A Theorem on Fourier coefficients

Theorem. *Let f in $L(\mathbb{T})$ be such that*

- *f is piecewise differentiable, i.e. f is differentiable on $[-\pi,\pi]$ except on a finite set of points,*
- *f' (defined a.e.) is integrable: $\|f'\|_1 = \frac{1}{2\pi}\int_{-\pi}^{\pi}|f'| < \infty$, and*
- *f is bounded, i.e. $\|f\|_\infty < \infty$.*

Then $\sup_{k\in\mathbb{Z}}|kc_k(f)| < \infty$.

Proof. We let $-\pi = s_1 < \cdots < s_m < s_{m+1} = \pi$ contain the points at which f fails to be differentiable. Then for any n in $\mathbb{N}$ we have

$$\int_{\bigcup_{j=1}^m (s_j-\frac{1}{n},s_j+\frac{1}{n})} |f| \leq \|f\|_\infty \sum_{j=0}^{m} \lambda\big((s_j-\tfrac{1}{n}, s_j+\tfrac{1}{n})\big) = \frac{2m\|f\|_\infty}{n}. \tag{$\dagger$}$$

Thus, given $\varepsilon > 0$, let n be so that $\frac{2}{n} \leq \max_{j=1,\dots,m}(s_{j+1}-s_j)$ and $\frac{2m\|f\|_\infty}{n} < \varepsilon$.

We remark that $f'(t) = \lim_{n\to\infty} n[f(t+\frac{1}{n})-f(t)]$ on its domain, so f' is measurable. Also

$$(fe^{-k})' = f'e^{-k} - ikfe^{-k} \quad \Rightarrow \quad fe^{-k} = \frac{1}{ik}[f'e^{-k} - (fe^{-k})'] \tag{$\ddagger$}$$

on the domain of f'. Hence, with our choice of n above, using first ($\ddagger$) then ($\dagger$) and FTofC we have

$$\begin{aligned} 2\pi|c_k(f)| &= \int_{-\pi}^{\pi} fe^{-k} \\ &\leq \int_{\bigcup_{j=1}^m (s_j-\frac{1}{n},s_j+\frac{1}{n})} |f| + \frac{1}{|k|}\left[\int_{-\pi}^{\pi}|f'| + \sum_{j=1}^{m}\left|\int_{s_j+\frac{1}{n}}^{s_{j+1}-\frac{1}{n}} (fe^{-k})'\right|\right] \\ &\leq \varepsilon + \frac{1}{|k|}\left[2\pi\|f'\|_1 + \sum_{j=1}^{m}\big(|f(s_{j+1}-\tfrac{1}{n})| + |f(s_j+\tfrac{1}{n})|\big)\right] \\ &\leq \varepsilon + \frac{1}{|k|}[2\pi\|f'\|_1 + 2m\|f\|_\infty]. \end{aligned}$$

Since ε can be choosen arbitrarily, we see that $|kc_k(f)| \leq \|f'\|_1 + \frac{m}{\pi}\|f\|_\infty < \infty$. $\square$

Notice that this proof is the usual "divide and conquor" argument for estimating integrals. Similar arguments were seen in the proofs of the Abstract Summability Kernel Theorem, Fejer's Theorem, the Localiasation Principle and Dini's Theorem. Seems like a pretty good technique!

We recall that the condition $\sup_{k\in\mathbb{Z}}|kc_k(f)| < \infty$ was the main assumed condition of Hardy's Tauberian Theorem. Hence we see that

$$\lim_{n\to\infty} s_n(f,t) = \begin{cases} \omega_f(t) & t = s_1,\dots,s_{m+1} \\ f(t) & \text{otherwise.} \end{cases}$$

In fact on any closed subinterval J of $\bigcup_{j=1}^m (s_j, s_{j+1})$ we have uniform convergence on J, i.e.

$$\lim_{n\to\infty}\sup_{t\in J}|s_n(f,t) - f(t)| = 0.$$