Pure Math 450/650, Assignment 2

Due: January 27.

1. **(a)** Show that for any subset E of $\mathbb{R}$, there is a G_δ set A such that $E \subset A$ and $\lambda(A) = \lambda^*(E)$.

   (b) Show that a subset E of $\mathbb{R}$ is Lebesgue measurable if and only if there is a G_δ set A such that $E \subset A$ and $\lambda^*(A \setminus E) = 0$.
   [Hint: Do this in the case that $\lambda^*(E) < \infty$, first.]

   (c) Show that if $E \subset B$, where B is a Lebesgue measurable set with $\lambda(B) < \infty$, then

   $$E \text{ is Lebesgue measurable } \Leftrightarrow \lambda(B) = \lambda^*(E) + \lambda^*(B \setminus E).$$

   Note that (c) is a vast improvement on Caratheodory's criterion, since it allows us to determine measurability by only checking how the outer measure splits over *one* set, rather than *all* sets.

2. Define the *Lebesgue inner measure* of a subset E of $\mathbb{R}$ by

   $$\lambda_*(E) = \sup\{\lambda(K) : K \subset E \text{ and } K \text{ is compact}\}.$$

   (a) Show that if E is bounded, i.e. $E \subset (a, b)$ for some bounded interval (a, b), then

   $$\lambda((a, b)) = \lambda_*(E) + \lambda^*((a, b) \setminus E).$$

   (b) Show that if E is Lebesgue measurable, then $\lambda_*(E) = \lambda(E)$.
   [Hint: Do this in the case that E is contained in a bounded interval, first.]

   (c) Show that if A is a bounded subset of $\mathbb{R}$ such that $\lambda_*(A) = \lambda^*(A)$, then A is measurable.

   Remark # 1. We note that Lebesgue's criterion for measurability of a subset A of $\mathbb{R}$ is that

   $$\lambda_*(A \cap (a, b)) = \lambda^*(A \cap (a, b)) \text{ for every bounded interval } (a, b).$$

   This is equivalent to Caratheodory's criterion. We also note that it can be deduced from (b) (or from 1.(b)) that every measurable set may be written as the union of an F_σ set and a null set.

   Remark #2. We saw in the last assignment that for any open set $G \subset \mathbb{R}$, with $G = \dot{\bigcup}_{i=1,2,\dots} J_i$, that

   $$\lambda(G) = \lambda^*(G) = \sum_{i=1,2,\dots} \ell(J_i)$$

   which we may take as the definition of $\lambda(G)$. If $K \subset \mathbb{R}$ is compact, it is bounded and we may find an open interval $(a, b) \supset K$. Then $(a, b) \setminus K$ is open, and we may define

   $$\lambda(K) = b - a - \lambda\big((a, b) \setminus K\big).$$

   By σ-additivity of $\lambda^* = \lambda$ on Borel sets, this defintion is is the same as $\lambda^*(K)$.

3. Fix $0 < \alpha \leq 1$. Let $C_{0,\alpha} = [0,1]$; $C_{1,\alpha}$ be $C_{0,\alpha}$ with the middle open interval of length $\frac{\alpha}{3}$ removed; $C_{2,\alpha}$ be $C_{1,\alpha}$ with the middle open intervals of length $\frac{\alpha}{9}$ from the 2 remaining closed intervals removed; ... $C_{n+1,\alpha}$ be $C_{n,\alpha}$ with the middle open intervals of length $\frac{\alpha}{3^{n+1}}$ from the 2^n remaining closed intervals removed. Then let

$$C_\alpha = \bigcap_{n=0}^{\infty} C_{n,\alpha}.$$

This set is called a *generalized Cantor set*.

(a) Show that C_α is closed and nowhere dense, i.e. contains no open sets.

(b) Determine $\lambda(C_\alpha)$.

(c) Show that we can realize $[0,1] = A \cup B$, where A is a set of first category (i.e. a countable union of sets whose closures are nowhere dense), and $\lambda(B) = 0$.

(d) Show that for any $0 < \alpha < 1$, C_α is homeomorphic to the usual Cantor set C.

[Hint: Show that there exists a continuous map $\varphi : C_\alpha \to C$ which is strictly increasing. You may need to remember your PM351.]

4. Given a set X, we denote its cardinality by $|X|$. Recall that $\mathcal{L}(\mathbb{R})$ and $\mathcal{B}(\mathbb{R})$ denote the σ-algebras of Lebesgue meaurable and Borel subsets of $\mathbb{R}$, respectively.

(a) Show that $|\mathcal{L}(\mathbb{R})| = 2^c$, where 2^c denotes the cardinality of the power set, $\mathcal{P}(\mathbb{R})$.

[Hint: What can Cantor do for you? Also Cantor-Bernstein-Schröder is okay.]

(b) (BONUS) Show that $|\mathcal{B}(\mathbb{R})| = c$, where $c = |\mathbb{R}|$.

[You may need to learn transfinite induction, for this one.]

Hence we conclude that $\mathcal{L}(\mathbb{R}) \supsetneq \mathcal{B}(\mathbb{R})$. It is suggested by 1.(b), that this difference is accounted for by non-Borel null sets.