Control of Systems Governed by Partial Differential Equations

Kirsten Morris
Dept. of Applied Mathematics
Faculty of Mathematics
University of Waterloo
Waterloo, CANADA

1 Introduction

In many applications, such as diffusion and structural vibrations, the physical quantity of interest depends on both position and time. Some examples are shown in Figures 1-3. These systems are modelled by partial differential equations (PDE's) and the solution evolves on an infinite-dimensional Hilbert space. For this reason, these systems are often called infinite-dimensional systems. In contrast, the state of a system modelled by an ordinary differential equation evolves on a finite-dimensional system, such as $\mathbb{R}^n$, and these systems are called finite-dimensional. Since the solution of the PDE reflects the distribution in space of a physical quantity such as the temperature of a rod or the deflection of a beam, these systems are often also called distributed-parameter systems (DPS). Systems modelled by delay differential equations also have a solution that evolves on an infinite-dimensional space. Thus, although the physical situations are quite different, the theory and controller design approach is quite similar to that of systems modelled by partial differential equations. However, delay differential equations will not be discussed directly in this article.

The purpose of controller design for infinite-dimensional systems is similar to that for finite-dimensional systems. Every controlled system must of course be stable. Beyond that, the goals are to improve the response in some well-defined manner, such as by solving a linear-quadratic optimal control problem. Another common goal is design the controller to minimize the system's response to disturbances.

Classical controller design is based on an input/output description of the system, usually through the transfer function. Infinite-dimensional systems have transfer functions. However, unlike the transfer functions of finite-dimensional systems, the transfer function is not a rational function. If a closed form expression of the transfer function of an infinite-dimensional system can be obtained, it may be possible to design a controller directly. This is known as the *direct controller design* approach. Generalizations of many well-known finite-dimensional stability results such as the small gain theorem and the Nyquist stability criterion exist, see [24, 25]. Passivity generalizes in a straightforward way to irrational transfer functions [9, Chapters V, VI] and just as for finite-dimensional systems, any positive real system can be stabilized by the static output feedback $u = -\kappa y$ for any $\kappa > 0$. For some of these results it is not required to know the transfer function in order to ensure stability of the controlled system. It is only required to know whether the transfer function lies in the appropriate class. PI-control solutions to tracking problems for irrational transfer functions and the internal model principle is covered in [25, 26, 35]. For a high-performance controlled system, a model of the system needs to be used. H_∞-controller design has been successfully developed as a method for robust control and disturbance rejection in finite-dimensional systems. A theory of robust H_∞-control designs for infinite-dimensional systems using transfer functions is described in [11]. More recent results on this approach can be found in [19] and references therein.

The chief drawback of direct controller design is that an explicit representation of the transfer function is required. Another drawback of direct controller design is that, in general, the resulting controller is infinite-dimensional and must be approximated by a finite-dimensional system. For this reason, direct controller design is sometimes referred to as *late lumping* since a last step in the controller design is to approximate the controller by a finite-dimensional, or lumped parameter, system.

For many practical examples, controller design based on the transfer function is not feasible, since a closed-form expression for the transfer function may not be available. Instead, a finite-dimensional approximation of the system is first obtained and controller design is based on this finite-dimensional approximation. This approach is known as *indirect controller design*, or *early lumping*. This is the most common method of controller design for systems

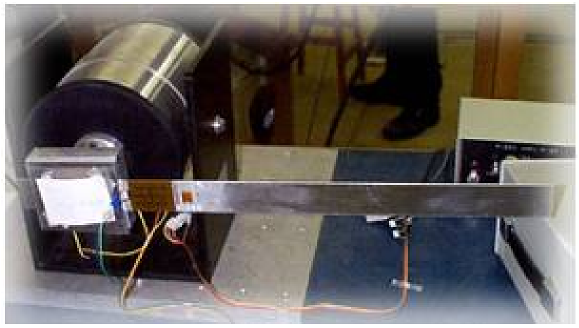


Figure 1: A flexible beam is the simplest example of transverse vibrations in a structure. It has relevance to control of flexible robots and space structures. This photograph shows a beam controlled by means of a motor at one end. (Photo by courtesy of Prof. M.F. Golnaraghi, Simon Fraser University.)

modeled by partial differential equations. The hope is that the controller has the desired effect on the original system. That this method is not always sucessful was first documented in Balas [1], where the term *spillover effect* was coined. Spillover refers to the phenomenon that a controller which stabilizes a reduced-order model need not necessarily stabilize the original model. Systems with infinitely many poles either on or asymptoting to the imaginary axis are notorious candidates for spillover effects. However, conditions under which this practical approach to controller design works have been obtained and are presented in this article.

In the next section a brief overview of the state-space theory for infinite-dimensional systems is given. Some issues associated with approximation of systems for the purpose of controller design are discussed in the following section. Results for the most popular methods for multi-input-multi-output controller design, linear-quadratic controller and H_∞-controller design, are then presented in sections 4 and 5.

2 State-space Formulation

Systems modelled by linear ordinary differential equations are generally written as a set of n first-order differential equations putting the system into the state-space form

$$\dot{z}(t) = Az(t) + Bu(t), \tag{1}$$

Figure 2: Acoustic noise in a duct. A noise signal is produced by a loudspeaker placed at one end of the duct. In this photo a loudspeaker is mounted midway down the duct where it is used to control the noise signal. The pressure at the open end is measured by means of a microphone as shown in the photo. (Photo by courtesy of Prof. S. Lipshitz, University of Waterloo.)

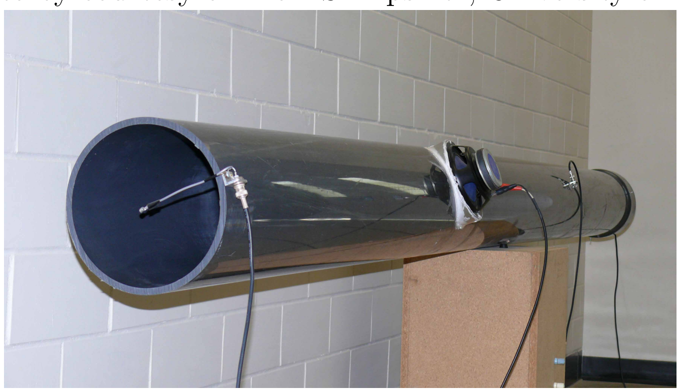
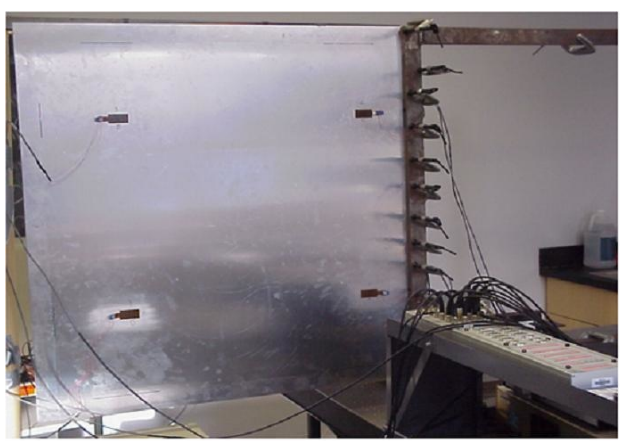


Figure 3: Vibrations in a plate occur due to various disturbances. In this apparatus the vibrations are controlled via the piezo-electric patches shown. (Photo by courtesy of Prof. M. Demetriou, Worcester Polytechnic University.)



where $z(t) \in \mathbb{R}^n$, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and m is the number of controls.

We write systems modelled by partial differential equations in a similar way. The main difference is that the matrices A becomes an operator acting, not on $\mathbb{R}^n$, but on an infinite-dimensional Hilbert space, $\mathcal{Z}$. Similarly, B maps the input space into the Hilbert space $\mathcal{Z}$. More detail on the systems theory described briefly in this section can be found in [8].

We first need to generalize the idea of a matrix exponential $\exp(At)$. Let $\mathcal{L}(\mathcal{X}_1, \mathcal{X}_2)$ indicate bounded linear operators from a Hilbert space $\mathcal{X}_1$ to a Hilbert space $\mathcal{X}_2$.

Definition 2.1 *A strongly continuous (C_0-) semigroup $S(t)$ on Hilbert space $\mathcal{Z}$ is a family $S(t) \in \mathcal{L}(\mathcal{Z}, \mathcal{Z})$ such that*

1. $S(0) = I$,
2. $S(t)S(s) = S(t+s)$,
3. $\lim_{t\downarrow 0} S(t)z = z$, *for all* $z \in \mathcal{Z}$.

Definition 2.2 *The infinitesimal generator A of a C_0-semigroup on $\mathcal{Z}$ is defined by*

$$Az = \lim_{t\downarrow 0} \frac{1}{t}(S(t)z - z)$$

with $D(A)$ the set of elements $z \in \mathcal{Z}$ for which the limit exists.

The matrix exponential $\exp(At)$ is a special case of a semigroup, defined on a finite-dimensional space. Its generator is the matrix A. Note that we only have strong convergence of $S(t)$ to the identity I in Definition 2.1(3). Uniform convergence implies that the generator is a bounded operator defined on the whole space and that the semigroup can be defined as

$$\sum_{i=0}^{\infty} \frac{(At)^i}{i!}$$

just as for a matrix exponential. However, for partial differential equations the generator A is an unbounded operator and only strong convergence of $S(t)$ to I is obtained.

If A is the generator of a C_0-semigroup $S(t)$ on a Hilbert space $\mathcal{Z}$, then for all $z_0 \in D(A)$,

$$\frac{d}{dt}S(t)z_0 = AS(t)z_0 = S(t)Az_0.$$

Figure 4: Heat Flow in a Rod. The regulation of the temperature profile of a rod is the simplest example of a control system modelled by a partial differential equation.

It follows that the differential equation on $\mathcal{Z}$

$$\frac{dz(t)}{dt} = Az(t), \qquad z(0) = z_0$$

has the solution

$$z(t) = S(t)z_0.$$

Furthermore, due to the properties of a semigroup, this solution is unique, and depends continuously on the initial data z_0.

Thus, for infinite-dimensional systems, instead of (1), we consider systems described by

$$\frac{dz}{dt} = Az(t) + Bu(t), \qquad z(0) = z_0 \tag{2}$$

where A with domain $D(A)$ generates a strongly continuous semigroup $S(t)$ on on a Hilbert space $\mathcal{Z}$ and $B \in \mathcal{L}(\mathcal{U}, \mathcal{Z})$. We will assume also that $\mathcal{U}$ is finite-dimensional (for instance, $\mathbb{R}^m$) as is generally the case in practice.

For many situations, such as control on the boundary of the region, typical models lead to a state-space representation where the control operator B is unbounded on the state-space. More precisely, it is a bounded operator into a larger space than the state space. However, this complicates the analysis considerably. To simplify the exposition, this paper will consider only bounded B. Appropriate references will be given for extension to unbounded operators where available. Note however, that including a model for the actuator often changes a simple model with an unbounded actuator or sensor to a more complex model with bounded control or sensing; see for instance [17, 43].

Example 2.3 *Diffusion.*

Consider the temperature in a rod of length L with constant thermal conductivity K_0, mass density ρ and specific heat C_p. (See Figure 4.) Applying the principle of conservation of energy to arbitrarily small volumes in the bar leads to the following partial-differential equation for the temperature $z(x,t)$ at time t at position x from the left-hand end [14, e.g. sect. 1.3]

$$C_p \rho \frac{\partial z(x,t)}{\partial t} = K_0 \frac{\partial^2 z(x,t)}{\partial x^2}, \qquad x \in (0,L), \qquad t \geq 0. \tag{3}$$

In addition to modeling heat flow, this equation also models other types of diffusion, such as chemical diffusion and neutron flux. To fully determine the temperature, one needs to specify the initial temperature profile $z(x,0)$ as well as the boundary conditions at each end. Assume Dirichlet boundary conditions:

$$z(0,t) = 0, \quad z(L,t) = 0.$$

and some initial temperature distribution $z(0) = z_0$, $z_0 \in \mathcal{L}_2(0,L)$. Define $Az = \partial^2 z/\partial x^2$. Since we can't take derivatives of all elements of $\mathcal{L}_2(0,L)$ and we need to consider boundary conditions, define

$$D(A) = \{z \in \mathcal{H}^2(0,L);\ z(0) = 0, z(L) = 0\}$$

where $\mathcal{H}^2(0,L)$ indicates the Sobolev space of functions with weak second derivatives [37]. We can rewrite the problem as

$$\dot{z}(t) = Az(t), \qquad z(0,t) = z_0.$$

The operator A generates a strongly continuous semigroup $S(t)$ on $\mathcal{Z} = \mathcal{L}_2(0,L)$. The state z, the temperature of the rod, evolves on the infinite-dimensional space $\mathcal{L}_2(0,L)$.

Suppose that the temperature is controlled using an input flux $u(t)$

$$\frac{\partial z}{\partial t} = \frac{\partial^2 z}{\partial x^2} + Bu(t), \qquad 0 < x < 1,$$

where $B \in \mathcal{L}(\mathbb{R}, \mathcal{Z})$ describes the distribution of applied energy, with the same Dirichlet boundary conditions. Since the input space is one-dimensional B can be defined by $Bu = b(x)u$ for some $b(x) \in \mathcal{L}_2(0,L)$. This leads to

$$\dot{z}(t) = Az(t) + b(x)u(t).$$

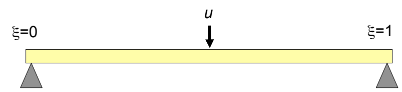


Figure 5: Simply supported beam with applied force.

Example 2.4 *Simply supported beam.* Consider a simply supported Euler-Bernoulli beam as shown in Figure 5 and let $w(x,t)$ denote the deflection of the beam from its rigid body motion at time t and position x. The control $u(t)$ is a force applied at the center with width δ. The analysis of beam vibrations is useful for applications such as flexible links in robots; but also in understanding the dynamics of more complex structures. If we normalize the variables, we obtain the partial differential equation (see, for example, [14, chap.6]):

$$\frac{\partial^2 w}{\partial t^2} + \frac{\partial^4 w}{\partial x^4} = b(x)u(t), \qquad t \geq 0,\ 0 < x < 1,$$

$$b(x) = \begin{cases} 1/\delta, & |x-0.5| < \frac{\delta}{2} \\ 0, & |x-0.5| \geq \frac{\delta}{2} \end{cases}$$

with boundary conditions

$$w(0,t) = 0,\ w_{xx}(0,t) = 0,\ w(1,t) = 0, w_{xx}(1,t) = 0. \tag{4}$$

This system is second-order in time, and analogously to a simple mass-spring system, we define the state as $z(t) = \begin{bmatrix} w(\cdot,t) & \dot{w}(\cdot,t) \end{bmatrix}$. Let

$$H_s(0,1) = \{w \in \mathcal{H}^2(0,1), w(0) = 0, w(1) = 0\}$$

and define the state-space $\mathcal{Z} = H_s(0,1) \times \mathcal{L}_2(0,1)$. A state-space formulation of the above partial differential equation problem is

$$\frac{d}{dt} z(t) = Az(t) + Bu(t),$$

where

$$B = \begin{bmatrix} 0 \\ \\ b(\cdot) \end{bmatrix}, \qquad A = \begin{bmatrix} 0 & I \\ \\ -\frac{d^4}{dx^4} & 0 \end{bmatrix},$$

with domain

$$D(A) = \{(\phi, \psi) \in H_s(0,1) \times H_s(0,1);\ \phi_{xx} \in H_s(0,1)\}.$$

The operator A with domain $D(A)$ generates a C_0-semigroup on $\mathcal{Z}$.

Even for finite-dimensional systems, the entire state cannot generally be measured. Measurement of the entire state is never possible for systems described by partial differential equations, and we define

$$y(t) = Cz(t) + Eu(t) \tag{5}$$

where $C \in \mathcal{L}(\mathcal{Z}, \mathcal{Y})$, $E \in \mathcal{L}(\mathcal{U}, \mathcal{Y})$ and $\mathcal{Y}$ is a Hilbert space. The expression (5) can also represent the cost in controller design. Note that as for the control operator, it is assumed that C is a bounded operator from the state-space $\mathcal{Z}$. The operator E is a feedthrough term that is non-zero in some control configurations.

The state at any time t and control $u \in L_2(0, t; U)$ is given by

$$z(t) = S(t)z_0 + \int_0^t S(t-s)Bu(s)ds$$

and the output is

$$y = CS(t)z_0 + C\int_0^t S(t-\tau)Bu(\tau)d\tau,$$

or defining

$$g(t) = CS(t)B,$$

$$y(t) = CS(t)Bz_0 + (g * u)(t)$$

where $*$ indicates convolution.

The Laplace transform G of g yields the transfer function of the system: If $z(0) = 0$,

$$\hat{y}(s) = G(s)\hat{u}(s).$$

The transfer function of a system modelled by a system of ordinary differential equations is always rational with real coefficients; for example $\frac{2s+1}{s^2+s+25}$. Transfer functions of systems modelled by partial differential equations are

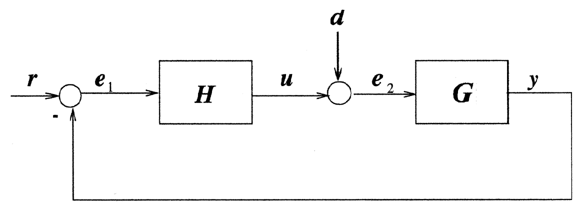


Figure 6: Standard feedback diagram

non-rational. There are some differences in the systems theory for infinite-dimensional systems [7]. For instance, it is possible for the transfer function to have different limits along the real and imaginary axes.

The definitions of stability for finite-dimensional systems generalize to infinite-dimensions.

Definition 2.5 *A system is* externally stable or $\mathcal{L}_2$-stable *if for every input* $u \in \mathcal{L}_2(0, \infty; \mathcal{U})$*, the output* $y \in \mathcal{L}_2(0, \infty; \mathcal{Y})$*. If a system is externally stable, the maximum ratio between the norm of the input and the norm of the output is called the* L_2-gain*.*

Define

$$\mathbb{H}_\infty = \{G : \mathbb{C}_0^+ \to \mathbb{C} | G \quad \text{analytic} \quad \text{and} \quad \sup_{\text{Re}s > 0} |G(s)| < \infty\}$$

with norm

$$\|G\|_\infty = \sup_{\text{Re}s > 0} |G(s)|.$$

Matrices with entries in $\mathbb{H}_\infty$ will be indicated by $M(\mathbb{H}_\infty)$. The H_∞-norm of matrix functions is

$$\|G\|_\infty = \sup_{\text{Re}s > 0} \sigma_{\max}\big(G(s)\big).$$

The theorem below is stated for systems with finite-dimensional input and output spaces $\mathcal{U}$ and $\mathcal{Y}$ but it generalizes to infinite-dimensional $\mathcal{U}$ and $\mathcal{Y}$.

Theorem 2.6 *A linear system is externally stable if and only if its transfer function matrix* $G \in M(\mathbb{H}_\infty)$*. In this case,* $\|G\|_\infty$ *is the* L_2-gain *of the system and we say that* G *is a* stable transfer function*.*

As for finite-dimensional systems, we need additional conditions to ensure that internal and external stability are equivalent.

Definition 2.7 *The semigroup $S(t)$ is* exponentially stable *if there is $M \geq 1$, $\alpha > 0$ such that $\|S(t)\| \leq Me^{-\alpha t}$ for all $t \geq 0$.*

Definition 2.8 *The system (A, B, C) is internally stable if A generates an exponentially stable semigroup $S(t)$.*

Definition 2.9 *The pair (A, B) is stabilizable if there exists $K \in \mathcal{L}(\mathcal{U}, \mathcal{Z})$ such that $A - BK$ generates an exponentially stable semigroup.*

Definition 2.10 *The pair (C, A) is detectable if there exists $F \in \mathcal{L}(\mathcal{Y}, \mathcal{Z})$ such that $A - FC$ generates an exponentially stable semigroup.*

Theorem 2.11 *[18, Thm. 26, Cor. 27] A stabilizable and detectable system is internally stable if and only if it is externally stable.*

Now, let G be the transfer function of a given plant and let H be the transfer function of a controller, of compatible dimensions, arranged in the standard feedback configuration shown in Figure 6. This framework is general enough to include most common control problems. For instance, in tracking, r is the reference signal to be tracked by the plant output y_1. Since r can also be regarded as modelling sensor noise and d as modelling actuator noise, it is reasonable to regard the control system in Figure 6 as externally stable if the four maps from r, d to e_1, e_2 are in $M(\mathbb{H}_\infty)$. (Stability could also be defined in terms of the transfer matrix from (r, d) to (y_1, y_2): both notions of stability are equivalent.) Let (A, B, C, E) be a state-space realization for G and similarly let (A_c, B_c, C_c, E_c) be a state-space realization for H. If $I + EE_c$ is invertible, then the 2×2 transfer matrix $\Delta(G, H)$ which maps the pair (r, d) into the pair (e_1, e_2) is given by

$$\Delta(G, H) = \begin{bmatrix} (I + GH)^{-1} & -G(I + HG)^{-1} \\ \\ H(I + GH)^{-1} & (I + HG)^{-1} \end{bmatrix}.$$

Definition 2.12 *The feedback system (Figure 6), or alternatively the pair (G, H), is said to be* externally stable *if $I + EE_c$ is invertible, and each of the four elements in $\Delta(G, H)$ belongs to $M(\mathbb{H}_\infty)$.*

Typically the plant feedthrough E is zero and so the invertibility of $I + EE_c$ is trivially satisfied. The above definition of external stability is sufficient to ensure that all maps from uncontrolled inputs to outputs are bounded. Furthermore, under the additional assumptions of stabilizability and detectability, external stability and internal stability are equivalent.

Theorem 2.13 *[18, Thm. 35] Assume that (A, B, C, E) is a jointly stabilizable/detectable control system and that a controller (A_c, B_c, C_c, E_c) is also jointly stabilizable/detectable.The closed loop system is externally stable if and only if it is internally stable.*

This equivalence between internal and external stability justifies the use of controller design techniques based on system input/output behaviour for infinite-dimensional systems.

3 Issues in Controller Design

For most practical examples, a closed form solution of the partial differential equation or of the transfer function is not available and an approximation needs to be used. This approximation is generally calculated using one of the many standard methods, such as finite elements, developed for simulation of partial differential equation models. The resulting system of ordinary differential equations is used in controller design. The advantage to this approach is that the wide body of synthesis methods available for finite-dimensional systems can be used.

The usual assumption made on an approximation scheme used for simulation are as follows. Suppose the approximation lies in some finite-dimensional subspace $\mathcal{Z}_n$ of the state-space $\mathcal{Z}$, with an orthogonal projection $P_n : \mathcal{Z} \to \mathcal{Z}_n$ where for each $z \in \mathcal{Z}$, $\lim_{n\to\infty} \|P_n z - z\| = 0$. The space $\mathcal{Z}_n$ is equipped with the norm inherited from $\mathcal{Z}$. Define $B_n = P_n B$, $C_n = C|_{\mathcal{Z}_n}$ (the restriction of C_n to $\mathcal{Z}_n$) and define $A_n \in \mathcal{L}(\mathcal{Z}_n, \mathcal{Z}_n)$ using some method. This leads to a sequence of finite-dimensional approximations

$$\begin{aligned} \frac{dz}{dt} &= A_n z(t) + B_n u(t), \qquad z(0) = P_n z_0, \\ y(t) &= C_n z(t). \end{aligned}$$

Let S_n indicate the semigroup (a matrix exponential) generated by A_n. The following assumption is standard.

(A1) For each $z \in \mathcal{Z}$, and all intervals of time $[t_1, t_2]$

$$\lim_{n\to\infty} \sup_{t\in[t_1,t_2]} \|S_n(t) P_n z - S(t) z\| = 0.$$

Assumption (A1) is required for convergence of the response to an initial condition. Assumption (A1) is often satisfied by ensuring that the conditions of the Trotter-Kato Theorem hold, see for instance, [34, sect. 3.4]. Assumption (A1) implies that $P_n z \to z$ for all $z \in \mathcal{Z}$. The strong convergence $P_n z \to z$ for all $z \in \mathcal{Z}$ and the definitions of $B_n = P_n B$ and $C_n = C|_{\mathcal{Z}_n}$ imply that for all $u \in \mathcal{U}$, $z \in \mathcal{Z}$, $\|B_n u - Bu\| \to 0$, $\|C_n P_n z - Cz\| \to 0$.

The following result on open loop convergence is straightforward.

Theorem 3.1 *Suppose that the approximating systems (A_n, B_n, C_n) satisfy assumption (A1). Then for each initial condition $z \in \mathcal{Z}$, the uncontrolled approximating state $z(t)$ converges uniformly on bounded intervals to the exact state. Also, for each $u \in \mathcal{L}_2(0,T;\mathcal{U})$, $y_n \to y$ in the norm on $\mathcal{L}_2(0,T;\mathcal{Y})$.*

Example 3.2 *Indirect Controller Design for Simply Supported Beam. (Eg. 2.4 cont.)* Consider a simply supported Euler-Bernoulli beam (Figure 5). As shown in Example 2.4, a state-space formulation with state space $\mathcal{Z} = H_s(0,1) \times \mathcal{L}_2(0,1)$ is

$$\frac{d}{dt}z(t) = Az(t) + Bu(t),$$

where

$$A = \begin{bmatrix} 0 & I \\ \\ -\frac{d^4}{dx^4} & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ \\ b(\cdot) \end{bmatrix},$$

with domain

$$D(A) = \{(\phi, \psi) \in H_s(0,1) \times H_s(0,1);\ \phi'' \in H_s(0,1)\}$$

and

$$b(x) = \begin{cases} 1/\delta, & |x - 0.5| < \frac{\delta}{2} \\ 0, & |x - 0.5| \geq \frac{\delta}{2} \end{cases} .$$

Let $\phi_i(x)$ indicate the eigenfunctions of $\frac{\partial^4 w}{\partial x^4}$ with simply supported boundary conditions (4). Defining $\mathcal{X}_n$ to be the span of ϕ_i, $i = 1..n$, we choose $\mathcal{Z}_n = \mathcal{X}_n \times \mathcal{X}_n$ and define P_n to be the projection onto $\mathcal{Z}_n$. Let $\langle \cdot, \cdot \rangle$ indicate the inner product on $\mathcal{Z}$ (and on $\mathcal{Z}_n$). Define the approximating system (A_n, B_n) by the Galerkin approximation

$$\langle \dot{z}(t), \phi_i \rangle = \langle Az(t), \phi_i \rangle + \langle b, \phi_i \rangle u(t), \qquad i = 1...n.$$

This leads to the standard modal approximation

$$\dot{z}(t) = A_n z(t) + B_n u(t),\ z_n(0) = P_n z(0). \tag{6}$$

This approximation scheme satisfies assumption (A1).

Consider linear-quadratic (LQ) controller design [30, e.g.]: find a control $u(t)$ so that the cost functional

$$J(u, z_0) = \int_0^\infty \langle z(t), z(t)\rangle + |u(t)|^2 dt$$

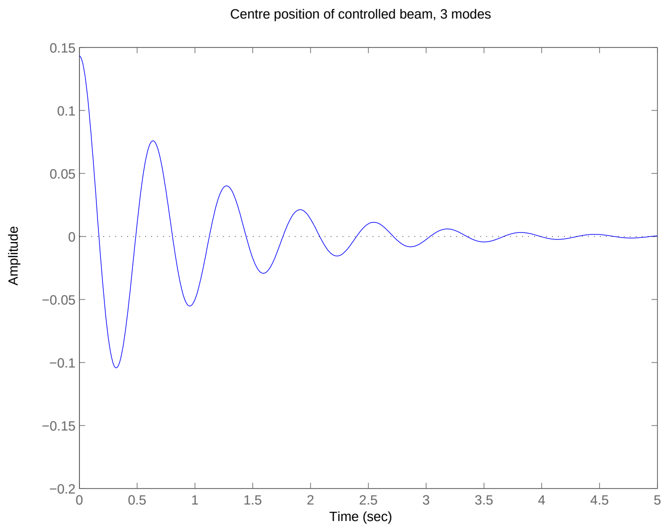
is minimized where $z(t)$ is determined by (6). The resulting optimal controller is $u(t) = -K_n z(t)$ where $K_n = B_n^* \Pi_n z(t)$ and Π_n solves the algebraic Riccati equation

$$A_n^* \Pi_n + \Pi_n A_n - \Pi_n B_n B_n^* \Pi_n + I = 0$$

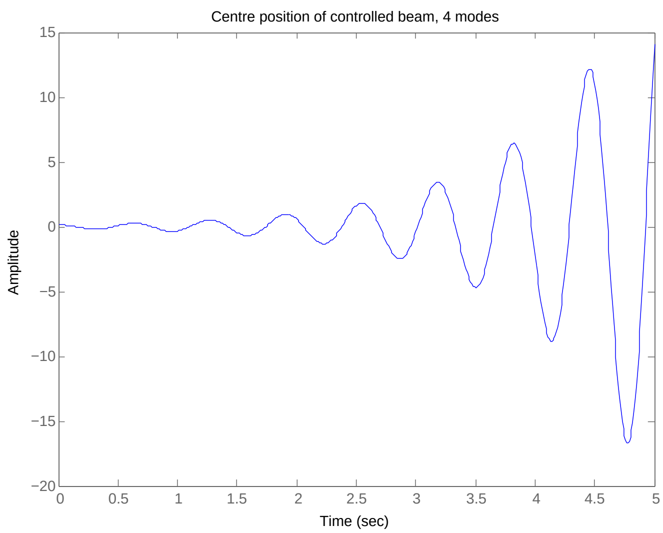
where M^* indicates the adjoint operator of M. For a matrix M, M^* is the complex conjugate transpose of M. Suppose we use the first 3 modes (or eigenfunctions) to design the controller. As expected, the controller stabilizes the model used in design. However, Figure 7b shows that if even one additional mode is added to the system model, the controller no longer stabilizes the system. This phenomenon is often called *spillover* [1]. Figure 8 shows the sequence of controllers obtained for increasing model order. The controller sequence is not converging to some controller appropriate for the original infinite-dimensional system. The increase in $\|K_n\|$ as approximation order increases suggests that the original system is not stabilizable. This is the case here. Although the approximations are stabilizable, the original model is not stabilizable [13].

As shown by the above example, and also by an example in [29], requirements additional to those sufficient for simulation are required when an approximation is used for controller design. The issues are illustrated in Figure 9. Possible problems that may occur are:

- The controlled system may not perform as predicted.
- The sequence of controllers for the approximating systems may not converge.
- The original control system may not be stabilizable, even if the approximations are stabilizable.



(a) with 3 modes



(b) with 4 modes

Figure 7: A linear-quadratic feedback controller K was designed for the beam in Example 3.2 using the first 3 modes (eigenfunctions). Simulation of the controlled system with (a) 3 modes and (b) 4 modes is shown. The initial condition in both cases is the first eigenfunction. Figure (b) illustrates that the addition of only one additional mode to the model leads to instability in the controlled system.

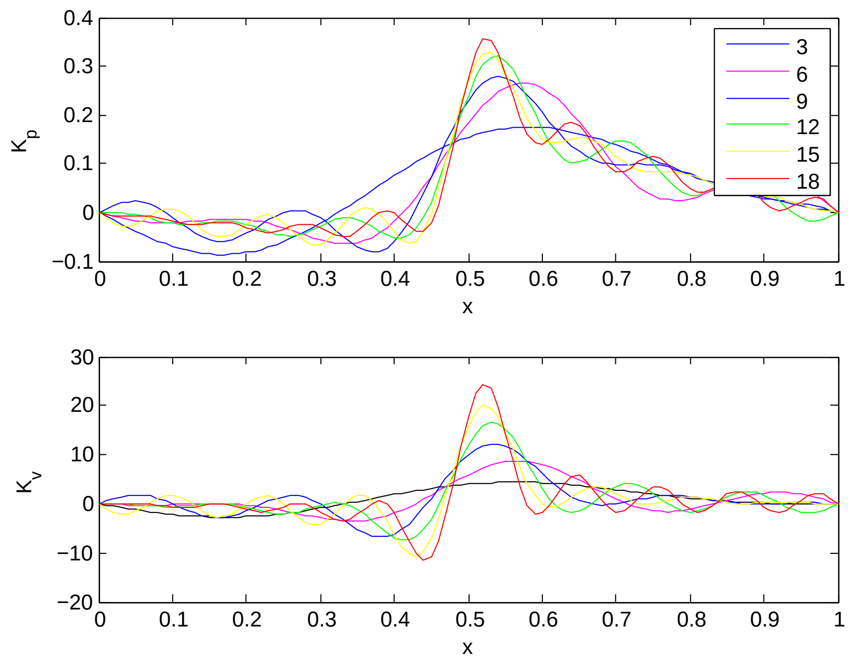
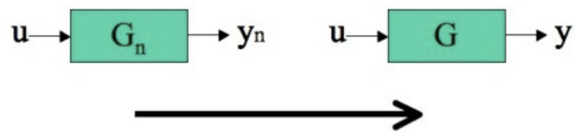
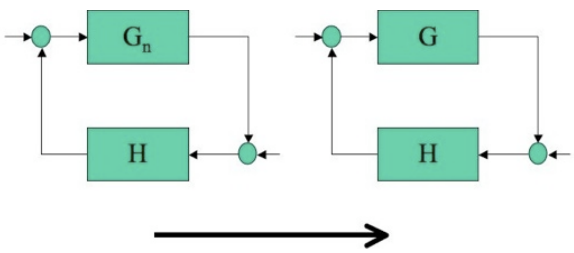


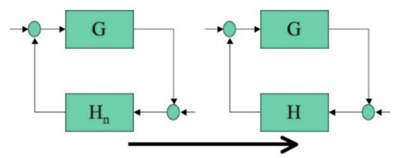
Figure 8: Linear-quadratic optimal feedback for modal approximations of the simply supported beam in Example 3.2. Since the input space $U = \mathbb{R}$, the feedback operator K_n is a bounded linear functional and hence can be uniquely identified with a function, called the gain. The upper figure shows the feedback gain for the position of beam; the lower figure shows the velocity gains. Neither sequence is converging as the approximation order increases.



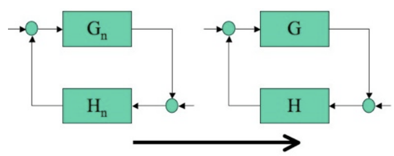
(a) Open Loop Convergence



(b) Closed Loop Convergence



(c) Convergence of controllers



(d) Convergence of controllers and plants

Figure 9: Typical approximation criteria ensure that the open loops converge. However, in controller design, the controller is generally implemented as a feedback controller and control of the resulting closed loop system is needed. Furthermore, a sequence of controllers is produced by applying a controller synthesis technique to the approximations. This sequence of controllers should converge so that closed loop performance with the original plant is similar to that predicted by simulations with the approximations.

– The performance of the controlled infinite-dimensional system may not be close to that predicted by simulations with approximations (and may be unstable).

Although (A1) guarantees open loop convergence, additional conditions are required in order to obtain closed loop convergence.

The *gap topology*, first introduced in [40], is useful in establishing conditions under which an approximation can be used in controller design. Consider a stable system G; that is $G \in M(\mathbb{H}_\infty)$. A sequence G_n converges to G in the gap topology if and only if $\lim_{n\to\infty} \|G_n - G\|_\infty = 0$. The extension to unstable systems uses *coprime factorizations*. Let G be the transfer function of a system, with right coprime factorization $G = \tilde{N}\tilde{D}^{-1}$:

$$\tilde{X}\tilde{N} + \tilde{Y}\tilde{D} = I, \qquad \tilde{X}, \tilde{N}, \tilde{Y}, \tilde{D} \in M(\mathbb{H}_\infty).$$

(If $G \in M(\mathbb{H}_\infty)$, then we can choose $\tilde{N} = G$, $\tilde{D} = I$, $\tilde{X} = 0$, $\tilde{Y} = I$.) A sequence G_n converges to G in the gap topology if and only if for some right coprime factorization $(\tilde{N}_n, \tilde{D}_n)$ of G_n, $\tilde{N}_n$ converges to $\tilde{N}$ and $\tilde{D}_n$ converges to $\tilde{D}$ in the $\mathbb{H}_\infty$-norm. For details on the gap (or graph) topology, see [39, 42]. The importance of the gap topology in controller design is stated in the following result.

Theorem 3.3 *[39] Let $G_n \in M(\mathbb{H}_\infty)$ be a sequence of system transfer functions.*

1. *Suppose G_n converges to G in the gap topology. Then if H stabilizes G, there is an N such that H stabilizes G_n for all $n \geq N$ and the closed loop transfer matrix $\Delta(G_n, H)$ converges to $\Delta(G, H)$ in the $\mathbb{H}_\infty$-norm.*

2. *Conversely, suppose that there exists an H that stabilizes G_n for all $n \geq N$ and so that $\Delta(G_n, H)$ converges to $\Delta(G, H)$ in the $\mathbb{H}_\infty$-norm. Then G_n converges to G in the gap topology.*

Thus, failure of a sequence of approximations to converge in the gap topology implies that for each possible controller H at least one of the following conditions holds:

– H does not stabilize G_n for all n sufficiently large,

– the closed loop response $\Delta(G_n, H)$ does not converge to $\Delta(G_n, H)$.

On the other hand, if a sequence of approximations does converge in the gap topology, then the closed loop performance $\Delta(G_n, H)$ converges and moreover, every H that stabilizes G also stabilizes G_n for sufficiently large approximation order.

The following condition, with assumption (A1) provides a sufficient condition for convergence of approximations in the gap topology. It was first formulated in [3] in the context of approximation of linear-quadratic regulators for parabolic partial differential equations.

Definition 3.4 *The control systems (A_n, B_n) are* uniformly stabilizable *if there exists a sequence of feedback operators $\{K_n\}$ with $\|K_n\| \leq M_1$ for some constant M_1 such that $A_n - B_nK_n$ generate $S_{Kn}(t)$, $\|S_{Kn}(t)\| \leq M_2 e^{-\alpha_2 t}$, $M_2 \geq 1$, $\alpha_2 > 0$.*

Since $\mathcal{U}$ is assumed finite-dimensional, B is a compact operator. Thus, (A1) and uniform stabilizability imply stabilizability of (A, B) [15, Thm 2.3] and in fact the existence of a uniformly stabilizing sequence K_n satisfying (A3) with $\lim_{n\to\infty} K_n P_n z = Kz$ for all $z \in \mathcal{Z}$. The beam in Example 3.2 is not stabilizable and therefore there is no sequence of uniformly stabilizable approximations.

Theorem 3.5 *[29, Thm. 4.2] Consider a stabilizable and detectable control system (A, B, C, E), and a sequence of approximations (A_n, B_n, C_n, E) that satisfy (A1). If the approximating systems are uniformly stabilizable then they converge to the exact system in the gap topology.*

Example 3.6 *Diffusion (Eg. 2.3 cont).*

$$
\begin{aligned}
\frac{\partial z}{\partial t} &= \frac{\partial^2 z}{\partial x^2} + b(x)u(t), \qquad 0 < x < 1, \\
z(0,t) = 0, &\qquad z(1,t) = 0, \\
y(t) &= \int_0^1 c(x)z(x,t)dx
\end{aligned}
$$

for some $b, c \in \mathcal{L}_2(0,1)$. This can be written as

$$
\begin{aligned}
\dot{z}(t) &= Az(t) + Bu(t), \\
y(t) &= Cz(t)
\end{aligned}
$$

where A and B are as defined in Example 2.3 and

$$Cz = \int_0^1 c(x)z(x)dx.$$

The operator A generates an exponentially stable semigroup $S(t)$ with $\|S(t)\| \leq e^{-\pi^2 t}$ on the state-space $\mathcal{L}_2(0,L)$ [8] and so the system is trivially stabilizable and detectable. The eigenfunctions $\phi_i(x) = \sqrt{2}\sin(\pi i x)$, $i = 1, 2, ...,$ of A form an orthonormal basis for $\mathcal{L}_2(0,L)$. Defining $\mathcal{Z}_n = \mathrm{span}_{i=1..n}\phi_i(x)$, and letting P_n be the projection onto $\mathcal{Z}_n$, define A_n by the Galerkin approximation

$$\langle A_n\phi_j, \phi_i\rangle = \langle A\phi_j, \phi_i\rangle \qquad i = 1..n,\ j = 1..n.$$

It is straightforward to show that this set of approximations satisfies assumption (A1) and that the semigroup generated by A_n has bound $S_n(t) \leq e^{-\pi^2 t}$. Hence the approximations are uniformly stabilizable (using $K_n = 0$). Thus, the sequence of approximating systems converges in the gap topology and will yield reliable results when used in controller design.

It is easy to show that if the original problem is exponentially stable, and the eigenfunctions of A form an orthonormal basis for $\mathcal{Z}$, then any approximation formed using the eigenfunctions as a basis, as in the previous example, will both satisfy assumption (A1) and be trivially uniformly stabilizable and detectable. However, in practice other approximation methods, such as finite elements, are often used.

Many generators A for partial differential equation models can be described as follows. Let V be a Hilbert space that is dense in $\mathcal{Z}$. The notation $\langle\cdot,\cdot\rangle$ indicates the inner product on $\mathcal{Z}$, and $\langle\cdot,\cdot\rangle_V$ indicates the inner product on V. The norm on $\mathcal{Z}$ is indicated by $\|\cdot\|$ while the norm on V will be indicated by $\|\cdot\|_V$. Let the bilinear form $a : V \times V \mapsto \mathcal{C}$ be such that for some $c_1 > 0$

$$|a(\phi,\psi)| \leq c_1\|\phi\|_V\|\psi\|_V \tag{7}$$

for all $\phi, \psi \in V$. An operator A can be defined through this form by

$$\langle -A\phi, \psi\rangle = a(\phi,\psi), \qquad \forall\psi \in V$$

with $D(A) = \{\phi \in V |\, a(\phi,\cdot) \in \mathcal{Z}\}$. Assume that in addition to (7), $a(\cdot,\cdot)$ satisfies Garding's inequality: there exists $k \geq 0$, such that for all $\phi \in V$

$$\mathrm{Re\,a}(\phi,\phi) + \mathrm{k}\langle\phi,\phi\rangle \ \geq \mathrm{c}\|\phi\|_{\mathrm{V}}^2. \tag{8}$$

For example, in the diffusion example above, $V = \mathcal{H}_0^1(0,1)$ and

$$a(\phi, \psi) = \int_0^1 \phi'(x)\psi'(x)dx.$$

The inequalities (7) and (8) guarantee that A generates a C_0- semigroup with bound $\|T(t)\| \leq e^{kt}$ [37, sect. 4.6]. This framework includes many problems of practical interest such as diffusion and beam vibrations with Kelvin-Voigt damping [29].

Defining a sequence of finite-dimensional subspaces $\mathcal{Z}_n \subset V$, the approximating generator A_n is defined by

$$\langle -A_n z_n, v_n \rangle = a(z_n, v_n), \quad \forall z_n, v_n \in \mathcal{Z}_n. \tag{9}$$

This type of approximation is generally referred to as a *Galerkin* approximation and includes finite-element as well as many other popular approximation methods. For such problems, the following result, which generalizes [3, Lem. 3.3] is useful. It applies to a number of common applications, such as the usual linear spline finite-elements for approximating the heat equation and other diffusion problems. Finite-element cubic spline approximations to damped beam vibrations are also included.

Theorem 3.7 *[29, Thm. 5.2,5.3] Let $\mathcal{H}_n \subset V$ be a sequence of finite-dimensional subspaces such that for all $z \in V$ there exists a sequence $z_n \in \mathcal{Z}_n$ with*

$$\lim_{n\to\infty} \|z_n - z\|_V = 0. \tag{10}$$

If the operator A satisfies the inequalities (7) and (8) then

1. *Assumption (A1) is satisfied with $\|S_n(t)\| \leq e^{kt}$;*

2. *If $K \in \mathcal{L}(\mathcal{Z}, U)$ is such that $A - BK$ generates a stable semigroup then the semigroups $S_{nK}(t)$ generated by $A_n - B_n K P_n$ are uniformly exponentially stable. In other words, there exists $M \geq 1$, $\alpha > 0$ such that*

$$\|S_{nK}(t)\| \leq Me^{-\alpha t} \qquad \forall n > N. \tag{11}$$

and the approximations (A_n, B_n) are thus uniformly stabilizable.

Example 3.8 *Damped String.* The wave equation

$$\frac{\partial^2 w(x,t)}{\partial t^2} = c^2 \frac{\partial^2 w(x,t)}{\partial x^2}, \qquad 0 < x < 1, \qquad t \geq 0,$$

describes the deflection $z(x,t)$ of a vibrating string of unit length as well as many other situations such as acoustic plane waves, lateral vibrations in beams, and electrical transmission lines [38, e.g., chap.1]. Suppose the ends are fixed:

$$w(0,t) = 0, \qquad w(1,t) = 0.$$

Including control and observation, as well as the effect of some light damping [36] leads to the model

$$\frac{\partial^2 w(x,t)}{\partial t^2} + \epsilon \langle \frac{\partial w(\cdot,t)}{\partial t}, b(\cdot) \rangle b(x) = \frac{\partial^2 w}{\partial x^2} + b(x)u(t), \qquad 0 < x < 1,$$

$$y(t) = \int_0^1 b(x) \frac{\partial w(x,t)}{\partial t} dx$$

where $\epsilon > 0$ and $b \in \mathcal{L}_2(0,1)$ describes both the control and observation action, which is a type of distributed colocation. The state-space is $\mathcal{Z} = \mathcal{H}_0^1(0,1) \times \mathcal{L}_2(0,1)$ and the state-space equations are

$$\begin{aligned} \frac{d}{dt}\begin{bmatrix} z \\ \frac{dz}{dt} \end{bmatrix} &= A \begin{bmatrix} z \\ \frac{dz}{dt} \end{bmatrix} + \begin{bmatrix} 0 \\ b(x) \end{bmatrix} u(t), \\ y(t) &= C \begin{bmatrix} z \\ \frac{dz}{dt} \end{bmatrix} \end{aligned}$$

where

$$\begin{aligned} A \begin{bmatrix} w \\ v \end{bmatrix} &= \begin{bmatrix} 0 & I \\ \frac{\partial^2 w}{\partial x^2} & -\epsilon \langle v, b \rangle b(x) \end{bmatrix}, \\ D(A) &= \left\{ (w,v) \in H_0^1(0,1) \times H_0^1(0,1) \right\}, \\ C \begin{bmatrix} w \\ v \end{bmatrix} &= \begin{bmatrix} 0 & \langle b(x), v \rangle \end{bmatrix}. \end{aligned}$$

Suppose that

$$b(x) = \begin{cases} 1, & 0 < x < \frac{1}{2}, \\ 0, & \frac{1}{2} < x < 1 \end{cases}$$

The eigenvalues λ_n of A have all negative real parts, but asymptote to the imaginary axis so that $\sup_n \mathrm{Re}(\lambda_n) = 0$. The results in [18] (see also [8, sect. 5.2]) imply that the system is not exponentially stabilizable. Thus, no sequence of approximations is uniformly stabilizable.

However, it is possible to construct a sequence of finite-dimensional approximations that converge in the gap topology. The transfer function is

$$G(s) = \frac{\frac{s}{2}\sinh(s) + 2\cosh\left(\frac{s}{2}\right) - 3\cosh^2\left(\frac{s}{2}\right) + 1}{s(s+\frac{\varepsilon}{2})\sinh(s) + \varepsilon(2\cosh\left(\frac{s}{2}\right) - 3\cosh^2\left(\frac{s}{2}\right) + 1)}.$$

The function $G \in \mathbb{H}_\infty$, so the system is L_2-stable, and furthermore,

$$\lim_{|s|\to\infty, \mathrm{Re}s>0} G(s) = 0.$$

Thus, we can find a sequence of rational functions G_n so that

$$\lim_{n\to\infty} \|G_n(s) - G(s)\|_\infty = 0.$$

The state-space realizations corresponding to $\{G_n\}$ are finite-dimensional and thus there are finite-dimensional approximations that converge in the gap topology.

The above example illustrates that uniform stabilizability is a sufficient, not necessary, condition for convergence of approximations in the gap topology.

Once an approximation scheme that converges in the gap topology is found (typically, by finding one that satisfies (A1) and is uniformly stabilizable), the next step is controller design. The sequence of controllers designed using the approximations should converge to a controller for the original infinite-dimensional system that yields the required performance, as well as stability. (See Figure 9.)

A common procedure for controller design is to first design a state feedback controller: $u(t) = -Kz(t)$. Then, since the full state is not available, an estimator is designed to obtain an estimate of the state using knowledge of the output y and the input u. The controller is formed by using the state estimate as input to a state feedback controller. Controller design of this type for infinite-dimensional systems is described in [8, sect. 5.3].

However, typically both the state feedback and the estimator are designed using a finite-dimensional approximation (A_n, B_n, C_n, E). Suppose

$F_n \in \mathcal{L}(\mathcal{Y}, \mathcal{Z})$ is found so that all the eigenvalues of $A_n - F_nC_n$ have negative real parts, and similarly $K_n \in \mathcal{L}(\mathcal{Z},\mathcal{U})$ is such that all the eigenvalues of $A_n - B_nK_n$ have negative real parts. The resulting finite-dimensional controller is

$$\begin{aligned} \dot{z}_c(t) &= A_n z_c(t) + B_n u(t) + F_n\big(y(t) - C_n z_c(t)\big) \\ y_c(t) &= -K_n z_c(t) \end{aligned}$$

This framework does not include the effect of disturbances to the controlled system, shown as r and v in Figure 6. To include these effects, and put the system into the standard framework, define an augmented system output

$$\begin{aligned} \tilde{y}(t) &= \begin{bmatrix} y(t) \\ u(t) \end{bmatrix} \\ &= \begin{bmatrix} C \\ 0 \end{bmatrix} z(t) + \begin{bmatrix} E \\ I \end{bmatrix} u(t) \end{aligned}$$

Letting $e_1(t) = r - \tilde{y}$ indicate the controller input, the controller equations are then

$$\begin{aligned} \dot{z}_c(t) &= (A_n - F_nC_n) z_c(t) - \begin{bmatrix} F_n & B_n \end{bmatrix} e(t) \\ y_c(t) &= -K_n z_c(t) \end{aligned} \tag{12}$$

and the plant input is $y_c + d$.

It is well known that such a controller stabilizes the approximation (A_n, B_n, C_n, E) [30, e.g.]. However, it must also stabilize the original system (A, B, C, E). For this to happen, the controller sequence must converge in some sense. For controller convergence, an assumption in addition to (A1) and uniform stabilizability is required.

Definition 3.9 *The observation systems* (A_n, C_n) *are* uniformly detectable *if there exists a sequence of operators* $\{F_n\}$ *with* $\|F_n\| \leq M_3$ *for some constant* M_3 *such that* $A_n - F_nC_n$ *generate* $S_{Fn}(t)$, $\|S_{Fn}(t)\| \leq M_4 e^{-\alpha_4 t}$, $M_4 \geq 1$, $\alpha_4 > 0$.

The approximating systems (A_n, C_n) are uniformly detectable if and only if (A_n^*, C_n^*) is uniformly stabilizable and thus uniform detectability can be established using conditions for uniform stabilizability. In particular, if A is defined through a bilinear form satisfying (7) and (8) and condition (10) in Theorem 3.7 is satisfied, then detectability of (A, C) implies uniform detectability of (A_n, C_n).

Theorem 3.10 *Assume that (A1) holds, that the operators K_n, F_n used to define the sequence of controllers (12) satisfy Definitions 3.4 and 3.9 respectively and there exists $K \in \mathcal{L}(\mathcal{U}, \mathcal{Z}), F \in \mathcal{L}(\mathcal{Y}, \mathcal{Z})$ such that $\lim_{n\to\infty} K_n P_n z = Kz$ for all $z \in \mathcal{Z}$, $\lim_{n\to\infty} F_n y = Fy$ for all $y \in \mathcal{Y}$. Indicating the controller transfer function by H_n, the controllers converge in the gap topology and so for sufficiently large n, the output feedback controllers (12) stabilize the infinite-dimensional system (2,5). Furthermore, the closed loop systems $\Delta(G_n, H_n)$ converge uniformly to the closed system $\Delta(G, H)$ where H indicates the transfer function of the infinite-dimensional controller*

$$\begin{aligned} \dot{z}_c(t) &= (A - FC)z_c(t) - \begin{bmatrix} F & B \end{bmatrix} e(t) \\ y_c(t) &= -Kz_c(t) \end{aligned} \tag{13}$$

Proof: The assumptions on the controller imply that, as for the plant (Thm. 3.10) the controllers converge in the gap topology. See [28] for details. This implies that closed loops $\Delta(G, H_n)$ converge to $\Delta(G, H)$ and that the controllers stabilize the original system for large enough n. Since the assumptions also imply that the approximating systems G_n converge in the gap topology to G, the closed loop systems $\Delta(G_n, H_n)$ converge uniformly to the closed system $\Delta(G, H)$. □

Controller design is explored further in the next two sections for the synthesis methods most commonly used for multi-input-multi-output systems: linear quadratic control and $\mathbb{H}_\infty$-control.

4 Linear-Quadratic Regulators

Consider the linear-quadratic (LQ) controller design objective of finding a control $u(t)$ so that the cost functional

$$J(u, z_0) = \int_0^\infty \langle C_1 z(t), C_1 z(t)\rangle + \langle u(t), Ru(t)\rangle dt \tag{14}$$

is minimized where $R \in \mathcal{L}(\mathcal{U},\mathcal{U})$ is a symmetric positive definite operator weighting the control, $C_1 \in \mathcal{L}(\mathcal{Z}, \mathcal{Y})$ (with Hilbert space $\mathcal{Y}$) weights the state, and $z(t)$ is determined by (2). The theoretical solution to this problem is similar in structure to that for finite-dimensional systems [8, 12, 22, 23].

Definition 4.1 *The system (2) with cost (14) is optimizable if for every $z_0 \in \mathcal{Z}$ there exists $u \in L_2(0, \infty; \mathcal{U})$ such that the cost is finite.*

Theorem 4.2 *[8, Thm 6.2.4, 6.2.7] If (2) with cost (14) is optimizable and (A, C_1) is detectable, then the cost function (14) has a minimum for every $z_0 \in \mathcal{Z}$. Furthermore, there exists a self-adjoint non-negative operator $\Pi \in \mathcal{L}(H, H)$ such that*

$$\min_{u \in L_2(0,\infty;\mathcal{U})} J(u, z_0) = \langle z_0, \Pi z_0 \rangle.$$

The operator Π is the unique non-negative solution to the operator equation

$$\langle Az_1, \Pi z_2 \rangle + \langle \Pi z_1, Az_2 \rangle + \langle C_1 z_1, C_1 z_2 \rangle - \langle B^* \Pi z_1, R^{-1} B^* \Pi z_1 \rangle = 0 \quad z_1, z_2 \in D(A). \tag{15}$$

Defining $K = R^{-1} B^* \Pi$, the corresponding optimal control is $u = -Kz(t)$ and $A - BK$ generates an exponentially stable semigroup.

It is straightforward to show that the assumption of optimizability in Theorem 4.2 is equivalent to stabilizability.

The Riccati operator equation (15) is equivalent to

$$\left(A^* \Pi + \Pi A - \Pi B R^{-1} B^* \Pi + C_1^* C_1\right) z = 0, \qquad \forall z \in D(A).$$

In practice, the operator equation (15) cannot be solved and the control is calculated using an approximation. The cost functional becomes

$$J(u, z_0) = \int_0^\infty \langle C_{1n} z(t), C_{1n} z(t) \rangle + \langle u(t), Ru(t) \rangle dt \tag{16}$$

where $z(t)$ is the state of the approximating system

$$\dot{z}(t) = A_n z(t) + B_n u(t), \quad z(0) = P_n z_0,$$

on $\mathcal{Z}_n$ and $C_{1n} = C_1|_{\mathcal{Z}_n}$. If (A_n, B_n) is stabilizable and (A_n, C_{1n}) is detectable, then the cost functional has the minimum cost $\langle P_n z_0, \Pi_n P_n z_0 \rangle$ where Π_n is the unique non-negative solution to the algebraic Riccati equation

$$A_n^* \Pi_n + \Pi_n A_n - \Pi_n B_n R^{-1} B_n^* \Pi_n + C_{1n}^* C_{1n} = 0 \tag{17}$$

on the finite-dimensional space $\mathcal{Z}_n$. The feedback control $K_n = R^{-1} B_n^* \Pi_n$ is used to control the original system (2).

The sequence of controllers K_n, along with the associated performance must converge in some sense in order for this approach to be valid. Assumption (A1), along with uniform stabilizability, guarantees convergence of the approximating systems. However, in order to obtain controller convergence a set of assumptions involving the dual system (A^*, B^*, C_1^*) is required.

(A1*) (i) For each $z \in \mathcal{Z}$, and all intervals of time $[t_1, t_2]$

$$\sup_{t \in [t_1, t_2]} \|S_n^*(t) P_n z - S^*(t) z\| \to 0;$$

(ii) For all $u \in U$, $y \in Y$, $\|C_{1n}^* y - C_1^* y\| \to 0$ and $\|B_n^* P_n z - B^* z\| \to 0$.

Theorem 4.3 *[3, Thm. 6.9],[15, Thm. 2.1, Cor. 2.2] If assumptions (A1), (A1*) are satisfied, (A_n, B_n) is uniformly stabilizable and (A_n, C_{1n}) is uniformly detectable, then for each n, the finite-dimensional ARE (17) has a unique nonnegative solution Π_n with $\sup \|\Pi_n\| < \infty$. There exists constants $M_1 \geq 1$, $\alpha_1 > 0$, independent of n, such that the semigroup $S_{nK}(t)$ generated by $A_n - B_n K_n$ satisfy*

$$\|S_{nK}(t)\| \leq M_1 e^{-\alpha_1 t}.$$

For sufficiently large n, the semigroups $S_{Kn}(t)$ generated by $A - BK_n$ are uniformly exponentially stable; that is there exists $M_2 \geq 1$, $\alpha_2 > 0$, independent of n, such that

$$\|S_{Kn}(t)\| \leq M_2 e^{-\alpha_2 t}.$$

Furthermore, letting Π indicate the solution to the infinite-dimensional Riccati equation (15), for all $z \in \mathcal{Z}$,

$$\lim_{n \to \infty} \|\Pi_n P_n z - \Pi z\| = 0$$

and

$$\lim_{n \to \infty} \|K_n P_n z - K z\| = 0,$$

and the cost with feedback $K_n z(t)$ converges to the optimal cost:

$$J(-K_n z(t), z_0) \to \langle \Pi z_0, z_0 \rangle.$$

The assumption (A1*) implies open-loop convergence of the dual systems (A_n^*, C_{1n}^*, B_n^*). It is required since the optimal control Kz relates to an optimization problem involving the dual system. Note that (A, C_1) is uniformly detectable if and only if (A^*, C_1^*) is uniformly stabilizable, and so (A1*) along with uniform detectability can be regarded as dual assumptions to (A1) and uniform stabilizability. Since the operators B and C_1 are bounded, (A1*ii) holds if both the input and output spaces are finite-dimensional. However, the satisfaction of (A1*i), strong convergence of the adjoint semigroups, is not automatic. A counter-example may be found in [5] where the assumptions except (A1*i) are satisfied and the conclusions of the above theorem do not hold. The conclusions of the above theorem, that is, uniform boundedness of Π_n and the uniform exponential stability of $S_{nK}(t)$, imply uniform stabilizability of (A_n, B_n). Although Example 3.8 illustrated that uniform stabilizability is not necessary for convergence of the approximating systems, it is necessary to obtain a linear-quadratic controller sequence that provides uniform exponential stability. The above result has been extended to unbounded control operators B for parabolic partial differential equations, such as diffusion problems [2, 22].

Example 4.4 *Damped Beam* As in Examples 2.4 and 3.2, consider a simply supported Euler-Bernoulli beam but now include viscous damping with parameter $c_d = .1$. We obtain the partial differential equation

$$\frac{\partial^2 w}{\partial t^2} + c_d \frac{\partial w}{\partial t} + \frac{\partial^4 w}{\partial x^4} = b_r(x)u(t), \qquad t \geq 0, 0 < x < 1,$$

with the same boundary conditions as before. However, we now consider an arbitrary location r for the control operator b so that

$$b_r(x) = \begin{cases} 1/\delta, & |x - r| < \frac{\delta}{2} \\ 0, & |x - r| \geq \frac{\delta}{2} \end{cases} .$$

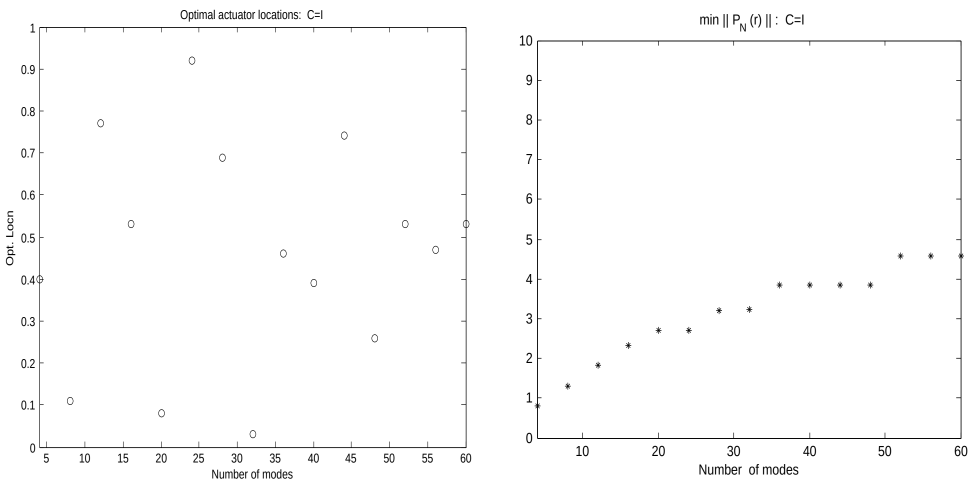
Recall that the state-space is $\mathcal{Z} = H_s(0,1)(0,1) \times \mathcal{L}_2(0,1)$ with state $z(t) = (w(\cdot,t), \frac{\partial}{\partial t} w(\cdot,t))$. An obvious choice of weight for the state is $C_1 = I$. Since there is only one control, choose control weight $R = 1$. We wish to choose the actuator location in order to minimize the response to the worst choice of initial condition. In other words, choose r in order to minimize

$$\max_{\substack{z_0 \in \mathcal{Z} \\ \|z_0\|=1}} \min_{u \in L_2(0,\infty;U)} J^r(u, z_o) = \|\Pi(r)\|.$$

The performance for a particular r is $\mu(r) = \|\Pi(r)\|$ and the optimal performance

$$\hat{\mu} = \inf_{r \in \Omega^m} \|\Pi(r)\|.$$

This optimal actuator location problem is well-posed and a optimal location $\hat{r}$ exists [31, Thm. 2.6].



(a) Optimal actuator location $\hat{r}_n$ (b) Optimal performance $\hat{\mu}_n = \|\Pi(\hat{r}_n)\|$

Figure 10: Optimal actuator location and performance for approximations of the viscously damped beam with weights $C_1 = I$, $R = 1$. No convergence of the optimal location or performance is seen as the approximation order is increased.

Let $\phi_i(x)$ indicate the eigenfunctions of $\frac{\partial^4 w}{\partial x^4}$ with simply supported boundary conditions. Defining X_n to be the span of ϕ_i, $i = 1..n$, we choose $\mathcal{Z}_n = X_n \times X_n$. This approximation scheme satisfies all the assumptions of Theorem 4.3 and so the sequence of solutions Π_n to the corresponding finite-dimensional ARE's converge strongly to the exact solution Π.

However, as shown in Figure 10, this does not imply convergence of the optimal actuator locations, or of the corresponding actuator locations.

The problem is that strong convergence of the Riccati operators is not sufficient to ensure that as the approximation order increases, the optimal cost $\hat{\mu}_n$ and a corresponding sequence of optimal actuator locations $\hat{r}_n$ converge. Since the cost is the norm of the Riccati operator, uniform convergence of the operators is required. That is,

$$\lim_{n \to \infty} \|\Pi_n P_n - \Pi\| = 0,$$

is needed in order to use approximations in determining optimal actuator location. The first point to consider is that since Π_n has finite rank, Π must be a compact operator in order for uniform convergence to occur, regardless of the choice of approximation method. Since the solution to an ARE is not always compact, it is not possible to compute the optimal actuator location for every problem.

Example 4.5 [8] Consider any A, B, C_1 such that $A^* = -A$ and $C_1 = B^*$. Then $\Pi = I$ is a solution to the ARE

$$A^*\Pi + \Pi A - \Pi BB^*\Pi + C_1^*C_1 = 0.$$

The identity I is not compact on any infinite-dimensional Hilbert space.

Example 4.6 This example is a generalization of [6, Example 1]. On the Hilbert space $\mathcal{Z} = \mathbb{R} \times H$ where H is any infinite-dimensional Hilbert space, define

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -I \end{bmatrix}, \qquad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad C_1 = \begin{bmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{2}M \end{bmatrix}$$

where M is a bounded operator on H. The solution to the ARE

$$A^*\Pi + \Pi A - \Pi BB^*\Pi + C_1^*C_1 = 0$$

is

$$\Pi = \begin{bmatrix} 1 & 0 \\ 0 & M^2 \end{bmatrix}.$$

This operator is not compact if M is not a compact operator; for instance if $M = I$. This example is particularly interesting because A is a bounded operator and also generates an exponentially stable semigroup.

Theorem 4.7 *[31, Thm. 2.9,3.3] If B and C_1 are both compact operators, then the Riccati operator Π is compact. Furthermore, if a sequence of approximations satisfy (A1), (A1*) and are uniformly stabilizable and detectable, then the minimal non-negative solution Π_n to (17) converges uniformly to the non-negative solution Π to (15):* $\lim_{n\to\infty} \|\Pi_n - \Pi\| = 0$.

Thus, if B and C_1 are compact operators, guaranteeing compactness of the Riccati operator, then any approximation method satisfying the assumptions of Theorem 4.3 will lead to a convergent sequence of optimal actuator locations. A finite-dimensional input space guarantees that B is a compact operator; and similarly a finite-dimensional output space will guarantee that C_1 is compact, although these assumptions are not necessary.

For an important class of problems the Riccati operator is compact, even if the observation operator C_1 is not compact.

Definition 4.8 *A semigroup $S(t)$ is analytic if $t \to S(t)$ is analytic in some sector $|\arg t| < \theta$.*

Analytic semigroups have a number of nice properties [34, 37]. Recall that the solution $S(t)z \in D(A)$, $t \geq 0$, if $z \in D(A)$. If S is an analytic semgroup, $S(t)z \in D(A)$ for all $z \in \mathcal{Z}$. Also, the eigenvalues of the generator A of an analytic semigroup lie in a sector $|\arg\lambda| < \pi - \epsilon$ where $\epsilon > 0$. The heat equation and other parabolic partial differential equations lead to an analytic semigroup. Weakly damped wave and beam equations are not associated with analytic semigroups.

If A generates an analytic semigroup, uniform convergence can be obtained without compactness of the state weight C_1. The result [22, Thm. 4.1], applies to operators B and C_1 that may be unbounded. It is stated below for bounded B and C_1.

Theorem 4.9 *Let A generate an analytic semigroup $S(t)$ with $\|S(t)\| \leq Me^{\omega_0 t}$ and define $\hat{A} = (\omega I - A)$ for $\omega > \omega_0$. Assume that the system (A, B, C_1) has the following properties:*

1. *(A, B) is stabilizable and (A, C_1) is detectable;*
2. *either $C_1^* C_1 \geq rI$, $r > 0$, or for some $F \in \mathcal{L}(\mathcal{Y}, \mathcal{Z})$ such that $A - FC_1$ generates an exponentially stable semigroup, $\hat{A}^{-1}FC_1$ is compact;*
3. *either $B^*\hat{A}^{-1}$ is compact or there exists a compact operator $K \in \mathcal{L}(\mathcal{Z}, \mathcal{U})$ such that $A - BK$ generates an exponentially stable semigroup;*

Assume the following list of properties for the approximation scheme, where γ is any number $0 \leq \gamma < 1$:

1. *For all $z \in \mathcal{Z}$, $\|P_n z - z\| \to 0$;*

2. *The approximations are uniformly analytic. That is, for some* $\epsilon > 0$

$$\|A_n^\theta e^{A_n t}\| \leq \frac{M_\theta e^{(\omega_0+\epsilon)t}}{t^\theta}, \qquad t > 0,\ 0 \leq \theta \leq 1;$$

3. *For some* s *and* γ *independent of* n, $0 \leq \gamma < 1$,
   (a) $\|\hat{A}^{-1} - \hat{A}_n^{-1} P_n\| \leq \frac{M}{n^s}$;
   (b) $\|B^* z - B_n^* P_n z\| \leq M n^{s(\gamma-1)} \|z\|_{[D(A^*)]},\ z \in D(A^*)$.

Then $\lim_{n\to\infty} \|\Pi_n P_n - \Pi\| = 0$.

The above conditions on the approximation scheme imply assumptions (A1) and (A1*) as well as uniform stabilizability and detectability.

Provided that Π_n converges to Π in operator norm at each actuator location, the sequence of optimal actuator locations for the approximations converges to the correct optimal location.

Theorem 4.10 *[31, Thm. 3.5] Let* Ω *be a closed and bounded set in* $\mathbb{R}^N$. *Assume that* $B(r)$, $r \in \Omega$, *is compact and such that for any* $r_0 \in \Omega$,

$$\lim_{r\to r_0} \|B(r) - B(r_0)\| = 0.$$

Assume also that $(A_n, B_n(r), C_{1n})$ *is a family of uniformly stabilizable and detectable approximations satisfying (A1) and (A1*) such that for each* r,

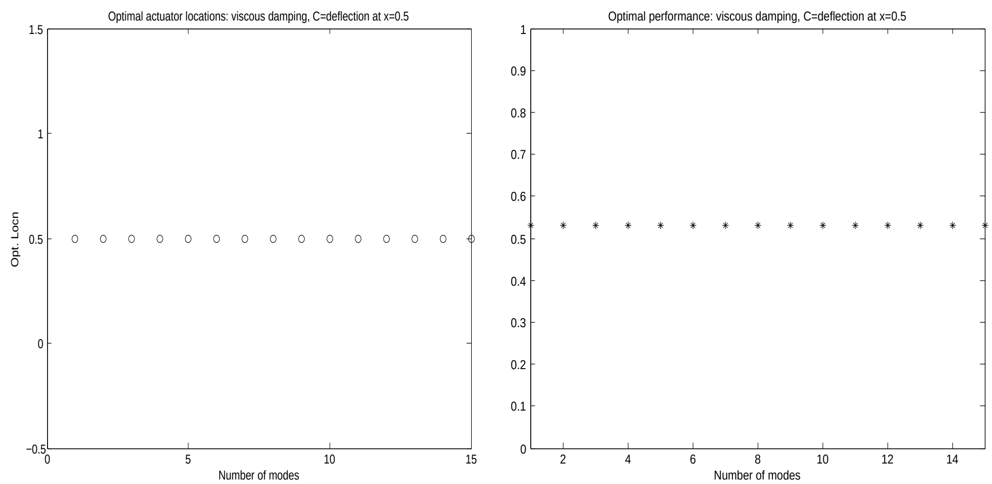
$$\lim_{n\to\infty} \|\Pi_n(r) - \Pi(r)\| = 0.$$

Letting $\hat{r}$ *be the optimal actuator location for* $(A, B(r), C_1)$ *with optimal cost* $\hat{\mu}$ *and defining similarly* $\hat{r}_n$, $\hat{\mu}_n$, *it follows that*

$$\mu_n \to \mu,$$

$$\hat{r}_n \to \hat{r}.$$

Example 4.11 *Viscously Damped Beam, cont.* Consider the same viscously damped bean system and control problem as in Example 4.4, except that now instead of trying to minimize the norm of the entire state, $C_1 = I$, we consider only the position. Choose the weight $C_1 = [I \;\; 0]$ where I here



(a) Optimal actuator location $\hat{r}_n$ (b) Optimal performance $\hat{\mu}_n = \|\Pi(\hat{r}_n)\|$

Figure 11: Optimal actuator location and performance for different approximations of the viscously damped beam with weights $C_1 = [I \ 0]$, $R = 1$. Although the output space is infinite-dimensional, C_1 is a compact operator. This implies uniform convergence of the Riccati operators and thus convergence of both the optimal actuator locations $\hat{r}_n$ and optimal costs $\hat{\mu}_n$.

indicates the mapping from $\mathcal{H}_0^2(0,1)$ into $\mathcal{L}_2(0,1)$. Although the semigroup is not analytic, both B and C_1 are compact operators on $\mathcal{Z}$. Using the same modal approximations as before, we obtain convergence of the approximating optimal performance and the actuator locations. This is illustrated in Figure 11.

Example 4.12 *Diffusion. (Eg. 2.3 cont.)*

$$\frac{\partial z}{\partial t} = \frac{\partial^2 z}{\partial x^2} + b_r(x)u(t) \qquad 0 < x < 1,$$

$$b_r(x) = \begin{cases} 1/\delta, & |x-r| < \frac{\delta}{2} \\ 0, & |x-r| \geq \frac{\delta}{2} \end{cases} .$$

$$\frac{\partial z}{\partial x}(0,t) = 0, \quad \frac{\partial z}{\partial x}(1,t) = 0.$$

We wish to determine the best location r of the actuator to minimize

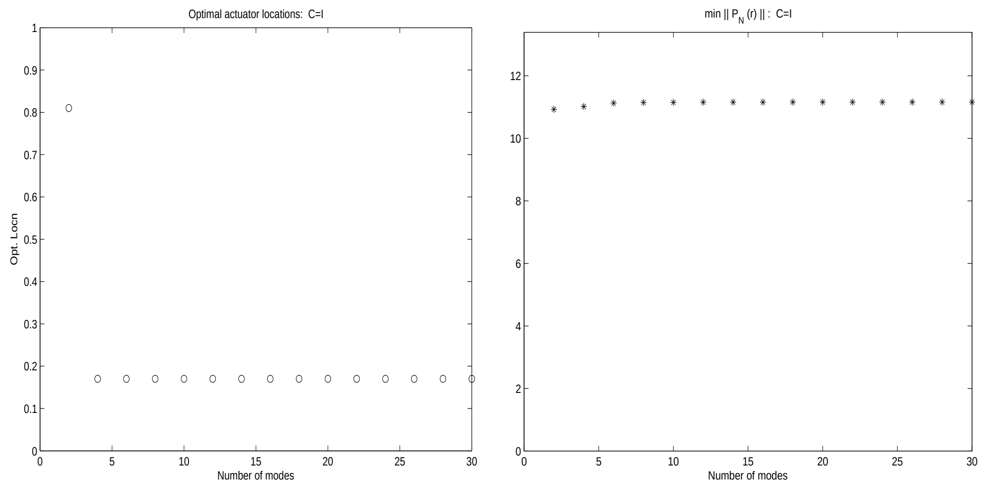
$$J^r(u, z_0) = \int_0^\infty 1000 \int_0^1 |z(x,t)|^2 dx + |u(t)|^2 dt$$

with respect to the worst possible initial condition. This means that we want to minimize $\|\Pi(r)\|$ where Π solves the ARE with $C_1 = \sqrt{1000}I$ and $R = 1$. Note that C_1 is not a compact operator. However, $A = \frac{\partial^2}{dx^2}$ with domain $D(A) = \{z \in H_2(0,1) | z'(0) = z'(1) = 0\}$ generates an analytic semigroup on $\mathcal{L}(0,1)$. Defining $\mathcal{Z}_n$ to be the span of the first n eigenfunctions and defining the corresponding Galerkin approximation as in Example 3.6 leads to an approximation that satisfies the assumptions of Theorem 4.9 [22] and so $\lim_{n\to\infty} \|\Pi_n(r) - \Pi(r)\| = 0$ for each location r. Convergence of the optimal performance and of a corresponding sequence of actuator locations is shown in Figure 12.

Figure 13 illustrates that this optimal actuator location problem is non-convex. We are only guaranteed to have convergence of a sequence of optimal actuator locations, not every sequence.

The discussion in this section has so far been concerned only with state feedback. However, in general the full state z is not available and a measurement

$$y(t) = C_2 z(t), \tag{18}$$



(a) Optimal actuator location $\hat{r}_n$ (b) Optimal performance $\hat{\mu}_n = \|\Pi(\hat{r}_n)\|$

Figure 12: Optimal actuator location and performance for approximations of the of the diffusion equation, $C_1 = \sqrt{1000}I$, $R = 1$. Since the semigroup is analytic, uniform convergence of Π_n to Π is obtained, even for a non-compact C_1 such as used here. This leads to convergence of the optimal performance and of a corresponding sequence of actuator locations.

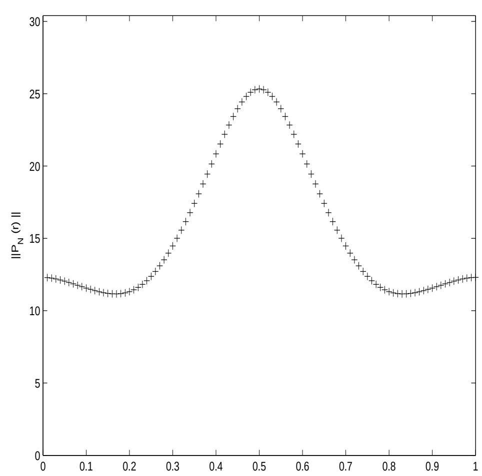


Figure 13: The cost $\|\Pi(r)\|$ as a function of r for the heat equation, $C = \sqrt{1000}I$, is not a convex function.

where $C_2 \in \mathcal{L}(\mathcal{Z}, \mathcal{W})$ and $\mathcal{W}$ is a finite-dimensional Hilbert space, is used to estimate the state.

As for finite-dimensional systems, we construct an estimate of the state. Choose some $F \in \mathcal{L}(\mathcal{W}, \mathcal{Z})$ so that $A - FC_2$ generates an exponentially stable semigroup. This can be done, for instance, by solving a Riccati equation dual to that for control: $F = \Sigma C_2^* R_e^{-1}$ where for some $B_1 \in \mathcal{L}(\mathcal{Z}, \mathcal{V})$, $\mathcal{V}$ a Hilbert space and $R_e \in \mathcal{L}(\mathcal{W}, \mathcal{W})$ with $R_e > 0$,

$$\big(\Sigma A^* + A\Sigma + B_1 B_1^* - \Sigma C_2^* R_e^{-1} C_2 \Sigma\big) z = 0, \qquad \forall z \in D(A^*). \tag{19}$$

The estimator is

$$\dot{z}_e(t) = A z_e(t) + Bu(t) + F\big(y(t) - C_2 z_e(t)\big).$$

Stability of $A - FC_2$ guarantees that $\lim_{t\to\infty} \|z_e(t) - z(t)\| = 0$.

The controller is formed by using the state estimate as input to a state feedback controller. As explained in section 3, this leads to the controller

$$\begin{aligned} \dot{z}_c(t) &= (A - FC_2) z_c(t) - \begin{bmatrix} F & B \end{bmatrix} e(t) \\ y_c(t) &= -K z_c(t) \end{aligned} \tag{20}$$

where $e(t) = r(t) - \tilde{y}(t)$ and

$$\begin{aligned} \tilde{y}(t) &= \begin{bmatrix} y(t) \\ u(t) \end{bmatrix} \\ &= \begin{bmatrix} C_2 \\ 0 \end{bmatrix} z(t) + \begin{bmatrix} 0 \\ I \end{bmatrix} u(t). \end{aligned}$$

As for finite-dimensional systems, if $A - FC_2$ and $A - BK$ each generate an exponentially stable semigroup the above controller stabilizes the infinite-dimensional system (2),(18) [8, sect. 5.3].

The controller (20) is infinite-dimensional. A finite-dimensional approximation to this controller can be calculated using a finite-dimensional approximation (A_n, B_n, C_{2n}) to the original system (A, B, C_2). Consider $F_n = \Sigma_n C_{2n}^* R_e^{-1}$, where Σ_n solves the ARE

$$\Sigma_n A_n^* + A_n \Sigma_n + B_{1n} B_{1n}^* - \Sigma_n C_{2n}^* R_e^{-1} C_{2n} \Sigma = 0. \tag{21}$$

Results for convergence of solutions Σ_n and the operators F_n follow from arguments dual to those for convergence of solutions to the control Riccati equation (17).

Theorem 4.13 *Assume that assumptions (A1) and (A1*) hold for approximations of* $(A, \begin{bmatrix} B & B_1 \end{bmatrix}, \begin{bmatrix} C_1 \\ C_2 \end{bmatrix})$ *and that the approximations are also uniformly stabilizable and uniformly detectable. Then the operators* K_n , F_n *obtained by solving the Riccati equations (17) and (21) respectively converge to the operators* K *and* F *obtained by solving (15) and (19). Furthermore, the sequence* K_n *uniformly stabilizes* (A_n, B_n) *and the sequence* F_n *is uniformly detectable for* (A_n, C_{2n}).

Defining

$$\begin{aligned} \tilde{y}(t) &= \begin{bmatrix} y(t) \\ u(t) \end{bmatrix} \\ &= \begin{bmatrix} C_2 \\ 0 \end{bmatrix} z(t) + \begin{bmatrix} 0 \\ I \end{bmatrix} u(t) \end{aligned}$$

and letting $e(t) = r - \tilde{y}$, the finite-dimensional controller is

$$\begin{aligned} \dot{z}_c(t) &= (A_n - F_n C_{2n}) z_c(t) - \begin{bmatrix} F_n & B_n \end{bmatrix} e(t) \\ y_c(t) &= -K_n z_c(t). \end{aligned} \tag{22}$$

It follows from Theorem 3.10 that the sequence of controllers (22) converge in the gap topology to (20), and that they stabilize the original system for large enough n. Furthermore, the corresponding closed loop systems converge.

5 $\mathbb{H}_\infty$ Control

Many applications involve a unknown and uncontrolled disturbance $d(t)$. An important objective of controller design in these situations is to reduce the system's response to the disturbance. The system equations (2,5) become

$$\frac{dz}{dt} = Az(t) + Bu(t) + Dd(t), \qquad z(0) = 0 \tag{23}$$

with cost

$$y_1(t) = C_1 z(t) + E_1 u(t). \tag{24}$$

Since we are interested in reducing the response to the disturbance, the initial condition $z(0)$ is set to zero. We assume that $d(t) \in L_2(0, \infty; \mathcal{V})$ where $\mathcal{V}$ is a Hilbert space and that $D \in \mathcal{L}(\mathcal{V}, \mathcal{Z})$ is a compact operator. (This last assumption follows automatically if $\mathcal{V}$ is finite-dimensional.) The measured signal, or input to the controller is

$$y_2(t) = C_2 z(t) + E_2 d(t) \tag{25}$$

The operator $C_2 \in \mathcal{L}(\mathcal{Z}, \mathcal{W})$ for some finite-dimensional Hilbert space $\mathcal{W}$ and $E_2 \in \mathcal{L}(\mathcal{V}, \mathcal{W})$.

Let G denote the transfer function from d to y_1 and let H denote the controller transfer function:

$$\hat{u}(s) = H(s)\hat{y}_2(s).$$

The map from the disturbance d to the cost y_1 is

$$\begin{aligned} \hat{y}_1 &= C_1(sI - A)^{-1}(D\hat{d} + B\hat{u}) \\ &= C_1(sI - A)^{-1}(D\hat{d} + BH\hat{y}_2). \end{aligned}$$

Using (23) and (25) to eliminate y_2, and defining

$$\begin{aligned} G_{11}(s) &= C_1(sI - A)^{-1}D, & G_{12}(s) &= C_1(sI - A)^{-1}B + E_1, \\ G_{21}(s) &= C_2(sI - A)^{-1}D + E_2, & G_{22}(s) &= C_2(sI - A)^{-1}B, \end{aligned}$$

we obtain the transfer function

$$\mathcal{F}(G, H) = G_{11}(s) + G_{12}(s)H(s)(I - G_{22}(s)H(s))^{-1}G_{21}(s)$$

from the disturbance d to the cost y_1.

The controller design problem is to find, for given $\gamma > 0$, a stabilizing controller H so that

$$\int_0^\infty \|y_1(t)\|^2 < \int_0^\infty \gamma^2 \|d(t)\|^2 dt.$$

If such a controller is found, the controlled system will then have L_2-gain less than γ.

Definition 5.1 *The system (23,24,25) is* stabilizable with attenuation γ *if there is a stabilizing controller H so that*

$$\|\mathcal{F}(G, H)\|_\infty < \gamma.$$

To simplify the formulae, we make the assumptions that

$$E_1^* \begin{bmatrix} C_1 & E_1 \end{bmatrix} = \begin{bmatrix} 0 & I \end{bmatrix}, \qquad \begin{bmatrix} D \\ E_2 \end{bmatrix} E_2^* = \begin{bmatrix} 0 \\ I \end{bmatrix}. \tag{26}$$

With these simplifying assumptions, the cost y_1 has norm

$$\int_0^\infty \|y_1(t)\|^2 dt = \int_0^\infty \|C_1 z(t)\|^2 + \|u(t)\|^2 dt$$

which is the linear quadratic cost (14) with normalized control weight $R = I$. The difference is that here we are considering the effect of the disturbance d on the cost, instead of the initial condition $z(0)$. Also, the problem formulation (23,24,25) can include robustness and other performance constraints. For details, see, for example, [30, 41].

We will assume throughout that (A, D) and (A, B) are stabilizable and that (A, C_1) and (A, C_2) are detectable. These assumptions ensure that an internally stabilizing controller exists; and that internal and external stability are equivalent for the closed loop if the controller realization is stabilizable and detectable.

Consider first the full information case:

$$\begin{aligned} y_2(t) &= \begin{bmatrix} x(t) \\ d(t) \end{bmatrix} \\ &= \begin{bmatrix} I \\ 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ I \end{bmatrix} d(t). \end{aligned} \tag{27}$$

An important characteristic of $\mathbb{H}_\infty$-disturbance attenuation is that, in general, a system is not stabilizable with attenuation γ for every γ. However, if it is stabilizable with attenuation γ, the attenuation (in the full-information case) can be achieved with constant state-feedback.

Definition 5.2 *The state feedback $K \in \mathcal{L}(\mathcal{Z}, \mathcal{U})$ is γ-admissible if $A - BK$ generates an exponentially stable semigroup and the feedback $u(t) = -Kz(t)$ is such that γ-attenuation is achieved.*

Theorem 5.3 *[4, 20] Assume that (A, B) is stabilizable and (A, C_1) is detectable. For $\gamma > 0$ the following are equivalent:*

- *the full-information system (23,24,27) is stabilizable with disturbance attenuation γ,*

– *there exists a nonnegative, self-adjoint operator* Σ *on* $\mathcal{Z}$ *such that for all* $z \in D(A)$,

$$\left(A^*\Pi + \Pi A + \Pi\left(\frac{1}{\gamma^2}DD^* - BB^*\right)\Pi + C_1^*C_1\right)z = 0 \tag{28}$$

and $A - BB^*\Pi + \frac{1}{\gamma^2}DD^*\Pi\frac{1}{\gamma^2}DD^*\Pi$ *generates an exponentially stable semigroup on* $\mathcal{Z}$.

In this case $K = B^*\Pi$ *is a* γ*-admissible state-feedback.*

Notice that as $\gamma \to \infty$, the Riccati equation for the linear quadratic problem is obtained. Since the operator Riccati equation (28) can't be solved, a finite-dimensional approximation to the original infinite-dimensional system is used to approximate the exact control $K = B^*\Pi$. As in previous sections, let $\mathcal{Z}_n$ be a finite-dimensional subspace of $\mathcal{Z}$ and P_n the orthogonal projection of $\mathcal{Z}$ onto $\mathcal{Z}_n$. Consider a sequence of operators $A_n \in \mathcal{L}(\mathcal{Z}_n, \mathcal{Z}_n)$, $B_n = P_nB$, $D_n = P_nD$, $C_{1n} = C_1|_{\mathcal{Z}_n}$. Assumptions similar to those used for linear quadratic control are required.

Theorem 5.4 *[16, Theorem 2.5, Cor. 2.6] Assume a sequence of approximations satisfy (A1),(A1*),* (A_n, B_n) *are uniformly stabilizable and* (A_n, C_{1n}) *are uniformly detectable. Assume that the original problem is stabilizable with attenuation* γ. *For sufficiently large* n *the Riccati equation*

$$A_n^*\Pi_n + \Pi_nA_n + \Pi_n\left(\frac{1}{\gamma^2}D_nD_n^* - B_nB_n^*\right)\Pi_n + C_{1n}^*C_{1n} = 0, \tag{29}$$

has a nonnegative, self-adjoint solution Π_n. *For such* n

– *There exist positive constants* M_1 *and* ω_1 *such that the semigroup* $S_{n2}(t)$ *generated by* $A_n + \frac{1}{\gamma^2}D_nD_n^*\Pi_n - B_nB_n^*\Pi_n$ *satisfies* $\|S_{n2}(t)\| \leq M_1e^{-\omega_1 t}$.

– $K_n = (B_n)^*\Pi_n$ *is a* γ*-admissible state feedback for the approximating system and there exists* $M_2, \omega_2 > 0$ *such that the semigroup* $S_{nK}(t)$ *generated by* $A_n + B_nK_n$ *satisfies* $\|S_{nK}(t)\| \leq M_2e^{-\omega_2 t}$.

Moreover, for all $z \in \mathcal{Z}$, $\Pi_nP_nz \to \Pi z$ *as* $n \to \infty$ *and* $K_n = (B_n)^*\Pi_n$ *converges to* $K = B_2^*\Pi$ *in norm. For* n *sufficiently large,* K_nP_n *is a* γ*-admissible state feedback for the infinite-dimensional system.*

The optimal disturbance attenuation problem is to find

$$\hat{\gamma} = \inf \gamma$$

over all γ such that (23,24,27) is stabilizable with attenuation γ. Let $\hat{\gamma}_n$ indicate the corresponding optimal disturbance attenuation for the approximating problems. Theorem 5.4 implies that $\limsup_{n\to\infty} \hat{\gamma}_n \leq \hat{\gamma}$ but in fact convergence of the optimal disturbance attenuation can be shown.

Corollary 5.5 *[16, Thm. 2.8] With the same assumptions as Theorem 5.4, it follows that*

$$\lim_{n\to\infty} \hat{\gamma}_n = \hat{\gamma}.$$

Example 5.6 *Flexible Slewing Beam.* Consider an Euler-Bernoulli beam clamped at one end and free at other end. Let $w(x,t)$ denote the deflection of the beam from its rigid body motion at time t and position x. The deflection can be controlled by applying a torque at the clamped end ($x = 0$). We assume that the hub inertia I_h is much larger than the beam inertia, so that, letting $\theta(t)$ indicate the rotation angle, $u(t) = I_h\ddot{\theta}(t)$ is a reasonable approximation to the applied torque. The disturbance $d(t)$ induces a uniformly distributed load $\rho_d d(t)$. The values of the physical parameters used in the simulations are listed in Table 1. Use of the Kelvin-Voigt model for damping leads to the following description of the beam vibrations:

$$\rho\frac{\partial^2 w}{\partial t^2}+c_v\frac{\partial w}{\partial t}+\frac{\partial^2}{\partial r^2}\left[EI\frac{\partial^2 w}{\partial x^2}+c_dI\frac{\partial^3 w}{\partial x^2\partial t}\right] = \frac{\rho x}{I_h}u(t)+\rho_d\, d(t), \qquad 0 < x < L.$$

The boundary conditions are

$$w(0,t) = 0, \qquad \frac{\partial w}{\partial x}(0,t) = 0,$$

$$\left[EI\frac{\partial^2 w}{\partial r^2}+c_dI\frac{\partial^3 w}{\partial x^2\partial t}\right]_{x=L} = 0, \qquad \left[EI\frac{\partial^3 w}{\partial x^3}+c_dI\frac{\partial^4 w}{\partial x^3\partial t}\right]_{x=L} = 0.$$

Let $z(t) = (w(\cdot,t), \frac{\partial}{\partial t}w(\cdot,t))$, $H_f(0,L)$ be the closed linear subspace of $\mathcal{H}^2(0,L)$ defined by

$$H_f(0,L) = \left\{ w \in \mathcal{H}^2(0,L) : w(0) = \frac{dw}{dx}(0) = 0 \right\}.$$

E	$2.1 \times 10^{11}\, N/m^2$
I	$1.2 \times 10^{-10}\, m^4$
ρ	$3.0\, kg/m$
c_v	$.0010\, Ns/m^2$
c_d	$.010\, Ns/m^2$
L	$7.0\, m$
I_h	$39. kgm^2$
ρ_d	$.12\, 1/m$

Table 1: Physical Constants

With the state space $\mathcal{Z} = H_f(0, L) \times \mathcal{L}^2(0, L)$, a state-space formulation of the above partial differential equation problem is

$$\tfrac{d}{dt} z(t) = Az(t) + Bu(t) + Dd(t),$$

where

$$B = \begin{bmatrix} 0 \\ \\ \frac{x}{I_h} \end{bmatrix}, \quad D = \begin{bmatrix} 0 \\ \\ \frac{\rho_d}{\rho} \end{bmatrix}, \qquad A = \begin{bmatrix} 0 & I \\ \\ -\frac{EI}{\rho}\frac{d^4}{dr^4} & -\frac{c_d I}{\rho}\frac{d^4}{dr^4} - \frac{c_v}{\rho} \end{bmatrix},$$

with, defining $M = EI\frac{d^2}{dr^2}\phi + c_d I \frac{d^2}{dr^2}\psi$, A has domain

$$\operatorname{dom}(A) = \left\{(\phi, \psi) \in X : \psi \in H_f(0, L);\ M \in \mathcal{H}^2(0, L) \text{ with } M(L) = \tfrac{d}{dr} M(L) = 0\right\}.$$

The operators B and D are clearly bounded operators from $\mathbb{R}$ to $\mathcal{Z}$.

Suppose the objective of the controller design is to reduce the effect of disturbances on the tip position:

$$y(t) = C_1 z(t) = w(L, t).$$

Sobolev's Inequality implies that evaluation at a point is bounded on $H_f(0, L)$ and so C_1 is bounded from $\mathcal{Z}$ to $\mathbb{R}$.

Define the bilinear form on $H_f(0, L) \times H_f(0, L)$

$$\sigma(\phi, \psi) = \int_0^L \frac{EI}{\rho} \frac{d^2}{dr^2}\phi(r) \frac{d^2}{dx^2}\psi(r)\, dx\,.$$

Define $\mathcal{V} = H_f(0, L) \times H_f(0, L)$ and define $a(\cdot, \cdot)$ on $\mathcal{V} \times \mathcal{V}$ by

$$a((\phi_1, \psi_1), (\phi_2, \psi_2)) = -\sigma(\psi_1, \phi_2) + \sigma(\phi_1 + \frac{c_d}{E}\,\psi_1,\, \psi_2) + \langle \frac{c_v}{\rho}\,\psi_1,\, \psi_2 \rangle$$

for $(\phi_i, \psi_i) \in V,\ i = 1, 2$. Then A can be defined by

$$\langle Ay, z \rangle_{\mathcal{V}^* \times \mathcal{V}} = -a(y, z), \quad \text{for } y,\ z \in \mathcal{V}.$$

The form $a(\cdot, \cdot)$ satisfies the inequality (7) and also (8) with $k < 0$ and so the operator A generates an exponentially stable semigroup on $\mathcal{Z}$.

Let $H_n \subset H_f(0, L)$ be a sequence of finite-dimensional subspaces formed by the span of n standard finite-element cubic $B-$spline approximations [33]. The approximating generator A_n on $\mathcal{Z}_n = H_n \times H_n$ is defined by the Galerkin approximation

$$\langle -A_n y_n, z_n \rangle = a(y_n, z_n), \quad \forall z_n,\ y_n \in \mathcal{Z}_n.$$

For all $\phi \in H_f(0, L)$ there exists a sequence $\phi_n \in H_n$ with $\lim_{n\to\infty} \|\phi_n - \phi\|_{H_f(0,L)} = 0$ [33]. It follows then from Theorem 3.7 and exponential stability of the original system that (A1) is satisfied and that the approximations are uniformly exponentially stabilizable (trivially, by the zero operator). The adjoint of A can be defined through $a(z, y)$ and (A1*) and uniform detectability also follow. Thus, Theorem 5.4 applies and convergence of the approximating feedback operators is obtained.

The corresponding series of finite-dimensional Riccati equations (29) were solved with $\gamma = 2.3$. Figure 14 compares the open and closed loop responses $w(L, t)$ to a disturbance consisting of a 100 second pulse, for the approximation with 10 elements. The feedback controller leads to a closed loop system which is able to almost entirely reject this disturbance. Figure 15 compares the open and closed loop responses to the periodic disturbance $\sin(\omega t)$ where ω is the first resonant frequency: $\omega = \min_i |\mathrm{Im}(\lambda_i(A_{10})|$. The resonance in the open loop is not present in the closed loop.

Figure 16 displays the convergence of the feedback gains predicted by Theorem 2.3. Since $\mathcal{Z}_n$ is a product space, the first and second components of the gains are displayed separately as displacement and velocity gains respectively.

In general, of course, the full state is not measured and the measured output is described by (25).

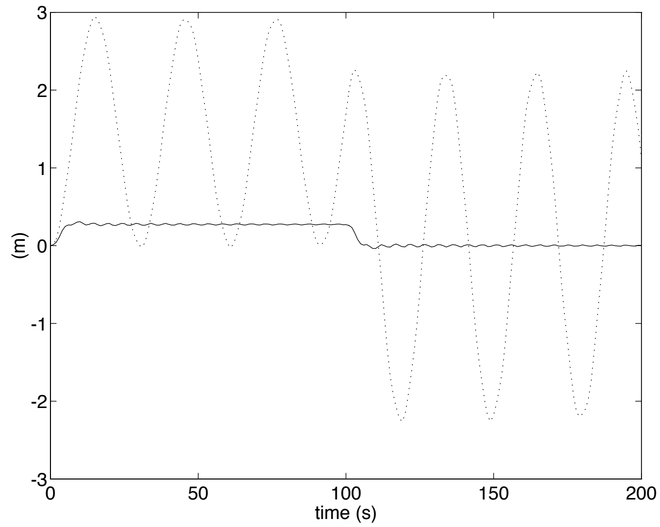


Figure 14: H_∞-state feedback for flexible slewing beam. Open (..) and closed loop (–) responses to a disturbance $d(t) = 1, t \leq 100s$: 10 elements.

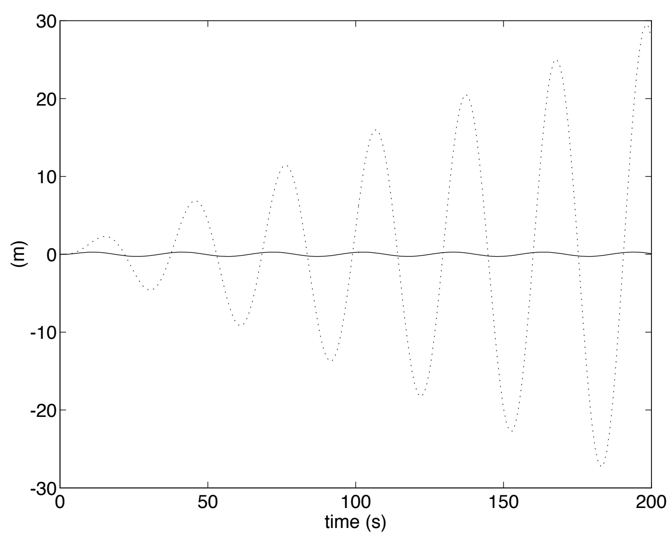


Figure 15: H_∞-state feedback for flexible slewing beam. Open (..) and closed loop (–) responses to a disturbance $d(t) = \sin(\omega t)$: 10 elements .

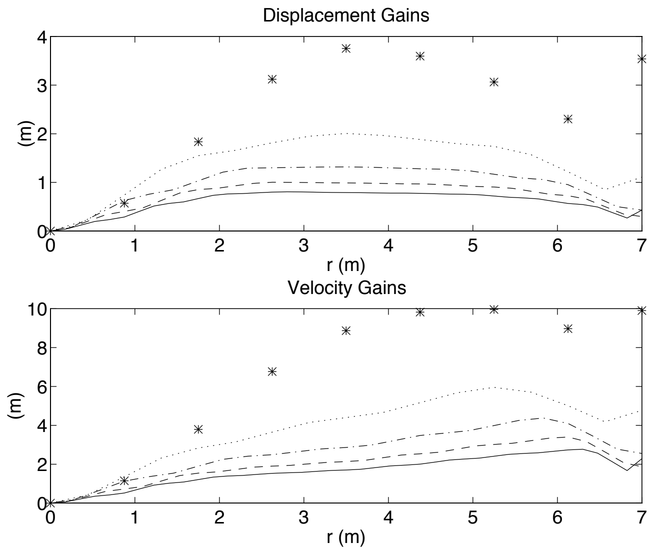


Figure 16: H_∞-state feedback for flexible slewing beam. Feedback gains for: 2 elements *, 4 elements ..., 6 elements _._., 8 elements _ _, 10 elements, __ are plotted. As predicted by the theory, the feedback operators are converging.

Theorem 5.7 *[4, 20] The system (23,24,25) is stabilizable with attenuation $\gamma > 0$ if and only if the following two conditions are satisfied:*

1. *There exists a nonnegative self-adjoint operator Π on $\mathcal{Z}$ satisfying the Riccati equation*

$$\left(A^*\Pi + \Pi A + \Pi\left(\frac{1}{\gamma^2}DD^* - BB^*\right)\Pi + C_1^*C_1\right)z = 0, \qquad \forall z \in D(A), \tag{30}$$

such that $A + (\frac{1}{\gamma^2}DD^* - BB^*)\Pi$ generates an exponentially stable semigroup on $\mathcal{Z}$;

2. *Define $\tilde{A} = A + \frac{1}{\gamma^2}DD^*\Pi$ and $K = B^*\Pi$. There exists a nonnegative self-adjoint operator $\tilde{\Sigma}$ on X satisfying the Riccati equation*

$$\left(\tilde{A}\tilde{\Sigma} + \tilde{\Sigma}\tilde{A}^* + \tilde{\Sigma}\left(\frac{1}{\gamma^2}K^*K - C_2^*C_2\right)\tilde{\Sigma} + DD^*\right)z = 0, \qquad \forall z \in D(A^*), \tag{31}$$

such that $\tilde{A} + \tilde{\Sigma}\left(\frac{1}{\gamma^2}K^*K - C_2^*C_2\right)$ *generates an exponentially stable semigroup on* X.

Moreover, if both conditions are satisfied, define $F = \tilde{\Sigma}C_2^*$ *and* $A_c = A + \frac{1}{\gamma^2}DD^*\Sigma - BK - FC_2$. *The controller with state-space description*

$$\begin{aligned} \dot{z}_c(t) &= A_c z_c(t) + F y_2(t) \\ u(t) &= -K z_c(t) \end{aligned} \tag{32}$$

stabilizes the system (23,24,25) with attenuation γ.

Condition (1) is simply the Riccati equation to be solved for the full-information state feedback controller. Condition (2) leads to an estimate of the state of the system controlled by $Kz(t)$. Unlike linear quadratic control, the design of the state-feedback and the estimator is coupled. Condition (2) above is more often written as the following two equivalent conditions:

a) There exists a nonnegative, self-adjoint operator Σ on $\mathcal{Z}$ satisfying the Riccati equation

$$\left(A\Sigma + \Sigma A^* + \Sigma\left(\frac{1}{\gamma^2}C_1^*C_1 - C_2^*C_2\right)\Pi + DD^*\right)z = 0, \qquad \forall z \in D(A^*) \tag{33}$$

such that $A + \Pi(\frac{1}{\gamma^2}C_1^*C_1 - C_2^*C_2)$ generates an exponentially stable semigroup on $\mathcal{Z}$, and

b) $r(\Sigma\Pi) < \gamma^2$ where r indicates the spectral radius.

In the presence of condition (1) in Theorem 5.7, condition (2) is equivalent to conditions (a) and (b). Also $\tilde{\Sigma} = (I - \frac{1}{\gamma^2}\Sigma\Pi)^{-1}\Sigma = \Sigma(I - \frac{1}{\gamma^2}\Pi\Sigma)^{-1}$. The advantage of replacing condition (2) by conditions (a) and (b) is numerical. The Riccati equation in (2) is coupled to the solution of (1) while the Riccati equation in (a) is independent of the solution of (1). This theoretical result has been extended to a class of control systems with unbounded control and observation operators [20].

For bounded control and observation operators, a complete approximation theory exists. Define a sequence of approximations on finite-dimensional spaces $\mathcal{Z}_N$, as for the full information case, with the addition of $C_{2n} = C_2|_{\mathcal{Z}_n}$.

Strong convergence of solutions Σ_n to Riccati equations approximating (33) will follow from Theorem 5.4 and a straightforward duality argument if (A1) and (A1*) hold, along with assumptions on uniform stabilizability of (A, D) and uniform detectability of (A, C_2). However, strong convergence of $\Pi_n \to \Pi$ and of $\Sigma_n \to \Sigma$ does not imply convergence (or even existence) of the inverse operator $(I - \frac{1}{\gamma^2}\Sigma_n \Pi_n)^{-1}$ so we cannot show controller convergence. Convergence of the solution $\tilde{\Sigma}_n$ to the estimation Riccati equation (31) can be proven.

Theorem 5.8 *[27, Thm. 3.5] Assume that (A1) and (A1*) hold, that (A_n, D_n) are uniformly stabilizable, and that (A_n, C_{2n}) are uniformly detectable. Let $\gamma > 0$ be such that the infinite-dimensional problem is stabilizable. Let N be large enough that approximations to the full-information problem are stabilizable with attenuation γ, and let Π_n indicate the solution to the ARE (29) for $n > N$. Define $K_n = B_n^* \Pi_n$ and $\tilde{A}_n = A_n + \frac{1}{\gamma^2} D_n D_n^* \Pi_n$.*

For sufficiently large n the Riccati equation

$$\tilde{A}_n \tilde{\Sigma}_n + \tilde{\Sigma}_n \tilde{A}_n^* + \tilde{\Sigma}_n \left(\frac{1}{\gamma^2} K_n^* K_n - C_{2n}^* C_{2n} \right) \tilde{\Sigma}_n + D_n D_n^* = 0 \qquad (34)$$

has a nonnegative, self-adjoint solution $\tilde{\Sigma}_n$. For such n there exist positive constants M_3 and ω_3 such that the semigroup $\tilde{S}_{n2}(t)$ generated by $\tilde{A}_n + \frac{1}{\gamma^2} \tilde{\Sigma}_n K_n^* K_n - \tilde{\Sigma}_n C_{2n}^* C_{2n}$ satisfies $\|\tilde{S}_{n2}(t)\| \leq M_3 e^{-\omega_3 t}$. Moreover, for each $z \in \mathcal{Z}$, $\tilde{\Sigma}_n P_n z \to \tilde{\Sigma} z$ as $n \to \infty$ and $F_n = \tilde{\Sigma}_n C_{2n}^*$ converges to $F = \tilde{\Sigma} C_2^*$ in norm.

Defining $A_{cn} = A_n + \frac{1}{\gamma^2} D_n D_n^* \Sigma - B_n K_n - F_n C_{2n}$, Theorems 5.4 and 5.8 imply convergence of the controllers

$$\begin{aligned} \dot{z}_c(t) &= A_{cn} z_c(t) + F_n y_2(t) \\ u(t) &= -K_n z_c(t) \end{aligned} \qquad (35)$$

to the infinite-dimensional controller (32) in the gap topology. The same assumptions imply convergence of the plants which leads to the following result.

Theorem 5.9 *[27, Thm. 3.6] Let γ be such that the infinite-dimensional system is stabilizable with attenuation γ. Assume that (A1) and (A1*) hold, that (A, B) and (A, D) are uniformly stabilizable, and that (A, C_1) and (A, C_2) are uniformly detectable. Then the finite-dimensional controllers (35) converge in the gap topology to the infinite-dimensional controller (32). For sufficiently large N, the finite-dimensional controllers (35) stabilize the infinite-dimensional system and provide γ-attenuation.*

Convergence of the optimal attenuation also holds for the output feedback problem.

Corollary 5.10 *[27, Thm. 3.7] Let $\hat{\gamma}$ indicate the optimal disturbance attenuation for the output feedback problem (23,24,25), and similarly indicate the optimal attenuation for the approximating problems by $\hat{\gamma}_n$. With the same assumptions as for Theorem 5.9,*

$$\lim_{n\to\infty} \hat{\gamma}_n = \hat{\gamma}.$$

6 Summary

For most practical systems, an approximation to the partial differential equation must be used in controller design. Similarly, control for delay-differential equations often proceeds by using an approximation, although delay-differential equations were not discussed directly in this paper. This has the advantage of making available the vast array of tools available for design of controllers for finite-dimensional systems. Since the underlying model is infinite-dimensional, this process is not entirely straightforward. However, there are a number of tools and techniques available for satisfactory controller design. This article has presented an overview of the main issues surrounding controller design for these systems. The key point is that the controller must control the original system. Sufficient conditions for satisfactory linear-quadratic controller design and H_∞-controller design were presented. Uniform stabilizability and detectability, along with convergence of the adjoint systems, are assumptions not required in simulation but key to obtaining satisfactory performance of a controller designed using a finite-dimensional approximation. There are results guaranteeing these assumptions for many problems. However, for problems where these assumptions are not known and proof is not feasible, the approximating systems should be checked numerically for uniform stabilizability and detectability. One test is to verify that the sequence of controlled systems is uniformly exponentially stable.

This paper only discussed systems with bounded control and observation operators B and C. Introducing a better model for an actuator sometimes converts a control system with an unbounded control operator to a more complex model with a bounded operator, and similarly for sensor models. For some systems, though, the most natural model leads to unbounded operators. There are considerably fewer results for controller design for these systems.

However, results do exist for linear-quadratic control of parabolic problems, such as diffusion [2, 22].

Once a suitable approximation scheme, and controller synthesis technique, has been chosen, the problem of computation remains. The primary difficulty is solving the large Riccati equations that arise in linear-quadratic and H_∞-controller design. For problems where an approximation of suitable accuracy is of relatively low order (less than about a hundred) direct methods can be used to solve the Riccati equations. However, for larger problems, an iterative method is required. Probably the most popular method for the Riccati equations that arise in linear-quadratic control is the Newton-Kleinman method [21] and its variants - see for example, [10, 32]. This method is guaranteed to converge, and has quadratic convergence. However, calculation of an H_∞-controller corresponds to calculation of a saddle point, not a minimum as in the case of linear-quadratic control. Suitable methods for design of H_∞-controllers for large-scale systems is an open problem at the present time.

References

[1] M. J. Balas. Active control of flexible systems. *J. Optim. Theory Appl.*, 23(3):415–436, 1976.

[2] H.T. Banks and K. Ito. Approximation in LQR problems for infinite-dimensional systems with unbounded input operators. *J. Math. Systems Estim. Control*, 7(1):1–34, 1997.

[3] H.T. Banks and K. Kunisch. The linear regulator problem for parabolic systems. *SIAM J. Control and Optim.*, 22(5):684–698, 1984.

[4] A. Bensoussan and P. Bernhard. On the standard problem of H_∞-optimal control for infinite dimensional systems. In *Identification and Control in Systems Governed by Partial Differential Equations*, pages 117–140. SIAM, 1993.

[5] J.A. Burns, K. Ito, and G.Propst. On non-convergence of adjoint semigroups for control systems with delays. *SIAM J. Control and Optim.*, 26(6):1442–1454, 1988.

[6] J.A. Burns, E. W. Sachs, and L. Zietsman. Mesh independence of Kleinman-Newton iterations for Riccati equations in Hilbert space. *SIAM J. Control and Optim.*, 47(5):2663–2692, 2008.

[7] R. F. Curtain and K. A Morris. Transfer functions of distributed parameter systems: A tutorial. *Automatica*, 45(5):1101–1116, 2009.

[8] R. F. Curtain and H. Zwart. *An Introduction to Infinite-Dimensional Linear Systems Theory*. Springer Verlag, Berlin, 1995.

[9] C. A. Desoer and M. Vidyasagar. *Feedback Systems: input-output properties*. Academic Press, 1975.

[10] F. Feitzinger, T. Hylla, and E. W. Sachs. Inexact Kleinman-Newton method for Riccati equations. *SIAM J. Matrix Anal. Appl.*, 21(2):272–288, 2009.

[11] C. Foias, H. H. Ozbay, and Tannenbaum. *Robust Control of Infinite Dimensional Systems: Frequency domain methods*. Springer Verlag, 1995. LNCIS vol. 209.

[12] J.S. Gibson. The Riccati integral equations for optimal control problems on Hilbert spaces. *SIAM J. Control and Optim.*, 17(4):637–665, 1979.

[13] J.S. Gibson. A note on stabilization of infinite-dimensional linear oscillators by compact linear feedback. *SIAM J. Control and Optim.*, 18(3):311–316, 1980.

[14] R. B. Guenther and J. W. Lee. *Partial Differential Equations of Mathematical Physics and Integral Equations*. Prentice-Hall, 1988.

[15] K. Ito. Strong convergence and convergence rates of approximating solutions for algebraic Riccati equations in Hilbert spaces. In W. Schappacher F. Kappel, K. Kunisch, editor, *Distributed Parameter Systems*. Springer-Verlag, 1987.

[16] K. Ito and K. A. Morris. An approximation theory for solutions to operator Riccati equations for H_∞ control. *SIAM J. Control and Optim.*, 36(1):82–99, 1998.

[17] B. Jacob and K. A. Morris. Second-order systems with acceleration measurements. *preprint*, 2009.

[18] C. A. Jacobson and C. N. Nett. Linear state-space systems in infinite-dimensional space: the role and characterization of joint stabilizability/detectability. *IEEE Trans. Auto. Cont.*, 33(6):541–549, 1988.

[19] K. Kashima and Y. Yamamoto. On standard H_∞ control problems for systems with infinitely many unstable poles. *Systems Control Lett.*, 57(4):309–314, 2008.

[20] B.V. Keulen. *$H^\infty-$Control for Distributed Parameter Systems: A State-Space Approach.* Birkhauser, Boston, 1993.

[21] D. Kleinman. On an iterative technique for Riccati equation computations. *IEEE Trans. Auto. Cont.*, 13:114–115, 1968.

[22] I. Lasiecka and R. Triggiani. *Control Theory for Partial Differential Equations: Continuous and Approximation Theories*, volume I. Cambridge University Press, 2000.

[23] I. Lasiecka and R. Triggiani. *Control Theory for Partial Differential Equations: Continuous and Approximation Theories*, volume II. Cambridge University Press, 2000.

[24] H. Logemann. Stabilization and regulation of infinite-dimensional systems using coprime factorizations. In R. F. Curtain, editor, *Analysis and Optimization of systems: State and frequency domain approaches for infinite-dimensional systems*, pages 102–139. Springer-Verlag, 1992.

[25] H. L. Logemann. Circle criteria, small-gain conditions and internal stability for infinite-dimensional systems. *Automatica*, 27:677–690, 1991.

[26] H. L. Logemann, E.P. Ryan, and S. Townley. Integral control of linear systems with actuator nonlinearities: lower bounds for the maximal regulating gain. *IEEE Trans. Auto. Cont.*, 44:1315–1319, 1999.

[27] K. A. Morris. H_∞ output feedback control of infinite-dimensional systems via approximation. *Systems and Control Letters*, 44(3):211–217, 2001.

[28] K.A. Morris. Convergence of controllers designed using state-space methods. *IEEE Trans. Auto. Cont.*, 39:2100–2104, 1994.

[29] K.A. Morris. Design of finite-dimensional controllers for infinite-dimensional systems by approximation. *Jour. of Mathematical Systems, Estimation and Control*, 4(1-30), 1994.

[30] K.A. Morris. *An Introduction to Feedback Controller Design*. Harcourt-Brace Ltd., 2001.

[31] K.A. Morris. Convergence of LQ-optimal actuator locations for systems governed by partial differential equations. *IEEE Trans. Auto. Cont.*, to appear.

[32] K.A. Morris and C. Navasca. Approximation of low rank solutions for linear quadratic feedback control of partial differential equations. *Comp. Opt. & App.*, to appear, 2009.

[33] N. Ottosen and H. Petersson. *Introduction to the Finite Element Method*. Prentice-Hall, 1992.

[34] A. Pazy. *Semigroups of Linear Operators and Applications to Partial Differential Equations*. Springer-Verlag, 1983.

[35] R. R. Rebarber and G. Weiss. Internal model-based tracking and disturbance rejection for stable well-posed systems. *Automatica*, 39(9):1555–1569, 2003.

[36] D. L. Russell. Linear stabilization of the linear oscillator in hilbert space. *Jour. of Math. Anal. and App.*, 25(3):663–675, 1969.

[37] R.E. Showalter. *Hilbert Space Methods for Partial Differential Equations*. Pitman Publishing Ltd., 1977.

[38] D. H. Towne. *Wave Phenomena*. Dover Publications, 1988.

[39] M. Vidyasagar. *Control System Synthesis: A Factorization Approach*. MIT Press, 1985.

[40] G. Zames and A. El-Sakkary. Unstable systems and feedback : the gap metric. In *Proc. Allerton Conf.*, pages 380–385, 1980.

[41] K. Zhou, J. C. Doyle, and K. Glover. *Robust and Optimal Control*. Prentice-Hall, Englewood Cliffs, NJ, 1996.

[42] S. Q. Zhu. Graph topology and gap topology for unstable systems. *IEEE Trans. Auto. Cont.*, 34:848–855, 1989.

[43] B. J. Zimmer, S. P. Lipshitz, K.A. Morris, J. Vanderkooy, and E.E E. E. Obasi. An improved acoustic model for active noise control in a duct. *ASME Journal of Dynamic Systems, Measurement and Control*, 125(3):382–395, 2003.