Essentially normal operators

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Abstract. This is a survey of essentially normal operators and related developments. There is an overview of Weyl–von Neumann theorems about expressing normal operators as diagonal plus compact operators. Then we consider the Brown–Douglas–Fillmore theorem classifying essentially normal operators. Finally we discuss almost commuting matrices, and how they were used to obtain two other proofs of the BDF theorem.

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1. Introduction

Problem 4 of Halmos's *Ten problems in Hilbert space* [**26**] asked whether every normal operator is the sum of a diagonal operator and a compact operator. I believe that this was the first of the ten problems to be solved. Indeed two solutions were independently produced by Berg [**6**] and Sikonia [**39**] almost immediately after dissemination of the question. But that is only the beginning of the story, as, like many of Paul's problems, the answer to the question is just a small step in the bigger picture.

The subsequent discussion in Halmos's article goes in several directions. In particular, he discusses operators of the form normal plus compact. It is not at all clear, *a priori*, if this set is even norm closed. It turned out that the characterization of this class by other invariants is very interesting.

It is immediately clear that if T is normal plus compact, then $T^*T - TT^*$ is compact. Operators with this latter property are called *essentially normal*. Not all essentially normal operators are normal plus compact. For example, the unilateral shift S acts on a basis $\{e_n : n \geq 0\}$ by $Se_n = e_{n+1}$. It evidently satisfies

$$S^*S - SS^* = I - SS^* = e_0 e_0^*,$$

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which is the rank one projection onto $\mathbb{C}e_0$. Thus the unilateral shift is essentially normal. However it has non-zero Fredholm index:

$$\operatorname{ind} S = \dim\ker S - \dim\ker S^* = -1.$$

For a normal operator N, one has $\ker N = \ker N^*$. Thus if N is Fredholm, then $\operatorname{ind} N = 0$. Index is invariant under compact perturbations, so the same persists for normal plus compact operators.

Brown, Douglas and Fillmore **[9]** raised the question of classifying essentially normal operators. The answer took them from a naive question in operator theory to the employment of new techniques from algebraic topology in the study of C*-algebras. They provided a striking answer to the question of which essentially normal operators are normal plus compact. They are precisely those essentially normal operators with the property that $\operatorname{ind}(T - \lambda I) = 0$ for every $\lambda \notin \sigma_e(T)$, namely the index is zero whenever it is defined. This implies, in particular, that the set of normal plus compact operators is norm closed.

Had they stopped there, BDF might have remained 'just' a tour de force that solved an interesting question in operator theory. However they recognized that their methods had deeper implications about the connection between topology and operator algebras. They defined an invariant $\operatorname{Ext}(\mathfrak{A})$ for any C*-algebra $\mathfrak{A}$, and determined nice functorial properties of this object in the case of separable, commutative C*-algebras. They showed that Ext has a natural pairing with the topological K-theory of Atiyah **[4]** which makes Ext a K-homology theory. This opened up a whole new world for C*-algebraists, and a new game was afoot.

At roughly the same time, George Elliott **[18]** introduced a complete algebraic invariant for AF C*-algebras. These C*-algebras, introduced by Bratteli **[8]**, are defined by the property that they are the closure of an (increasing) union of finite dimensional sub-algebras. It was soon recognized **[19]** that this new invariant is the K_0 functor from ring theory. A very striking converse to Elliott's theorem was found by Effros, Handelmann and Shen **[17]** which characterized those groups which arise as the K_0 group of an AF algebra.

The upshot was that two very different results almost simultaneously seeded the subject of C*-algebras with two new topological tools that provide interesting new invariants, namely Ext and K_0. These results created tremendous excitement, and launched a program which continues to this day to classify amenable C*-algebras. It revitalized the subject, and has led to a sophisticated set of tools which describe and distinguish many new algebras. It is fair to say that the renaissance of C*-algebras was due to these two developments. Indeed, not only are they related by the spirit of K-theory, they are in fact two sides of the same coin. Kasparov **[30]** introduced his bivariant KK-theory shortly afterwards which incorporates the two theories into one.

It is not my intention to survey the vast literature in C*-algebras which has developed as a consequence of the introduction of K-theory. I mention it to highlight the fallout of the pursuit of a natural problem in operator theory by three very insightful investigators. I will limit the balance of this article to the original operator theory questions, which have a lot of interest in their own right.

2. Weyl-von Neumann Theorems

In 1909, Hermann Weyl [**46**] proved that every self-adjoint bounded operator A on a *separable* Hilbert space can be written as $A = D + K$ where D is a diagonal operator with respect to some orthonormal basis and K is compact. Hilbert's spectral theorem for Hermitian operators says, as formulated by Halmos [**27**], that every Hermitian operator on Hilbert space is unitarily equivalent to a multiplication operator M_φ on $L^2(\mu)$, where φ is a bounded real-valued measurable function. The starting point of a proof of Weyl's theorem is the observation that if f is any function in $L^2(\mu)$ supported on $\varphi^{-1}\big([t, t+\varepsilon)\big)$, then f is almost an eigenvector in the sense that $\|M_\varphi f - tf\| < \varepsilon\|f\|$. One can carefully extract an orthonormal basis for $L^2(\mu)$ consisting of functions with increasingly narrow support.

To make this more precise, suppose for convenience that A has a cyclic vector. Then the spectral theorem produces a measure μ on the spectrum $\sigma(A) \subset \mathbb{R}$ so that A is identified with M_x on $L^2(\mu)$. Let P_n be the span of the characteristic functions of diadic intervals of length 2^{-n}. Then the previous observation can be used to show that $\|P_n A - AP_n\| < 2^{-n}$. So a routine calculation shows that A is approximated within 2^{1-n} by the operator

$$D_n = P_n A P_n + \sum_{k \geq n} (P_{k+1} - P_k) A (P_{k+1} - P_k),$$

and $A - D_n$ is compact. The operator D_n is a direct sum of finite rank self-adjoint operators, and so is diagonalizable—providing the desired approximant.

Weyl observed that if A and B are two Hermitian operators such that $A - B$ is compact, then the limit points of the spectrum of A and B must be the same. We now interpret this by saying that the essential spectra are equal, $\sigma_e(A) = \sigma_e(B)$, where $\sigma_e(\cdot)$ denotes the spectrum of the image in the Calkin algebra $\mathcal{B}(\mathcal{H})/\mathfrak{K}$. Von Neumann [**45**] established the converse: if two Hermitian operators have the same essential spectrum, then they are unitarily equivalent modulo a compact perturbation.

Halmos's questions asks for an extension to normal operators. It seems to require a new approach, because the trick of compressing a Hermitian operator to the range of an almost commuting projection $P_{k+1} - P_k$ yields a Hermitian matrix, but the same argument fails for normal operators. Also the spectrum is now a subset of the plane. David Berg [**6**] nevertheless answered Halmos's question by adapting this method. Sikonia gave a similar proof at the same time in his doctoral thesis (see [**39**]). Other proofs came quickly afterwards (for example Halmos [**27**]).

A proof that does the job simultaneously for a countable family of commuting Hermitian operators $\{A_i\}$ works by building a single Hermitian operator A so that $\mathrm{C}^*(A)$ contains every A_i. To accomplish this, first observe that the spectral theorem shows that $\mathrm{C}^*(\{A_i\})$ is contained in a commutative C*-algebra spanned by a countable family $\{E_j\}$ of commuting projections. Consider the Hermitian operator $A = \sum_{j\ge 1} 3^{-j} E_j$. It is easy to see that

$$\tfrac{1}{3}E_1 \le E_1 A \le \tfrac{1}{2}E_1 \quad \text{and} \quad 0 \le E_1^\perp A \le \tfrac{1}{6}E_1^\perp .$$

Thus it follows that the spectrum of A is contained in $[0,1/6]\cup[1/3,1/2]$, and that the spectral projection for $[1/3,1/2]$ is E_1. Similarly, each projection E_j belongs to $\mathrm{C}^*(A)$. It follows that each A_i is a continuous function of A. Diagonalizing A modulo compacts then does the same for each A_i, although one cannot control the norm of the compact perturbations for more than a finite number at a time.

At this point, we introduce a few definitions to aid in the discussion.

Definition 2.1. Let $\mathfrak{J}$ be a normed ideal of $\mathcal{B}(\mathcal{H})$. Two operators A and B are *unitarily equivalent modulo* $\mathfrak{J}$ if there is a unitary U so that $A - UBU^* \in \mathfrak{J}$. We say that A and B are *approximately unitarily equivalent modulo* $\mathfrak{J}$ if there is a sequence of unitaries U_k so that $A - U_k B U_k^* \in \mathfrak{J}$ and

$$\lim_{k\to\infty} \|A - U_k B U_k^*\|_{\mathfrak{J}} = 0.$$

We write $A \sim_{\mathfrak{J}} B$. When $\mathfrak{J} = \mathcal{B}(\mathcal{H})$, we simply say that A and B are *approximately unitarily equivalent* and write $A \sim_a B$.

A careful look at Weyl's proof shows that the perturbation will lie in certain smaller ideals, the Schatten classes $\mathfrak{S}_p$ with norm $\|K\|_p = \mathrm{Tr}(|K|^p)^{1/p}$, provided that $p > 1$; and this norm can also be made arbitrarily small. Kuroda **[32]** improved on this to show that one can obtain a small perturbation in any unitarily invariant ideal $\mathfrak{J}$ that strictly contains the trace class operators. So any Hermitian operator is approximately unitarily equivalent modulo $\mathfrak{J}$ to a diagonalizable operator. Berg's proof works in a similar way via a process of dividing up the plane, and actually yields small perturbations in $\mathfrak{S}_p$ for $p > 2$. Likewise, for a commuting n-tuple, one can obtain small perturbations in $\mathfrak{S}_p$ for $p > n$. It is natural to ask whether this is sharp. Elementary examples show that one cannot obtain perturbations in $\mathfrak{S}_p$ when $p < n$.

In the case of $n = 1$, there is an obstruction found by Kato **[31]** and Rosenblum **[37]**. If the spectral measure of A is not singular with respect to Lebesgue measure, then there is no trace class perturbation which is diagonal. In particular, M_x on $L^2(0,1)$ is such an operator. For $n \ge 2$, Voiculescu **[41, 42]** showed that every commuting n-tuple of Hermitian operators is approximately unitarily equivalent to a diagonalizable n-tuple modulo the Schatten class $\mathfrak{S}_n$. (See **[14]** for an elementary argument.) Moreover, Voiculescu identified a somewhat smaller ideal $\mathfrak{S}_n^-$ which provides an obstruction when the n-tuple has a spectral measure that is not singular with respect to Lebesgue measure on $\mathbb{R}^n$. Bercovici and Voiculescu **[5]** strengthened this to the analogue of Kuroda's theorem, showing that if a unitarily invariant ideal is not included in $\mathfrak{S}_n^-$, then a small perturbation to a diagonal operator is possible.

The ideas involved in Voiculescu's work mentioned above build on a very important theorem of his that preceded these results, and had a direct bearing on the work of Brown, Douglas and Fillmore and on subsequent developments in C*-algebras. This is known as Voiculescu's Weyl-von Neumann Theorem [**40**]. Rather than state it in full generality, we concentrate on some of its major corollaries. The definition of approximate unitary equivalence can readily be extended to two maps from a C*-algebra $\mathfrak{A}$ into $\mathcal{B}(\mathcal{H})$: say $\rho \sim_a \sigma$ for two maps ρ and σ if there is a sequence of unitary operators U_n such that

$$\lim_{n\to\infty} \|\rho(a)U_n - U_n\sigma(a)\| = 0 \quad \text{for all} \quad a \in \mathfrak{A}.$$

One similarly defines approximate unitary equivalence relative to an ideal.

Voiculescu showed that if $\mathfrak{A}$ is a separable C*-algebra acting on $\mathcal{H}$ and ρ is a $*$-representation which annihilates $\mathfrak{A}\cap\mathfrak{K}$, then $\mathrm{id} \sim_{\mathfrak{K}} \mathrm{id}\oplus\rho$, where id is the identity representation of $\mathfrak{A}$. Let π denote the quotient map of $\mathcal{B}(\mathcal{H})$ onto the Calkin algebra $\mathcal{B}(\mathcal{H})/\mathfrak{K}$. Voiculescu's result extends to show for two representations ρ_1 and ρ_2, one has $\rho_1 \sim_a \rho_2$ if and only if $\rho_1 \sim_{\mathfrak{K}} \rho_2$. In particular, this holds provided that

$$\ker \rho_1 = \ker \pi\rho_1 = \ker \rho_2 = \ker \pi\rho_2.$$

If N is normal, then it may have countably many eigenvalues of finite multiplicity which do not lie in the essential spectrum. However they must be asymptotically close to the essential spectrum. One can peel off a finite diagonalizable summand, and make a small compact perturbation on the remainder to move the other eigenvalues into $\sigma_e(N)$. One now has a normal operator N' with $\sigma(N') = \sigma_e(N')$. If one applies Voiculescu's Theorem to $\mathrm{C}^*(N')$, one recovers the Weyl-von Neumann-Berg Theorem.

Another consequence of this theorem was a solution to Problem 8 of Halmos's ten problems, which asked whether the reducible operators are dense in $\mathcal{B}(\mathcal{H})$. One merely takes any representation ρ of $\mathrm{C}^*(T)$ which factors through $\mathrm{C}^*(T) + \mathfrak{K}/\mathfrak{K}$ to obtain $T \sim_a T \oplus \rho(T)$. Moreover, one can show that $\mathrm{id} \sim_{\mathfrak{K}} \sigma$ where σ is a countable direct sum of irreducible representations.

The implications of Voiculescu's theorem for essentially normal operators will be considered in the next section. We mention here that a very insightful treatment of Voiculescu's theorem is contained in Arveson's paper [**3**]. In particular, it provides a strengthening of the results for normal operators. Hadwin [**25**] contains a further refinement which shows that $\rho_1 \sim_{\mathfrak{K}} \rho_2$ if and only if $\operatorname{rank} \rho_1(a) = \operatorname{rank} \rho_2(a)$ for all $a \in \mathfrak{A}$. All of these ideas are treated in Chapter 2 of [**15**].

3. Essentially normal operators

We return to the problem of classifying essentially normal operators. Let T be essentially normal. Then $t = \pi(T)$ is a normal element of the Calkin algebra. So $\mathrm{C}^*(t) \simeq \mathrm{C}(X)$ where $X = \sigma(t) = \sigma_e(T)$. This determines a $*$-monomorphism τ of $\mathrm{C}(X)$ into $\mathcal{B}(\mathcal{H})/\mathfrak{K}$ determined by $\tau(z) = t$. Evidently τ determines T up to a compact perturbation. Two essentially normal operators T_1 and T_2 are unitarily equivalent modulo $\mathfrak{K}$ if and only if $\sigma_e(T_1) = \sigma_e(T_2) =: X$ and the associated monomorphisms τ_1 and τ_2 of $\mathrm{C}(X)$ are strongly unitarily equivalent, meaning that there is a unitary U so that $\operatorname{ad}\pi(U)\tau_1 = \tau_2$. (The weak version would allow equivalence by a unitary in the Calkin algebra. This turns out to be equivalent for commutative C*-algebras.) A monomorphism τ is associated to an *extension* of the compact operators. Let $\mathfrak{E} = \pi^{-1}(\tau(\mathrm{C}(X)) = \mathrm{C}^*(T) + \mathfrak{K}$. Then $\tau^{-1}\pi$ is a $*$-homomorphism of the C*-algebra $\mathfrak{E}$ onto $\mathrm{C}(X)$ with kernel $\mathfrak{K}$. We obtain the short exact sequence

$$0 \to \mathfrak{K} \to \mathfrak{E} \to \mathrm{C}(X) \to 0.$$

For example, the unilateral shift S is unitarily equivalent to the Toeplitz operator of multiplication by z on H^2. So one readily sees that $\mathrm{C}^*(S)$ is unitarily equivalent to the Toeplitz C*-algebra

$$\mathfrak{T}(\mathbb{T}) = \{T_f + K : f \in \mathrm{C}(\mathbb{T}) \text{ and } K \in \mathfrak{K}\}.$$

The map $\tau_1(f) = \pi(T_f)$ is a monomorphism of $\mathrm{C}(\mathbb{T})$ into the Calkin algebra. It is not hard to use the Fredholm index argument to show that this extension does not split; i.e. there is no $*$-monomorphism of $\mathrm{C}(\mathbb{T})$ into $\mathfrak{T}$ taking z to a compact perturbation of T_z. Therefore this extension is not equivalent to the representation of $\mathrm{C}(\mathbb{T})$ on $L^2(\mathbb{T})$ given by $\tau_2(f) = \pi(M_f)$ where M_f is the multiplication operator.

Turning the problem around, Brown, Douglas and Fillmore consider the class of all $*$-monomorphisms of $\mathrm{C}(X)$ into $\mathcal{B}(\mathcal{H})/\mathfrak{K}$ for any compact metric space X; or equivalently they consider all extensions of $\mathfrak{K}$ by $\mathrm{C}(X)$. Two extensions are called equivalent if the corresponding monomorphisms are strongly unitarily equivalent. The collection of all equivalence classes of extensions of $\mathrm{C}(X)$ is denoted by $\mathrm{Ext}(X)$. One can turn this into a commutative semigroup by defining $[\tau_1]+[\tau_2] = [\tau_1 \oplus \tau_2]$, which uses the fact that we can identify the direct sum $\mathcal{H}\oplus\mathcal{H}$ of two separable Hilbert spaces with the original space $\mathcal{H}$.

More generally, one can define $\mathrm{Ext}(\mathfrak{A})$ for any C*-algebra. The theory works best if one sticks to separable C*-algebras. Among these, things work out particularly well when $\mathfrak{A}$ is nuclear.

The Weyl–von Neumann–Berg Theorem is exactly what is needed to show that $\mathrm{Ext}(X)$ has a zero element. An extension is *trivial* if it splits, i.e. there is a $*$-monomorphism σ of $\mathrm{C}(X)$ into $\mathcal{B}(\mathcal{H})$ so that $\tau = \pi\sigma$. The generators of $\sigma(\mathrm{C}(X))$ can be perturbed by compact operators to commuting diagonal operators. The converse of von Neumann is adapted to show that any two trivial extensions are equivalent. An elementary argument can be used to construct approximate eigenvectors. Repeated application yields $\tau \sim_{\mathfrak{K}} \tau\oplus\sigma$, where σ is a trivial extension. So the equivalence class of all trivial extensions forms a zero element for $\mathrm{Ext}(X)$.

In fact, $\mathrm{Ext}(X)$ is a group. Brown, Douglas and Fillmore gave a complicated proof, which required a number of topological lemmas. The proof was significantly simplified by Arveson [**2**] by pointing out the crucial role of completely positive maps. The map τ may be lifted to a completely positive unital map σ into $\mathcal{B}(\mathcal{H})$, meaning that $\tau = \pi\sigma$. Then the Naimark dilation theorem dilates this map to a

$*$-representation ρ of $\mathrm{C}(X)$ on a larger space $\mathcal{K} \supset \mathcal{H}$; say

$$\rho(f) = \begin{bmatrix} \tau(f) & \rho_{12}(f) \\ \rho_{21}(f) & \rho_{22}(f) \end{bmatrix}.$$

Since the range of τ commutes modulo compacts, it is not hard to see that the ranges of ρ_{12} and ρ_{21} consist of compact operators. It follows that $\pi\rho_{22}$ is a $*$-homomorphism of $\mathrm{C}(X)$. The map $\pi\rho_{22} \oplus \tau_0$, where τ_0 is any trivial extension, yields an inverse.

More generally, Choi and Effros **[12]** showed that $\mathrm{Ext}(\mathfrak{A})$ is a group whenever $\mathfrak{A}$ is a separable nuclear C*-algebra. The argument uses nuclearity and the structure of completely positive maps to accomplish the lifting. The dilation follows from Stinespring's theorem for completely positive maps. Voiculescu's Theorem provides the zero element consisting of the class of trivial extensions. (See Arveson **[3]** where all of this is put together nicely.) When $\mathfrak{A}$ is not nuclear, Anderson **[1]** showed that $\mathrm{Ext}(\mathfrak{A})$ is generally not a group. Recently Haagerup and Thorbjornsen **[24]** have shown that Ext of the reduced C*-algebra of the free group $\mathbb{F}_2$ is not a group.

Next we observe that Ext is a covariant functor from the category of compact metric spaces with continuous maps into the category of abelian groups. Suppose that $p : X \to Y$ is a continuous map between compact metric spaces, and τ is an extension of $\mathrm{C}(X)$. Build an extension of $\mathrm{C}(Y)$ by fixing a trivial extension σ_0 in $\mathrm{Ext}(Y)$ and defining

$$\sigma(f) = \tau(f \circ p) \oplus \sigma_0(f) \quad \text{for all} \quad f \in \mathrm{C}(Y).$$

So far, we have seen little topology, although the original BDF proof used more topological methods to establish these facts. Now we discuss some of those aspects which are important for developing Ext as a homology theory. If A is a closed subset of X, j is the inclusion map and p is the quotient map of X onto X/A, then

$$\mathrm{Ext}(A) \xrightarrow{j_*} \mathrm{Ext}(X) \xrightarrow{p_*} \mathrm{Ext}(X/A)$$

is exact. Ext also behaves well with respect to projective limits of spaces. If X_n are compact metric spaces and $p_n : X_{n+1} \to X_n$ for $n \geq 1$, define $X = \operatorname{proj\,lim} X_n$ to be the subset of $\prod_{n\geq 1} X_n$ consisting of sequences (x_n) such that $p_n(x_{n+1}) = x_n$ for all $n \geq 1$. There are canonical maps $q_n : X \to X_n$ so that $q_n = p_n q_{n+1}$. One can likewise define $\operatorname{proj\,lim} \mathrm{Ext}(X_n)$. Since q_{n*} defines a compatible sequence of homomorphisms of $\mathrm{Ext}(X)$ into $\mathrm{Ext}(X_n)$, one obtains a natural map

$$\kappa : \mathrm{Ext}(\operatorname{proj\,lim} X_n) \longrightarrow \operatorname{proj\,lim} \mathrm{Ext}(X_n).$$

The key fact is that this map is always surjective. Moreover, it is an isomorphism when each X_n is a finite set. This latter fact follows by noting that when each X_n is finite, X is totally disconnected. From our discussion of the Weyl–von Neumann Theorem, $\mathrm{C}(X)$ is generated by a single self-adjoint element, and every extension is diagonalizable and hence trivial. So $\mathrm{Ext}(X) = \{0\}$.

In **[10]**, it is shown that any covariant functor from compact metric spaces into abelian groups satisfying the properties established in the previous paragraph is a homotopy invariant. That is, if f and g are homotopic maps from X to Y, then $f_* = g_*$. In particular, if X is contractible, then $\mathrm{Ext}(X) = \{0\}$.

There is a pairing between $\mathrm{Ext}(X)$ and $K^1(X)$ which yields a map into the integers based on Fredholm index. First consider the group $\pi^1(X) = \mathrm{C}(X)^{-1}/\mathrm{C}(X)_0^{-1}$. An element $[f]$ of $\pi^1(X)$ is a homotopy class of invertible functions on X. Thus if $[\tau] \in \mathrm{Ext}(X)$, one can compute $\mathrm{ind}\,\tau(f)$ independent of the choice of representatives. Moreover, this determines a homomorphism $\gamma[\tau]$ of $\pi^1(X)$ into $\mathbb{Z}$. Hence we have defined a homomorphism

$$\gamma : \mathrm{Ext}(X) \to \mathrm{Hom}(\pi^1(X), \mathbb{Z}).$$

Similarly, by considering the induced monomorphisms of $\mathfrak{M}_n(\mathrm{C}(X))$ into $\mathcal{B}(\mathcal{H}^{(n)})/\mathfrak{K}$, we can define an analogous map

$$\gamma^n : \mathrm{Ext}(X) \to \mathrm{Hom}(\mathrm{GL}_n(X)/\,\mathrm{GL}_n(X)_0, \mathbb{Z}).$$

The direct limit of the sequence of groups $\mathrm{GL}_n(X)/\,\mathrm{GL}_n(X)_0$ is called $K^1(X)$. The inductive limit of γ^n is a homomorphism $\gamma^\infty : \mathrm{Ext}(X) \to \mathrm{Hom}(K^1(X), \mathbb{Z})$. This is the *index map*, and is the first pairing of Ext with K-theory.

When X is a planar set, an elementary argument shows that $\pi^1(X)$ is a free abelian group with generators $[z - \lambda_i]$ where one chooses a point λ_i in each bounded component of $\mathbb{C} \setminus X$. An extension τ is determined by the essentially normal element $t = \tau(z) \in \mathcal{B}(\mathcal{H})/\mathfrak{K}$. The index map is given by

$$\gamma([\tau])([z - \lambda_i]) = \mathrm{ind}\,(t - \lambda_i).$$

The Brown–Douglas–Fillmore Theorem classifies essentially normal operators by showing that when X is planar, the map γ is an isomorphism. The hard part of the proof is a lemma that shows that when $\gamma[\tau] = 0$, one can cut the spectrum in half by a straight line, and split τ as the sum of two elements coming from Ext of the two halves. Repeated bisection eventually eliminates all of the holes.

An immediate consequence is the fact that an essentially normal operator T is normal plus compact if and only if $\mathrm{ind}\,(T - \lambda I) = 0$ whenever $\lambda \notin \sigma_e(T)$. More generally, two essentially normal operators T_1 and T_2 are unitarily equivalent modulo $\mathfrak{K}$ if and only if $\sigma_e(T_1) = \sigma_e(T_2) =: X$ and

$$\mathrm{ind}\,(T_1 - \lambda I) = \mathrm{ind}\,(T_2 - \lambda I) \text{ for all } \lambda \in \mathbb{C} \setminus X.$$

Another immediate corollary is that the set of normal plus compact operators is norm closed, since the essentially normal operators are closed, the set of Fredholm operators is open, and index is continuous.

More information on the BDF theory with an emphasis on the K-theoretical aspects is contained in the monograph **[16]** by Douglas. Most of the results mentioned here are treated in Chapter 9 of **[15]**.

4. Almost commuting matrices

Peter Rosenthal [**38**] asked whether nearly commuting matrices are close to commuting. To make sense of this question, one says that A and B nearly commute if $\|AB - BA\|$ is small, while close to commuting means that there are *commuting* matrices A' and B' with $\|A - A'\|$ and $\|B - B'\|$ both small. He makes it clear that to be an interesting problem, one must obtain estimates independent of the dimension of the space on which the matrices act. We may also limit A and B to have norm at most one. Peter recalled, in a private communication, that he discussed this problem with Paul Halmos when he was his student at the University of Michigan. The most interesting case, and the hardest, occurs when the matrices are all required to be Hermitian. Halmos mentions the problem specifically in this form in [**28**].

This 'finite dimensional' problem is closely linked to the Brown–Douglas–Fillmore Theorem, as we shall see. I put finite dimensional in quotes because problems about matrices which ask for quantitative answers independent of dimension are really infinite dimensional problems, and can generally be stated in terms of the compact operators rather than matrices of arbitrary size.

If A and B are Hermitian matrices in $\mathfrak{M}_n$, then $T = A + iB$ is a matrix satisfying

$$[T^*, T] = T^*T - TT^* = 2i(AB - BA).$$

So if A and B almost commute, then T is almost normal; and they are close to commuting if and only if T is close to a normal matrix.

Halmos [**26**] defines an operator T to be *quasidiagonal* if there is a sequence P_n of finite rank projections increasing to the identity so that $\|P_nT - TP_n\|$ goes to 0. The quasidiagonal operators form a closed set which is also closed under compact perturbations. Normal operators are quasidiagonal, and thus so are normal plus compact operators. Fredholm quasidiagonal operators have index 0.

Now suppose that T is an essentially normal operator which is quasidiagonal. Then one can construct a sequence of projections P_n increasing to the identity so that $\sum_{n\geq 1} \|P_nT - TP_n\|$ is small. A small compact perturbation of T yields the operator $\sum_{n\geq 1} \oplus T_n$ where

$$T_n = (P_n - P_{n-1})T(P_n - P_{n-1})|_{(P_n - P_{n-1})\mathcal{H}}.$$

The essentially normal property means that $\lim_n \|[T_n^*, T_n]\| = 0$. So a positive solution to the nearly commuting problem would show that T can be perturbed by a block diagonal compact operator to a direct sum of normal operators. This provides a direct link to the BDF theorem.

If one fixes the dimension n and limits A and B to the (compact) unit ball, then a compactness argument establishes the existence of a function $\delta(\varepsilon, n)$ such that if A and B are in the unit ball of $\mathfrak{M}_n$ and $\|AB - BA\| < \delta(\varepsilon, n)$, then A and B are within ε of a commuting pair. (See [**35**].) For this reason, the problem is much less interesting for fixed n. Pearcy and Shields [**36**] obtain the explicit estimate $\delta(\varepsilon, n) = 2\varepsilon^2/n$ when A and A' are Hermitian but B is arbitrary.

After these initial results, a variety of counterexamples were found to various versions of the problem. Voiculescu [**43**] used very deep methods to establish the existence of triples A_n, B_n, C_n of norm one Hermitian $n \times n$ matrices which asymptotically commute but are bounded away from commuting Hermitian triples. An explicit and somewhat stronger example due to the author [**13**] provides matrices $A_n = A_n^*$ and normal matrices B_n in the unit ball of $\mathfrak{M}_{n^2+1}$ with $\|A_nB_n - B_nA_n\| = n^{-2}$, but bounded away from commuting pairs A_n' and B_n' with A_n' Hermitian but B_n' are arbitrary. Voiculescu [**44**] also constructs asymptotically commuting unitary matrices which are bounded away from commuting unitaries. Exel and Loring [**20**] provide a very slick example in which the pairs of unitaries are actually bounded away from arbitrary commuting pairs. Finally, we mention a paper of Choi [**11**] who also found pairs of arbitrary matrices which asymptotically commute but are bounded away from commuting pairs.

We sketch the Exel–Loring example [**20**]. Let U_n be the cyclic shift on a basis $e_1, \dots, e_n$ and let V_n be the diagonal unitary with eigenvalues ω^j, $1 \le j \le n$, where $\omega = e^{2\pi i/n}$. Then $U_nV_nU_n^{-1}V_n^{-1} = \omega I_n$. In particular,

$$\|U_nV_n - V_nU_n\| = |1-\omega| = 2\sin\pi/n.$$

Now if A_n and B_n are commuting matrices within $1/3$ of U_n and V_n, define $A_s = (1-s)U_n + sA_n$ and $B_s = (1-s)V_n + sB_n$. One can check that for $0 \le s, t \le 1$,

$$\gamma(s,t) = \det\big((1-t)I_n + tA_sB_sA_s^{-1}B_s^{-1}\big)$$

is never 0; and $\gamma(s,0) = \gamma(s,1) = 1$. For fixed s and $0 \le t \le 1$, $\gamma(s,\cdot)$ determines a closed loop in $\mathbb{C} \setminus \{0\}$. When $s = 0$, it reduces to the loop $(1-t+t\omega)^n$, which has winding number 1. But at $s = 1$, it is the constant loop 1, which has winding number 0. This establishes a contradiction.

With all this negative evidence, one might suspect that the Hermitian pair question would also have a negative answer. However, the examples all use some kind of topological obstruction, which Loring [**34**] and Loring–Exel [**21**] make precise. The case of a pair of Hermitian matrices is different.

This author [**13**] provided a partial answer to the Hermitian case by proving an absorption result. If $T \in \mathfrak{M}_n$ is an arbitrary matrix, there is a normal matrix N in $\mathfrak{M}_n$ with $\|N\| \le \|T\|$ and a normal matrix N' in $\mathfrak{M}_{2n}$ so that

$$\|T \oplus N - N'\| \le 75\|T^*T - TT^*\|^{1/2}.$$

Thus if T has a small commutator, one obtains a normal matrix close to $T \oplus N$. While this does not answer the question exactly, it can take advantage of an approximate normal summand of T—and in the case of an essentially normal operator T, such a summand is available with spectrum equal to $\sigma_e(T)$. The real problem is that the normal matrix N may have too much spectrum, in some sense.

This approach was pursued in a paper by Berg and the author [**7**] in order to provide an operator theoretic proof of the BDF Theorem. The key was to establish a variant of the absorption theorem for the annulus. Specifically, if T is an invertible operator with $\|T\| \le R$ and $\|T^{-1}\| \le 1$, then there are normal operators N and N'

satisfying the same bounds (so the spectrum lies in the annulus $\{z : 1 \le |z| \le R\}$) such that

$$\|T \oplus N - N'\| \le 95\|T^*T - TT^*\|^{1/2}.$$

We established this by showing that the polar decomposition of T almost commutes, and using the normal summand to provide room for the perturbation. Combining this with an elementary extraction of approximate eigenvectors allows one to show that, with T as above, there is a normal operator N with spectrum in the annulus so that $\|T - N\| \le 100\|T^*T - TT^*\|^{1/2}$.

The second important step of our proof of BDF is to establish that if T is essentially normal and $\operatorname{ind}(T - \lambda I) = 0$ for $\lambda \notin \sigma_e(T)$, then T is quasidiagonal. Since the set of quasidiagonal operators is closed, it suffices to work with a small perturbation. Now $T \sim_a T \oplus N$ where N is a diagonal normal operator with $\sigma(N) = \sigma_e(N) = \sigma_e(T)$. We fatten up the spectrum of N with a small perturbation so that it is a nice domain with finitely many smooth holes. An approximation technique replaces $T \oplus N$ by a finite direct sum of operators with topologically annular spectra. Essentially one cuts the spectrum into annular regions without cutting through any holes. Then the Riesz functional calculus and the case for the annulus do the job. This results in a proof of the BDF theorem for essentially normal operators (i.e. planar X).

The almost commuting matrix question was finally solved by Huaxin Lin [**33**]. This paper is a tour de force. It starts with an idea of Voiculescu's. Suppose that there is a counterexample, namely asymptotically commuting $n \times n$ Hermitian matrices A_n and B_n of norm 1 which are bounded away from commuting pairs. Let $T_n = A_n + iB_n$. Then $T = T_1 \oplus T_2 \oplus \dots$ is a block diagonal, essentially normal operator which is not a *block diagonal* compact perturbation of a normal operator. One should consider T as an element of the von Neumann algebra $\mathfrak{M} := \prod \mathfrak{M}_n$ with commutator $T^*T - TT^*$ lying in the ideal $\mathfrak{J} = \sum \mathfrak{M}_n$ of sequences which converge to 0. The image $t = T + \mathfrak{J}$ is a normal element of the quotient algebra. Lin succeeds in proving that t can be approximated by a normal element having finite spectrum. Then it is an easy matter to lift the spectral projections to projections in $\mathfrak{M}$, and so approximate T by a normal element in $\mathfrak{M}$, which yields a good normal approximation to all but finitely many T_n. Unfortunately this is an extremely difficult proof.

Lin's Theorem was made much more accessible by Friis and Rordam [**22**], who provide a short, slick and elementary proof. They begin with the same setup. Observe that by the spectral theorem, every self-adjoint element of any von Neumann algebra can be approximated arbitrarily well be self-adjoint elements with finite spectrum. This property, called *real rank zero* (RR0), passes to quotients like $\mathfrak{M}/\mathfrak{J}$. In $\mathfrak{M}_n$, the invertible matrices are dense. If you prove this by modifying the positive part of the polar decomposition, the estimates are independent of dimension. Thus the argument can be readily extended to show that the invertible elements are dense in $\mathfrak{M}$. This property, called *topological stable rank one* (tsr1), also passes to quotients. Moreover, we observe that a normal element can perturbed to an invertible *normal* operator,

Now let t be a normal element of $\mathfrak{M}/\mathfrak{J}$, and fix $\varepsilon > 0$. Cover the spectrum $\sigma(t)$ with a grid of lines spaced ε apart horizontally and vertically. Use the tsr1 property to make a small perturbation which is normal and has a hole in the spectrum inside each square of the grid. Then use the continuous functional calculus to obtain another perturbation to a normal element nearby that has spectrum contained in the grid.

The RR0 property says that self-adjoint elements can be approximated by self-adjoint elements with finite spectrum. In particular, if $a = a^*$ has $\sigma(a) = [0, 1]$, this property allows us to find $b = b^*$ with $\|a - b\| < \varepsilon$ such that $b - 1/2$ is invertible. A modification of this idea works on each line segment of the grid. So another small perturbation yields a normal operator with spectrum contained in the grid minus the mid-point of each line segment. A further use of the functional calculus collapses the remaining components of the spectum to the lattice points of the grid. This produces the desired normal approximation with finite spectrum.

In their sequel **[23]**, Friis and Rordam use similar methods in the Calkin algebra provided that the index data is trivial. They establish quasidiagonality for essentially normal operators with zero index data, and thus provide a third proof of the BDF theorem.

Finally we mention that a paper has recently been posted on the arXiv by Hastings **[29]** which claims a constructive proof that almost commuting Hermitian matrices are close to commuting, with explicit estimates. This is a welcome addition since the soft proof provides no norm estimates at all. It is still an open question whether a perturbation of size $O(\|T^*T - TT^*\|^{1/2})$ is possible as in the case of the absorption results.

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