{array}{ll} \max & b^T y \\ \text{s.t.} & A^T y \leq c \in \mathbb{R}^n \\ & y \in \mathbb{R}^m \end{array}$$

(equivalently $A^T y + s = c, s \geq 0$ slack)

History: Kantorovich; Dantzig, Karmarkar

Kantorovich '39, USSR, WWII

- transportation models and optimal solutions (algorithm)
- helped NKVD with transportation problems

Dantzig '47, USA, SIMPLEX METHOD

- following duality/game-theory by Von Neumann
- Hotelling: "but the world is nonlinear"
- Von Neumann: "if you have a linear model, you can now solve it"
- SIAM survey 1970's: 70% of ALL world computer time is spent on the simplex method

Karmarkar '84, Interior Point Revolution

- Lustig-Marsten-Shanno OB1 code '90; large went
  from: $(m = 1e3 \times n = 1e4)$ to $(m = 1e5 \times n = 1e7)$
- to modern day: $(m = 1e6 \times n = 1e10)$

Strict Feasibility, Slater, Mangasarian-Fromovitz CQ

Feasible LPs; standard form (with FINITE opt. value)

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there exists $\hat{x}$ with $A\hat{x} = b, \hat{x} > 0$ (MFCQ)

Dual LP

$$(\mathcal{D}) \quad p^* = d^* = \begin{array}{ll} \max & b^T y \\ \text{s.t.} & A^T y \leq c \in \mathbb{R}^n \\ & y \in \mathbb{R}^m \end{array}$$

there exists $\hat{y}$ with $A^T \hat{y} < c$ (Slater CQ)

Stability: MFCQ/Slater $\iff$

stability wrt RHS perturbations

$\iff$ compact set of dual variables

Basic (Feasible/Degenerate) Solutions

Definition (basic (feasible) solution)

- Given: $x \in \mathbb{R}^n, Ax = b$ and $\mathcal{B} \subset \{1, \ldots, n\}$, $|\mathcal{B}| = m$; let $\mathcal{N} = \{1 \ldots n\} \backslash B$.
  Then x is a basic solution if
  $A(:, \mathcal{B})$ is nonsingular and $x_i = 0, \ \forall i \in \mathcal{N}$
- x is a basic feasible solution, BFS, if in addition $x \geq 0$. It is degenerate, if $\exists i \in \mathcal{B}, x_i = 0$

Equivalently, if $Ax = b, x \geq 0$ (feasible):

x is basic if there exists
$\mathcal{N} \subset \{1, \ldots, n\}, |\mathcal{N}| = n - m, x_i = 0, \forall i \in \mathcal{N}$;
and the corresponding matrix of active constraints

$$\begin{bmatrix} A \\ I_{\mathcal{N}} \end{bmatrix} \quad \text{is nonsingular.}$$

It is degenerate if there are redundant active constraints.

Two Kinds of Degeneracy

Definition (Degenerate BFS)

$$x \text{ BFS is } \begin{cases} \text{nondegenerate}, & \text{if } x_i > 0,\ \forall i \in \mathcal{B}, \\ \underline{\text{degenerate}}, & \text{otherwise} \end{cases}$$

Definition (variable fixed at 0)

Let $i_0 \in \mathcal{I} = \{1, \ldots, n\}$. x_{i_0} is <u>fixed at 0</u> if $x_{i_0} = 0, \forall x \in \mathcal{F}$. Let

$$\mathcal{I}^{=} = \{i \in \mathcal{I} : x_i \text{ is fixed at } 0\},\ \mathcal{I}^{<} = \mathcal{I} \backslash \mathcal{I}^{=}$$

$\bar{x}$ a <u>degenerate</u> BFS with basis $\mathcal{B}$ is of type:

1. if: $i \in \mathcal{B}, \bar{x}_i = 0 \implies i \in \mathcal{I}^{<}$
2. if: there exists $i \in \mathcal{B} \cap \mathcal{I}^{=}$

Below we see that:
if $\mathcal{I}^{=} \neq \emptyset$, then ALL BFS are of Type 2.

Facial Reduction, FR, for LPs that fail Strict Feasibility

Two Steps

- obtain an equivalent problem with strict feasibility;
- recover full-row rank for the constraint matrix (always needed for MFCQ)

Definition (Face of a convex set K)

A convex set $F \subseteq K \subseteq \mathbb{R}^n$ is a face of K, denoted $F \trianglelefteq K$, if

$$y, z \in K, x = \frac{1}{2}(y+z) \in F \implies y, z \in F.$$

The minimal face for F, $\mathrm{face}(F)$, is the intersection of all faces of K containing C.

faces of $\mathbb{R}^n_+$, nonnegative orthant

for fixed indices $\hat{\mathcal{I}} \subseteq \{1, \ldots, n\}$

$$F = \{x \in \mathbb{R}^n_+ : x_i = 0, \ \forall i \in \hat{\mathcal{I}}\}$$

Facial Reduction; Basics

Theorem (DW: [15, Theorem 3.1.3] Theorem of the Alternative)

For the feasible system $\mathcal{F}$ of the LP, exactly one of the following statements holds:

1. *There exists $x \in \mathbb{R}^n_{++}$ with $Ax = b$, i.e., strict feasibility holds;*
2. *There exists $y \in \mathbb{R}^m$ such that*

$$(*) \quad 0 \neq z := A^T y \in \mathbb{R}^m_+, \quad \text{and} \quad \langle b, y \rangle = 0,$$

exposing vector $z \in \mathbb{R}^n_+$

(*) is equivalent to:
exposing vector $0 \neq z \geq 0$ exists for the minimal face containing the feasible set, i.e.,

$$\begin{aligned} x \in \mathcal{F} \quad &\iff \quad Ax = b, x \geq 0 \\ &\implies \quad \langle z, x \rangle = \langle A^T y, x \rangle = \langle y, Ax \rangle = \langle y, b \rangle = 0 \end{aligned}$$

Facial Reduction two steps; Outline

suppose strict feasibility fails; i.e., get exposing vector z

1. Thm of Alternative implies: $\exists 0 \lneq z = A^T y \in \mathbb{R}^m$:

$$\begin{array}{rcl} x \in \mathcal{F} & \implies & 0 \leq \langle x, z \rangle = \langle x, A^T y \rangle = \langle Ax, y \rangle = \langle b, y \rangle = 0 \\ & \implies & 0 = x \circ z \\ & \iff & 0 = x_j z_j = 0, \forall j \end{array}$$

yields complementary unit vectors e_k

cardinality of support of z: $s_z = |\{i : z_i > 0\}|$

2. $z = \sum_{j=1}^{s_z} z_{t_j} e_{t_j}$, t_j nondecreasing order

$x = \sum_{j=1}^{n-s_z} x_{s_j} e_{s_j}$, s_j nondecreasing order.

$V = \begin{bmatrix} e_{s_1} & e_{s_2} & \dots & e_{s_{n-s_z}} \end{bmatrix} \in \mathbb{R}^{n \times (n-s_z)}$, $\quad Vz = 0$.

3. $\mathcal{F} = \{x \in \mathbb{R}^n_+ : Ax = b\} = \{x = Vv \in \mathbb{R}^n : AVv = b, v \in \mathbb{R}^{n-s_z}_+\}$
4. Recover full row rank: $A \leftarrow P_{\bar{m}} AV, b \leftarrow P_{\bar{m}} b$

Facial Reduction, FR; Two Steps

matrix $V \in \mathbb{R}^{n\times(n-s_z)}$, facial range vector

Every facial reduction step yields at least one redundant constraint, BW: [8],IW: [21, Lemma 2.7],S: [36, Section 3.5].

Lemma (step 2: redundant constraint)

Consider the facially reduced feasible set

$$\mathcal{F}_r = \left\{ v : AVv = b, v \in \mathbb{R}^{n-s_z}_{+} \right\}.$$

Then at least one linear constraint of the LP *is redundant.*

Proof.

Let: $0 \neq z = A^T y \geq 0$ *exposing vector;* V *corresponding facial range vector; Then:*

$$0 = V^T z = V^T A^T y = (AV)^T y = \sum_{i=1}^m y_i((AV)^T)_i$$

Since $0 \neq y \in \mathbb{R}^m$*, the rows of* AV *are linearly dependent.* □

Summary FR

Result of full two step FR: strict feas.; full rank

$$\begin{array}{rcl} \mathcal{F} & = & \{x \in \mathbb{R}^n_+ : Ax = b\} \\ & = & \{x = Vv \in \mathbb{R}^n : \bar{A}v := (P_{\bar{m}}AV)v = (P_{\bar{m}}b) =: \bar{b}, \\ & & \quad v \in \mathbb{R}^{n-s_z}_+\} \end{array}$$

- after substit: $\min (V^T c)^T v$ s.t. $\bar{A}v = \bar{v}$, $v \in \mathbb{R}^{n-s_z}_+$
- $\exists \hat{v} > 0, \bar{A}\hat{v} = \bar{b}$ (MFCQ)
- full rank $\bar{A} = P_{\bar{m}}AV$: $P_{\bar{m}} : \mathbb{R}^m \to \mathbb{R}^{\bar{m}}$, $\bar{m} = \operatorname{rank}(AV) < m$. $P_{\bar{m}}$ is projection that chooses the linearly independent rows of AV.
- BOTH # variables, # constraints are strictly reduced.

This emphasizes the ILL-CONDITIONING of problems where strict feasibility fails, i.e., Implicit singularity is eliminated using FR.

Two-Step Facial Reduction; $Ax = b,\ x \geq 0$

Facial Reduction, FR

a journey to reformulate a problem until strict feasibility is met

Two-Step Facial Reduction; $Ax = b$, $x \geq 0$

Facial Reduction, FR

a journey to reformulate a problem until strict feasibility is met

Solve the auxiliary system:

$$\text{Find } y \in \mathbb{R}^m \text{ s.t. } A^T y \in \mathbb{R}^n_+ \setminus \{0\}, \quad \langle b, y \rangle = 0$$

Set $V = I(:, \operatorname{supp}(A^T y)^c)$

$x \leftarrow Vv$

$\mathcal{F} \leftarrow \{v \geq 0 : (AV)v = b\}$

Two-Step Facial Reduction; $Ax = b,\ x \geq 0$

Facial Reduction, FR

a journey to reformulate a problem until strict feasibility is met

[STEP 1]
Solve the auxiliary system:

$$\text{Find } y \in \mathbb{R}^m \text{ s.t. } A^T y \in \mathbb{R}^n_+ \setminus \{0\},\quad \langle b, y\rangle = 0$$

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[STEP 2]

Any nontrivial FR

$\Downarrow$

discovery of redundant equalities

Use $P_{\bar{m}}$ to discard redundancies

$\mathcal{F} \leftarrow \{v \geq 0 : P_{\bar{m}}AV(v) = P_{\bar{m}}b\}$

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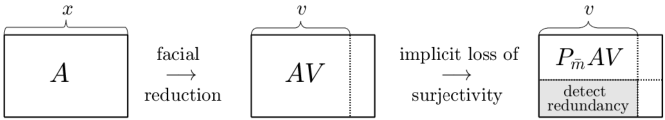

Example

Consider $\mathcal{F}$ with the data

$$A = \begin{bmatrix} 1 & 1 & 3 & 5 & 2 \\ 0 & 1 & 2 & -2 & 2 \end{bmatrix} \text{ and } b = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

$$\text{Set } y = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \implies A^T y = \begin{pmatrix} 1 & 0 & 1 & 7 & 0 \end{pmatrix}^T \geq 0 \text{ and } \langle b, y \rangle = 0.$$

$$V = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad x \leftarrow Vv = \begin{pmatrix} 0 \\ v_1 \\ 0 \\ 0 \\ v_2 \end{pmatrix}, \quad Ax = b \leftarrow AVv = b \equiv \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

(*) Side note

There are exactly six feasible bases in $\mathcal{F}$; (BFS all degenerate).

- $\mathcal{B} \in \{\{1,2\},\{2,3\},\{2,4\}\}$ is $x = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \end{pmatrix}^T$;
- $\mathcal{B} \in \{\{1,5\},\{3,5\},\{4,5\}\}$ is $x = \begin{pmatrix} 0 & 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}^T$.

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Detect Redundancy

Recall:

Lemma (AV is rank deficient)

Consider the facially reduced feasible set

$$\mathcal{F}_r = \left\{ v : AVv = b, v \in \mathbb{R}_+^{n-s_z} \right\}.$$

Then at least one linear equality of $AVv = b$ is redundant.

(proof) Let $z = A^T y$ be the exposing vector, V be a facial range vector induced by z. Then

$$0 = V^T z = V^T A^T y = (AV)^T y.$$

Found a nontrival row combination of AV, i.e., detected redundancy

Definition (implicit problem singularity)

The implicit problem singularity (**ips**) = The number of implicit redundant equalities of $\mathcal{F}$

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Singularity Degree $sd(\mathcal{F})$, Sturm '20 [37]

Definition ($d = sd(\mathcal{F}) = \min |FR\ steps|$)

Definition (Hölder regularity)

the pair of closed, convex subsets A, B is γ-*Hölder regular* if $\forall U$ compact, $\exists c > 0$ with:

$$\operatorname{dist}(x, A \cap B) \leq c \cdot \Big(\operatorname{dist}^{\gamma}(x, A) + \operatorname{dist}^{\gamma}(x, B) \Big) \qquad \text{for all } x \in U.$$

Sturm [37] error bound Theorem for SDP, $\mathcal{F} = \mathcal{L} \cap \mathbb{S}^n_+$

$(\mathcal{L}, \mathbb{S}^n_+)$ is $\frac{1}{2^d}$-Hölder regular. ($\mathcal{L}$ linear manifold)

- for LPs, FR in *one* iteration using maximal exposing vector, i.e., $d = \mathbf{sd}(\mathcal{F}) \leq 1$
- FR for LPs does not alter sparsity pattern of A. (only involves discarding columns of A; rows of A, b)

A Theoretical Result on degenerate BFS $\leftrightarrow$ MFCQ

Theorem

[a] *Suppose that strict feasibility of $\mathcal{F}$ fails. Then every basic feasible solution, BFS, $x \in \mathcal{F}$ with basis $\mathcal{B}$ has $\mathcal{B} \cap \mathcal{I}^= \neq \emptyset$ and thus is degenerate.*

[a] Contrapositive found in Bertsimas-Tsitsiklis book [4, Exer. 2.19].

Proof.

- $\mathcal{F} = \{x \in \mathbb{R}^n \ : \ AVv = b, \ v \in \mathbb{R}_+^{n-s_z}\}$, facial range vctr V
- wlog $V = \begin{bmatrix} I_r \\ 0 \end{bmatrix}$ and $r = n - s_z$;
- recall by redundant constraint lemma: $\operatorname{rank} AV < m$
- implies $\operatorname{rank} A(:, \{1, \ldots, r\}) < m$
- BFS implies $\operatorname{rank} A(:, \mathcal{B}) = m$; implies $\exists i \in \mathcal{B}, i > r$
- implies $\exists i \in \mathcal{B} \cap \mathcal{I}^=, x_i = 0$ (degeneracy) □

Corollary, Stability, Converse

Corollary (contrapositive motivates phase I part 2)

If there exists a nondegenerate basic feasible solution, then there exists a strictly feasible point in $\mathcal{F}$.

Stability from above corollary

Recall: strict feasibility (and full rank, MFCQ) is equivalent to stability wrt RHS perturbations.

Example (converse fails; all BFS degenerate $\not\Longrightarrow$ MFCQ fails)

$A = \begin{bmatrix} 1 & 0 & 2 & 0 & -2 \\ 1 & -3 & 2 & 1 & -2 \end{bmatrix}; b = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad 0 < x = \frac{1}{10}\begin{pmatrix} 1 & 1 & 5.5 & 3 & 1 \end{pmatrix}^T$

4 deg. feas. bases: $\mathcal{B} = \{\{1,2\},\{1,4\} : x = (1,0,0,0,0)^T$

$\mathcal{B} = \{2,3\},\{3,4\} : x = (0,0,1/2,0,0)^T$

(Also, the linear assignment problem is highly degenerate but has a strictly feasible point (average).)

Empirics for FR Preprocessing

We want to avoid implicit singularity

- improve conditioning, number of iterations

interior point methods

- Condition number of normal equation system
- stopping criteria

$$\text{KKT} = \left(\frac{\|Ax^* - b\|}{1 + \|b\|}, \frac{\|A^T y^* + s^* - c\|}{1 + \|c\|}, \frac{\langle x^*, s^* \rangle}{n} \right).$$

simplex methods (NETLIB data set)

- percentage of degenerate iterations

Interior Point Methods

Optimality Conditions at current $(x > 0, y, s > 0), \mu > 0$

$X = \mathrm{Diag}\,(x),\ S = \mathrm{Diag}\,(s)$.

$$\begin{array}{rcll} A^T\Delta y + \Delta s - c & = & 0 & \text{dual feasibility} \\ A\Delta x - b & = & 0 & \text{primal feasibility} \\ S\Delta x + X\Delta s & = & \mu e & \text{complementary slackness} \end{array}$$

After block elimination, solve normal equations for Δy

- Use Δs in eqn 1 to eliminate Δs in eqn 3.
- Solve for Δx in eqn 3 and eliminate it in eqn 2.
- We get the normal equations

$$AS^{-1}XA^T\Delta y = RHS.$$

- Backsolve for $\Delta x, \Delta s$ to get the Newton direction.

Numerical Experiments with Interior Point Methods

condition numbers of normal matrix; x^*, s^* near optimal

$$\kappa\left(AD^*A^T\right), \quad \text{where } D^* = \mathrm{Diag}\,(x^*)\mathrm{Diag}\,(s^*)^{-1} \tag{1}$$

three families of instances

1. $(\mathcal{P}_{(A,b,c)})$ do not have strictly feasible points;
2. $(\bar{\mathcal{P}}_{(A,\bar{b},c)})$ have strictly feasible points;
3. $(\mathcal{P}_{(A_{FR},b_{FR},c_{FR})})$ facially reduced instances of $(\mathcal{P}_{(A,b,c)})$.

Condition Numbers of Normal Matrix Near Optimum

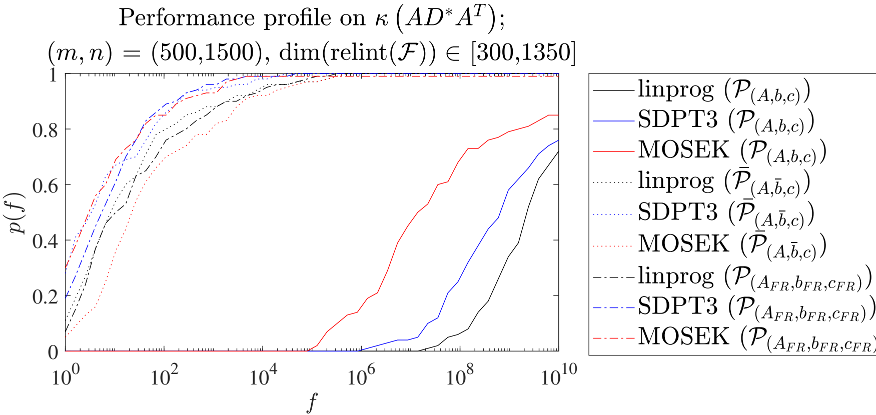

Figure: Performance profile on $\kappa\left(ADA^T\right)$ with(out) strict feasibility near optimum; various solvers

Empirics on Stopping Criteria

test the average performance of 10 instances of size $(n, m, r) = (3000, 500, 2000)$

$$\text{KKT} = \left(\frac{\|Ax^* - b\|}{1 + \|b\|}, \frac{\|A^T y^* + s^* - c\|}{1 + \|c\|}, \frac{\langle x^*, s^* \rangle}{n} \right)$$

		Non-Facially Reduced System	Facially Reduced System
linprog	KKT	(9.58e-16, 1.80e-12, 5.17e-09)	(5.78e-16, 1.51e-15, 5.57e-08)
	iter	23.30	17.60
	time	1.10	0.76
SDPT3	KKT	(1.51e-10, 1.49e-12, 4.67e-03)	(8.54e-12, 3.75e-16, 4.19e-06)
	iter	25.40	19.80
	time	0.82	0.53
MOSEK	KKT	(8.40e-09, 7.54e-16, -5.16e-06)	(5.16e-09, 3.81e-16, -2.03e-08)
	iter	35.90	10.10
	time	0.58	0.31

Table: Average of KKT conditions, iterations and time of (non)-facially reduced problems

Numerical Experiments with (Dual) Simplex Method

Empirics on the Number of Degenerate Iterations

- MOSEK (values in the table) reports percentage of degenerate iterations i.e„ ‘DEGITER(%)’ is ratio of degenerate iterations. (smaller value is better).
- $r = |\mathrm{supp}(s)|$; smaller value $(r/n)\%$ means entries of s are identically 0; 100% means strict feasibility holds.
- note significant decrease in ‘DEGITER(%)’.

		$(r/n)\%$				
		60%	70%	80%	90%	100%
(n, m)	(1000, 250)	36.62	10.18	0.01	0.02	0.00
	(2000, 500)	39.72	18.28	0.07	0.15	0.01
	(3000, 750)	25.99	10.66	0.32	0.75	0.02
	(4000, 1000)	29.78	18.25	0.25	0.53	0.02

Table: Average of ratio of degenerate iterations DEGITER(%)

Phase I(b): Towards Strict Feasibility

- $\bar{x}, \mathcal{B}$ degenerate BFS/basis; Wlog basic variables located first $\bar{x}$ as are degenerate variables. Solve (using basis from phase I simplex method)

$$p_1^* = \max\{x_1 \ : \ Ax = b, \ x \geq 0\}.$$

1. Suppose that $p_1^* > 0$. Then, the the variable x_1 is not an identically 0 variable, i.e., $1 \notin \mathcal{I}_0$.
2. Suppose that $p_1^* = 0$. Then, the variable x_1 is an identically 0 variable, i.e., $1 \in \mathcal{I}_0$. Let $\mathcal{B}^*$ be an optimal basis. Then we have an exposing vector

$$y^* = A(:, \mathcal{B}^*)^T e_1, \ \langle b, y^* \rangle = 0 \ \text{ and } A^T y^* \geq e_1.$$

- Add up certificates: $y^\circ = \sum_j y^j$ to get exposing vector

$$A^T y^\circ = \sum_j A^T y^j \geq 0, A^T y^\circ \neq 0, \langle b, y^\circ \rangle = \sum_j \langle b, y^j \rangle = 0.$$

Conclusion

- loss of strict feasibility has many applications recent survey Drusvyatskiy-W. [15].
- though not needed theoretically in LP, loss of MFCQ results in stability/numerical issues.
- In the paper we introduced new concept: Implicit Singularity Degree, maximum number of FR steps, and presented an algorithm, phase I (b), that regularizes an LP, for strict feasibility holding.

Regularized Nonsmooth Newton Algorithms for Best Approximation with Applications to Large Scale LP

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Main Problem/Best Approximation

Given $v \in \mathbb{R}^n$ and $P \subset \mathbb{R}^n$ a polyhedral set,
find the nearest point to v from the set P

Nonsmooth Algorithms

• Application of Moreau Decomposition/elegant equation
• present regularized nonsmooth method; singular Jacobian
• compare computational performance to classical projection methods (e.g., HLWB projection method)

Applications

solving large scale linear programs; triangles from branch and bound methods; generalized constrained linear least squares.

Notation

best approximation problem to polyhedral set $P \subset \mathbb{R}^n$

find the nearest point $x^* \in P$ to a given point $v \in \mathbb{R}^n$

uniquely attained optimum (projection of v onto P)

$$\text{optimum: } x^*(v) = \operatorname{argmin}_{x \in P} \frac{1}{2}\|x - v\|^2$$

optimal value: $p^*(v) = \frac{1}{2}\|x^*(v) - v\|^2$

Nonsmooth Newton Method

We apply a (regularized/scaled) nonsmooth Newton method to a special form of the optimality conditions based on a Moreau decomposition.

- The special Moreau decomposition for the optimality conditions comes from work in infinite dimensional Hilbert space e.g., [11, 12, 27, 9], where the projection is actually differentiable, and typically P is the intersection of a cone and a linear manifold of finite co-dimension (finite # constraints).
- parametrized quadratic problem to solve finite dimensional linear programs [35] applied in our work here below. (In this finite dimensional case differentiability was lost.)
- infinite dimensional applications appear in the theory of *partially finite programs* in [6, 7] Further references in [34, 22, 2].

Semismoothness

- differentiability is lost in finite dimensional; this led to application of semismoothness [28, 30, 29].
- More recently: applications for nearest Euclidean distance matrices and nearest doubly stochastic in [1, 20].
- The optimum $x^*(v)$ is often called the *projection onto the polyhedral set* and is known to be unique. Differentiability properties are nontrivial as discussed in e.g., [19]. A characterization of differentiability in terms of normal cones is given in [16]. Further results and connections to semismoothness is in e.g., [19, 18]. A survey presentation is at [33].

Projection onto a Polyhedral Set

$$
\text{(P)} \qquad
\begin{array}{rll}
x^*(v) := & \operatorname{argmin}_x & \frac{1}{2}\|x - v\|^2 \\
& \text{s.t.} & Ax = b \in \mathbb{R}^m \\
& & x \in \mathbb{R}^n_+,
\end{array}
$$

$$
\text{optimal value: } p^*(v) \quad = \quad \frac{1}{2}\|x^*(v) - v\|^2,
$$

Assumptions: A full row rank; feasible set nonempty

Optimality Conditions

Theorem ($F : \mathbb{R}^m \to \mathbb{R}^m$; find root y^*; Newton)

The optimum $x^*(v)$ *exists and is unique. Let*

$$(*) \quad \boxed{F(y) := A(v + A^T y)_+ - b}, \quad f(y) := \frac{1}{2}\|F(y)\|^2$$

Then $F(y) = 0$ *has a root* y^*, $F(y^*) = 0 \iff y \in \operatorname{argmin} f(y^*)$

$$x^*(v) = (v + A^T y^*)_+, \text{ for any root } F(y^*) = 0.$$

Moreover, strong duality holds and the dual problem is

$$\begin{aligned} p^*(v) &= d^*(v) \\ &:= \max_{z \geq 0, y} \phi(y, z) \quad (= \min_x L(x, y, z)) \\ &:= -\frac{1}{2}\left\|z - A^T y\right\|^2 + y^T(Av - b) - z^T v. \end{aligned}$$

AND

At each iteration, we get a provable/calculable lower bound

$$\max_{z \geq 0, y} \phi(y, z) = -\frac{1}{2}\left\|z - A^T y\right\|^2 + y^T(Av - b) - z^T v$$

Proof of Optimality Conditions

Proof.

$L(x, y, z) = \frac{1}{2}\|x - v\|^2 + y^T(b - Ax) - z^T x;$
$\nabla_x L(x, y, z) = x - v - A^T y - z;$
stationarity: $0 = \nabla_x L(x, y, z) \implies x = (v + A^T y) + z$
$\implies L(x, y, z) = -\frac{1}{2}\|z + A^T y\|^2 + y^T(b - Av) - z^T v.$

KKT optimality conditions

$$
\begin{array}{llcll}
\frac{\partial}{\partial x} L(x, y, z) & = & x - v - A^T y - z & = 0 & \text{(dual feasibility)} \\
\frac{\partial}{\partial y} L(x, y, z) & = & Ax - b & = 0 & \text{(primal feasibility)} \\
\frac{\partial}{\partial z} L(x, y, z) & \cong & x \in (\mathbb{R}^n_+ - z)^+ & & \text{(compl. slackness,} \\
& & & & z^T x = 0 \text{ or} \\
& & & & z \circ x = 0)
\end{array}
$$

□

Proof continued...

(cont... Solve opt. cond.

$$\begin{bmatrix} x - v - A^T y - z \\ Ax - b \\ z^T x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad x, z \in \mathbb{R}^n_+, y \in \mathbb{R}^m.$$

Moreau Decomposition:

$$v + A^T y = x - z \quad = x + (-z), x^T z = 0$$

$$x = (v + A^T y)_+; \; z = -(v + A^T y)_-$$

$$F : \mathbb{R}^m \to \mathbb{R}^m; \quad \boxed{F(y) = A(v + A^T y)_+ - b = 0, \; y \in \mathbb{R}^m}$$

□

Apply Newton at current y_c; Newton direction Δy

$$F'(y_c)\Delta y = -F(y_c); \qquad y_p = y_c + \Delta y$$

Compare Interior Point Methods

Block Elimination on Perturbed KKT Conditions

$$\begin{bmatrix} r_d \\ r_p \\ r_c \end{bmatrix} := \begin{bmatrix} x - v - A^T y - z \\ Ax - b \\ Zx - \mu e \end{bmatrix}, \quad x, z \in \mathbb{R}^n_+, y \in \mathbb{R}^m.$$

$$F'_\mu \Delta s = \begin{bmatrix} \underline{\Delta x} - A^T \Delta y - \Delta z \\ A\Delta x - b \\ X\Delta z + Z\Delta x \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{bmatrix} = - \begin{bmatrix} r_d \\ r_p \\ r_c \end{bmatrix}, \quad x, z \in \mathbb{R}^n_+, y \in \mathbb{R}^m.$$

Normal Equations Reduction to Δy

Currently, normal equations are not considered efficient. But the Newton equation was a percursor and appears to be efficient?

$$F : \mathbb{R}^m \to \mathbb{R}^m; \qquad \boxed{F(y) = A(v + A^T y)_+ - b = 0, \ \ y \in \mathbb{R}^m}$$

$$F'(y_c)\Delta y = -F(y_c); \qquad y_p = y_c + \Delta y$$

minimize squared residual $f(y) = \frac{1}{2}\|F(y)\|^2$

differentiable case $\{i : (v + A^T y)_i = 0\} = \emptyset$:
$\nabla f(y) = (F'(y))^* F(y)$

Definition ((local) Lipschitz Continuity)

Let $\Omega \subseteq \mathbb{R}^n$. A function $F : \Omega \to \mathbb{R}^n$ is *Lipschitz continuous* on Ω if there exists $K > 0$ such that

$$|F(y) - F(z)| \leq K\|y - z\|, \; \forall y, z \in \Omega.$$

F is *locally Lipschitz continuous* on Ω if for each $x \in \Omega$ there exists a neighbourhood U of x such that F is Lipschitz continuous on U.

Generalized Jacobian

Rademacher's Theorem [31, 17]

$F : \Omega \to \mathbb{R}^n$ locally Lipschitz on Ω implies that it is Frechét differentiable almost everywhere on Ω.

Definition (Clarke [13] Generalized Jacobian)

Suppose that $F : \mathbb{R}^m \to \mathbb{R}^m$ be locally Lipschitz. Let D_F be the set of points such that F is differentiable. Let $F'(y)$ be the usual Jacobian matrix at $y \in D_F$. The *generalized Jacobian of F at y*, $\partial F(y)$ is

$$\partial F(y) = \text{conv}\left\{ \lim_{\substack{y_i \to y \\ y_i \in D_F}} F'(y_i) \right\}.$$

In addition, $\partial F(y)$ is nonsingular if every $V \in \partial F(y)$ is nonsingular.

Case: Differentiable and $F'(y)$ invertible

Newton Direction; Newton Equation

$$(F'(y))^*(F'(y))\Delta y = -(F'(y))^* F(y) \iff F'(y)\Delta y = -F(y).$$

$$\Delta y = -\left((F'(y))^*(F'(y))\right)^{-1}(F'(y))^* F(y) = -(F'(y))^\dagger F(y)$$

directional derivative: $\Delta y^T \nabla f(y) = \dots$

$$-\left[(F'(y))^* F(y)\right]^T \boxed{\left((F'(y))^*(F'(y))\right)^{-1}} \left[(F'(y))^* F(y)\right]$$
$$< 0$$

We now see that we maintain a descent direction.

Lemma (for handling singularity in $(F'(y))^*(F'(y))$)

LM *direction is always a descent direction.*

Proof.

$(J \cong F'(y))$

$$(J^*J + \lambda I)\Delta y = -J^*F.$$

$$\Delta y = -\left(J^TJ + \lambda I\right)^{-1}(J^TF).$$

Therefore, the directional derivative is

$$\begin{aligned}\Delta y^T \nabla f(y) &= -\left(\left(J^TJ + \lambda I\right)^{-1}(J^TF)\right)^T (J^TF) \\ &= -(J^TF)^T\left(\left(J^TJ + \lambda I\right)^{-1}\right)(J^TF) \\ &< 0.\end{aligned}$$

□

Max. Rank Generalized Jacobian

Cols chosen $\cong$ pos. variables of w

$$Aw_+ = A(\mathcal{P}_{\mathcal{N}} w) = (A\mathcal{P}_{\mathcal{N}})w_+ = \sum_{w_i > 0} A(:, i) w_i$$

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Note: $v + A^T y \geq 0 \implies F'(\Delta y) = AIA^T \Delta y = AA^T \Delta y$

$$\mathcal{U}(y) := \left\{ u \in \mathbb{R}^n \mid u_i \in \begin{cases} 1 & \text{if } (v + A^T y)_i > 0 \\ [0, 1] & \text{if } (v + A^T y)_i = 0 \\ 0 & \text{if } (v + A^T y)_i < 0 \end{cases} \right\}$$

generalized Jacobian at y; after convex hull

$\partial F(y) = \{A \operatorname{Diag}(u) A^T \mid u \in \mathcal{U}(y)\}$
(max-rank: choose $u_i = 1$ when possible)

Semismooth Newton Method solving $F(y) = 0$

Solve $(V_k + \lambda I)d_{Newton} = -F(y^k)$, with
$V_k \in \partial F(y^k), \lambda > 0, c \in (0, 1)$

$y^{k+1} = y^k + d_{Newton}; \ \left(\text{or avging } y^{k+1} = (1-c)y^k + +cd_{Newton}\right)$

Max-rank Jacobian

$$
\begin{aligned}
AMA^T &:= A\mathrm{Diag}(u)A^T \\
&= \sum_{i\in\mathcal{I}_+} A_{:i}A_{:i}^T + \sum_{i\in\mathcal{I}_0} \alpha_i A_{:i}A_{:i}^T, \ \alpha_i \in [0,1], \forall i \in \mathcal{I}_0
\end{aligned}
$$

maximum (resp. minimum) rank for AMA:
$\alpha_i = 1, \forall i \in \mathcal{I}_0$ ($\alpha_i = 0, \forall i \in \mathcal{I}_0$, resp.)

Vertices and Polar Cones

Choosing the optima for the tests; (nondegenerate) vertex

In our tests we can decide on the characteristics of the optimal solution using the properties of (degenerate) vertices.
Recall: x optimal iff $x - v \in \mathcal{F}(x)^+$

Lemma (vertex and polar cone)

$y \in \mathbb{R}^m, x(y) = (v + A^T y)_+ \in \mathcal{F}$. Then:
$x(y)$ vertex $\iff A_{\mathcal{I}_+}$ nonsingular
$\iff$ corresp. gen. Jac. nonsingular.
$x = x(y) \in \mathcal{F} \implies$
$\mathcal{F}(x)^+ = \{w : w = A^T u + z, u \in \mathbb{R}^m, z \in \mathbb{R}^n_+, x^T z = 0\}$

Proof of Lemma

Proof.

wlog $A = [A_{\mathcal{I}_+} \ A_{\mathcal{I}_0}]$ implies active set is $\begin{bmatrix} A_{\mathcal{I}_+} & A_{\mathcal{I}_0} \\ 0 & I \end{bmatrix} x = \begin{pmatrix} b \\ 0 \end{pmatrix}$;
This has unique solution $x(y)$ iff $A_{\mathcal{I}_+}$ is nonsingular.
gradient of objective satisfies

$$x - v = A^T y + \sum_{j \in \mathcal{I}_0} z_j e_j.$$

Optimality conditions yield polar cone at a vertex. □

degeneracy of optimal solutions

Let $x \in \operatorname{bdry} \mathcal{F}$;
x is optimal iff $x - v \in \mathcal{F}(x)^+$, i.e., we can choose v with
$v = x - A^T u + z$, $z \geq 0$, $z^T x = 0$.
and
$x^*(v)$ is differentiable at $v \iff (x^*(v) - v) \in \operatorname{ri}(\mathcal{F} - x^*(v))^+$

Algorithm 1 Best Approx. of v in P; Exact Newton

Require: $v \in \mathbb{R}^n, y_0 \in \mathbb{R}^m$, $(A \in \mathbb{R}^{m\times n}, \operatorname{rank}(A) = m)$, $\varepsilon > 0$, maxiter

Output. Primal-dual opt: $x_{k+1}, (y_{k+1}, z_{k+1})$

Initialization. $k \leftarrow 0$, $x_0 \leftarrow (v + A^T y_0)_+$, $z_0 \leftarrow (x_0 - (v + A^T y_0))_+$,
$F_0 = Ax_0 - b$, stopcrit $\leftarrow \|F_0\|/(1 + \|b\|)$

while ((stopcrit $> \varepsilon$) & ($k \le$ maxiter)) **do**

  $\lambda = \min(1e^{-3}, \text{ stopcrit})$

  $\bar{V} = (V_k + \lambda I_m)$

  solve pos. def. $\bar{V} d = -F_k$ for Newton direction d

  updates

  $y_{k+1} \leftarrow y_k + d$

  $x_{k+1} \leftarrow (v + A^T y_{k+1})_+$

  $z_{k+1} \leftarrow (x_{k+1} - (v + A^T y_k))_+$

  $F_{k+1} \leftarrow Ax_{k+1} - b$ (residual)

  stopcrit $\leftarrow \|F_{k+1}\|/(1 + \|b\|)$

  $k \leftarrow k + 1$

end while

Halpern-Lions-Wittmann-Bauschke [3]

Halpern-Lions-Wittmann-Bauschke [3] (HLWB)

Algorithm 2 Extended HLWB algorithm

Require: $v \in \mathbb{R}^n$, ($A \in \mathbb{R}^{m \times n}$, $\mathrm{rank}\,(A) = m$), $\varepsilon > 0$, maxiter $\in \mathcal{N}$.

Output. x_{k+1}
Initialization. $k \leftarrow 0$, $msweeps \leftarrow 0$ $x_0 \leftarrow max(v, 0)$, $y_0 \leftarrow x_0$, $i_0 = 1$
  stopcrit $\leftarrow \|Ay_0 - b\|/(1 + \|b\|)$ ($= \|F_0\|/(1 + \|b\|)$)
while ((stopcrit $> \varepsilon$) & ($k \leq$ maxiter)) **do**
  if $1 \leq i(k) \leq m$ **then**
    $y_k = x_k + \frac{b_{i_k} - \langle a_{i_k}, x^k \rangle}{\|a_{i_k}\|^2} a_{i_k}$
  else
    $y_k = \max(0, x_k)$
  end if
  updates
  $\sigma_k = \frac{1}{k+1}$ (change to $\sigma_k = \frac{1}{msweeps+1}$??)
  $x^{k+1} \leftarrow \sigma_k v + (1 - \sigma_k) y^k$
  stopcrit $\leftarrow \|Ay_0 - b\|/(1 + \|b\|)$
  $k \leftarrow k + 1$
  if $k \mod (m+1) == 0$ **then**
    $msweeps = msweeps + 1$
  end if
  $i_k = k(\mathrm{mod}\ m) + 1$
end while

Numerical Tests BAP; varying sizes m, n

Table: $n = 3000$, % density=.81; varying $m = 100, 600, 1100, 1600$

Time (s)					Rel. Resids.				
Exact	Inexact	HLWB	LSQ	QPPAL	Exact	Inexact	HLWB	LSQ	QPPAL
2.13e-03	1.98e-02	1.89e+01	3.22e+00	8.04e-01	2.55e-16	2.41e-15	2.29e-04	4.12e-17	-8.43e-16
8.35e-02	3.03e-01	1.94e+02	4.28e+00	1.27e+00	5.10e-16	5.10e-18	2.19e-04	5.12e-17	-1.53e-16
7.02e-01	1.29e+00	4.16e+02	6.18e+00	2.53e+00	5.20e-16	8.71e-16	2.08e-04	3.80e-17	9.05e-17
1.40e+00	3.59e+00	6.57e+02	7.65e+00	5.13e+00	9.84e-18	1.11e-15	2.27e-04	3.82e-17	-8.61e-17

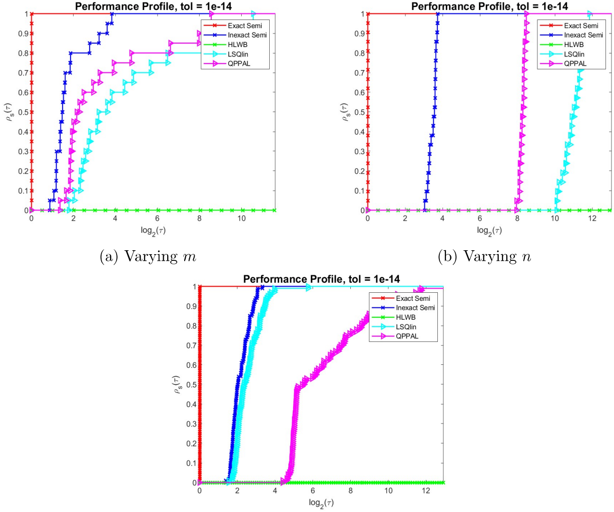
Table: $m = 200$, % density=.81, varying $n = 3000, 3500, 4000, 4500, 5000$

Time (s)					Rel. Resids.				
Exact	Inexact	HLWB	LSQ	QPPAL	Exact	Inexact	HLWB	LSQ	QPPAL
3.12e-03	3.69e-02	4.45e+01	3.50e+00	8.66e-01	8.64e-18	7.39e-17	2.56e-04	6.52e-16	5.89e-17
3.08e-03	4.05e-02	5.17e+01	4.93e+00	1.00e+00	9.07e-18	1.26e-17	2.78e-04	1.23e-15	2.15e-17
3.24e-03	3.70e-02	5.82e+01	7.31e+00	1.09e+00	1.46e-16	8.91e-16	2.80e-04	3.21e-16	-9.18e-18
3.99e-03	4.17e-02	6.58e+01	1.01e+01	1.18e+00	1.80e-15	2.05e-16	3.13e-04	4.61e-17	1.71e-16
3.93e-03	3.42e-02	7.30e+01	1.45e+01	1.26e+00	4.09e-17	1.80e-15	3.16e-04	5.27e-17	-6.28e-17

Numerical Tests BAP varying density

Table: $m = 300$, $n = 1000$, Varying % density=1, 6, 1.1, 1.6

Time (s)					Rel. Resids.				
Exact	Inexact	HLWB	LSQ	QPPAL	Exact	Inexact	HLWB	LSQ	QPPAL
5.65e-03	5.69e-02	1.67e+01	3.02e-01	5.32e-01	7.48e-16	7.27e-16	1.54e-04	3.33e-17	-8.29e-17
4.80e-02	2.52e-01	4.58e+01	3.15e-01	1.22e+00	3.44e-17	1.18e-16	1.51e-04	2.04e-15	-1.43e-17
6.18e-02	2.49e-01	5.41e+01	3.07e-01	2.10e+00	5.65e-17	1.54e-17	1.44e-04	1.09e-16	1.09e-16
7.79e-02	2.60e-01	5.34e+01	3.03e-01	2.11e+01	6.92e-17	7.98e-17	1.61e-04	4.19e-16	-2.88e-16



(a) Varying m

(b) Varying n

(c) Varying density

Figure 5.1: Performance profiles for problems with varying m, n, and densities for nondegenerate vertex solutions

Applications: Solving (maximization) Large Scale LP

primal (maximization) LP in standard form

$$
\text{(PLP)} \qquad \begin{array}{rll} p^*_{LP} := & \max & c^T x \\ & \text{s.t.} & Ax = b \in \mathbb{R}^m \\ & & x \in \mathbb{R}^n_+. \end{array}
$$

dual LP

$$
\text{(DLP)} \qquad \begin{array}{rll} d^*_{LP} := & \min & b^T y \\ & \text{s.t.} & A^T y - z = c \in \mathbb{R}^n \\ & & z \in \mathbb{R}^n_+. \end{array} \tag{2}
$$

Assumptions: full rank; finite optimal value

A full row rank;

$p^*_{\mathrm{LP}} \in \mathbb{R}$ (so $p^*_{\mathrm{LP}} = d^*_{\mathrm{LP}} \in \mathbb{R}$ and both attained)

Geometric Algorithm

solution can be found from the limit as $R \uparrow \infty$ of the projection of the vector $v_R = Rc \in \mathbb{R}^n$ onto the feasible set.

Lemma ([23, 24, 25, 35])

Let the given LP data be A, b, c with finite optimal value p^*_{LP}. For each $R > 0$ define

$$\begin{array}{rccl} x(R) := & \operatorname{argmin}_x & \frac{1}{2}\|x - Rc\|^2 \\ & s.t. & Ax = b \in \mathbb{R}^m \\ & & x \in \mathbb{R}^n_+. \end{array}$$

Then x^* is the minimum norm solution of (PLP) if, and only if, there exists $\bar{R} > 0$ such that

$$R \geq \bar{R} \implies x^* \in \operatorname{argmin}\left\{\frac{1}{2}\|x - Rc\|^2 \,:\, Ax = b,\, x \in \mathbb{R}^n_+\right\}.$$

□

We use the estimate $R = \min\left\{50, \frac{\sqrt{mn}\|b\|}{1+\|c\|}\right\}$

Avoid numerical/roundoff from large numbers

Corollary (scaling $\frac{1}{R}b$)

$A, b, c, R, x(R)$ as in Lemma. Then

$$\frac{1}{R}x(R) = w(R) := \begin{array}{cl} \operatorname{argmin}_w & \frac{1}{2}\|w - c\|^2 \\ s.t. & Aw = \frac{1}{R}b \in \mathbb{R}^m \\ & w \in \mathbb{R}^n_+. \end{array}$$

Proof.

From

$$\|x - Rc\|^2 = R^2\left\|\frac{1}{R}x - c\right\|^2 = R^2\|w - c\|^2, \; x = Rw,$$

we substitute for x and obtain $A(Rw) = b \iff Aw = \frac{1}{R}b$. The result follows from the observation that argmin does not change after discarding the constant R^2. □

Warm Start; Stepping Stone External Path Following

consider scaled problem with:

$$x(R) = Rw(R).$$

Recall the optimality conditions for $w = w(R)$:

$$\begin{bmatrix} w - c - A^T y - z \\ Aw - \frac{1}{R} b \\ z^T w \end{bmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad w, z \in \mathbb{R}^n_+, y \in \mathbb{R}^m.$$

We conclude that

$$\lim_{R\to\infty} \mathcal{P}_{\text{Range}(A^T)} w(R) = 0, \ \lim_{R\to\infty} Rw(R) = x^*, \ \text{the optimum of the LP.}$$

The optimality conditions are now

$$w = c + A^T y + z, \ b = ARw = AR(c + A^T y)_+, \quad w^T z = 0, x, z \geq 0.$$

Warm Start from current R: find new R_n, y_n

Theorem

Suppose triple (w, y, z) optimal for scaled problem; let

$$\mathcal{N} = \mathcal{N}(z) = \{i \,:\, z_i > 0\}, \quad \mathcal{B} = \mathcal{B}(w) = \{1 : n\} \backslash \mathcal{N};$$

$$b_{\mathcal{B}} = A_{\mathcal{B}}^T \left(A_{\mathcal{B}} A_{\mathcal{B}}^T\right)^{\dagger} b, \quad b_{\mathcal{N}} = A_{\mathcal{N}}^T \left(A_{\mathcal{B}} A_{\mathcal{B}}^T\right)^{\dagger} b;$$

$$e = \begin{pmatrix} (b_{\mathcal{B}} - R w_{\mathcal{B}}) \\ -(b_{\mathcal{N}} + R z_{\mathcal{N}}) \end{pmatrix}, \quad f = \begin{pmatrix} R b_{\mathcal{B}} \\ -R b_{\mathcal{N}} \end{pmatrix}.$$

Then max. value R without changing basis is

$$R_n = \min\{f_i/e_i : e_i > 0, f_i > 0,\ i = 1, \ldots |\mathcal{B}|\}.$$

Moreover, $R_n = \infty$ implies optimal solution found.

$R_n < \infty \implies$ corresponding changes are:

$$\Delta y_p = \left(A_{\mathcal{B}} A_{\mathcal{B}}^T\right)^{\dagger} b; \ \Delta y = \left(\frac{R - R_n}{R R_n}\right) \Delta y_p.$$

$$\Delta w_{\mathcal{B}} = A_{\mathcal{B}}^T \left(\frac{R - R_n}{R R_n}\right) \Delta y_p$$

$$\Delta z_N = -A_{\mathcal{N}}^T \left(\frac{R - R_n}{R R_n}\right) \Delta y_p$$

LP Numerical Tests

Specifications			Time (s)						Rel. Resids.					
m	n	% density	RNNM	Linprog DS	Linprog IPM	MOSEK DS	MOSEK IPM	SNIPAL	RNNM	Linprog DS	Linprog IPM	MOSEK DS	MOSEK IPM	SNIPAL
2e+03	5e+03	1.0e-01	7.93e-02	3.59e-02	4.74e-02	1.32e-01	1.65e-01	4.47e-01	3.38e-17	3.38e-17	4.88e-09	1.31e-16	1.53e-16	3.63e-03
2e+03	1e+04	1.0e-01	9.84e-02	4.98e-02	7.64e-02	1.52e-01	1.93e-01	5.86e-01	2.82e-17	2.82e-17	1.60e-04	1.31e-16	2.89e-16	3.30e-03
2e+03	1e+05	1.0e-01	1.69e-01	4.00e-01	7.56e-01	5.37e-01	6.45e-01	2.51e+00	1.48e-17	1.48e-17	1.72e-05	8.84e-17	3.68e-16	1.54e-03
5e+03	1e+04	1.0e-01	9.72e+01	2.09e-01	1.41e+01	4.29e-01	2.67e+00	5.75e+00	5.55e-17	5.55e-17	5.02e-07	1.67e-14	3.20e-16	3.94e-03
5e+03	1e+05	1.0e-01	7.76e+01	7.34e-01	1.43e+02	1.09e+00	7.95e+00	1.36e+01	2.36e-17	2.36e-17	6.38e-05	3.13e-16	3.91e-15	1.82e-03
5e+03	5e+05	1.0e-01	2.31e+02	7.04e+00	6.56e+02	7.02e+00	1.56e+01	3.01e+01	1.52e-17	1.52e-17	3.73e-05	3.92e-16	3.56e-16	1.13e-03
2e+04	1e+05	1.0e-02	6.16e-01	9.55e-01	5.73e+00	1.06e+00	2.51e+00	4.50e+00	1.36e-17	1.36e-17	4.33e-07	1.99e-06	1.28e-16	3.58e-03
2e+04	5e+05	1.0e-02	6.68e-01	4.50e+00	3.77e+01	5.68e+00	9.29e+00	1.69e+01	8.48e-18	8.48e-18	8.83e-07	1.36e-06	1.15e-15	1.47e-03
2e+04	1e+06	1.0e-02	1.52e+00	9.37e+00	6.52e+01	1.18e+01	1.58e+01	2.99e+01	7.08e-18	7.08e-18	6.27e-06	1.76e-06	4.20e-16	1.20e-03
1e+05	1e+07	1.0e-03	5.76e+00	1.14e+01	6.24e+00	9.47e+01	9.72e+01	2.33e+02	1.39e-18	1.39e-18	1.39e-18	1.76e-17	1.76e-17	1.04e-03

Table 5.4: LP application results averaged on 5 randomly generated problems per row

Performance Profile LP

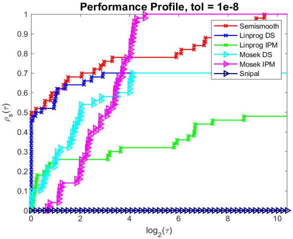


Figure 5.2: Performance Profiles for LP application wrt all problems

Conclusion for BAP and LP Algorithm

- efficient, robust algorithm for projection of a point onto a polyhedral set.
- One of may applications is to solving large scale LPs; we get a finite converging stepping stone exterior path following algorithm (mixture of simplex/interior-point)

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Thanks for your attention!

Regularized Nonsmooth Newton Algorithms for Best Approximation with Applications to Large Scale LP

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at: School of Mathematical and Statistical Sciences, Clemson® University

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