Pure Math 450, Assignment 4

Due: February 29.

1. Let $[a, b]$ be a compact interval in $\mathbb{R}$ and $1 < p < \infty$. [All of the results (a)-(d) below are also true for $p = 1$, with similar proofs.]

   (a) Show that if $f \in L_p[a, b]$ and $\varepsilon > 0$, then there is a measurable simple function $\varphi : [a, b] \to \mathbb{R}$ such that $\|f - \varphi\|_p < \varepsilon$.

   A function $\psi : [a, b] \to \mathbb{R}$ is said to be a *step function* if there is a family $I_1, \ldots, I_m$ of intervals, each contained in $[a, b]$, such that $\psi = \sum_{j=1}^{m} b_j \chi_{I_j}$, where $b_1, \ldots, b_m \in \mathbb{R}$.

   (b) Show that if $\varphi : [a, b] \to \mathbb{R}$ is a measurable simple function, then $\varphi \in L_p[a, b]$, and, given $\varepsilon > 0$, there exists a step function $\psi : [a, b] \to \mathbb{R}$ such that $\|\varphi - \psi\|_p < \varepsilon$. [Hint: Start with the case that $\varphi = \chi_E$.]

   (c) Show that if $\psi : [a, b] \to \mathbb{R}$ is a step function, and $\varepsilon > 0$, then there is a continuous function $h : [a, b] \to \mathbb{R}$ such that $\|\psi - h\|_p < \varepsilon$.

   Let $C[a, b]$ denote the space of continuous real-valued functions on $[a, b]$. Recall from PMATH 351 that $C[a, b]$ is a Banach space under the norm $\|h\|_u = \max_{x \in [a,b]} |h(x)|$. [We use this temporary notation $\|\cdot\|_u$, because we don't want to get it confused with the L_∞-norm – at least not yet.]

   (d) Show that there is a constant $K > 0$ such that for any h in $C[a, b]$, $\|h\|_p \leq K \|h\|_u$; and show that for such h, if $\|h\|_p = 0$ then $\|h\|_u = 0$ too. Hence each element f in $C[a, b]$ identifies uniquely an element of $L_p[a, b]$; we write $C[a, b] \subset L_p[a, b]$.

   (e) Show that $C[a, b]$ is a dense subset of $L_p[a, b]$.

   (f) Show that for any h in $C[a, b]$, $\|h\|_\infty = \|h\|_u$, and conclude that $C[a, b]$ is a closed subspace of $L_\infty[a, b]$.

   In light of (e), we may now conflate the notations $\|\cdot\|_\infty$ and $\|\cdot\|_u$ on $C[a, b]$.

Don't forget next page ...

2. A continuous function $h : \mathbb{R} \to \mathbb{R}$ is *compactly supported* if its support,

$$\text{supp}(h) = \overline{\{x \in \mathbb{R} : h(x) \neq 0\}}$$

is compact. [By Heine-Borel, this is equivalent to having that $\{x \in \mathbb{R} : h(x) \neq 0\}$ is bounded.]

(a) Show that the space $C_c(\mathbb{R})$, of continuous real-valued functions on $\mathbb{R}$ which are compactly supported, is dense in $L_p(\mathbb{R})$, $1 \leq p < \infty$.

[Hint: We can identify $L_p[-n, n]$ as a subspace of $L_p(\mathbb{R})$ in an obvious way. Show first that $\bigcup_{n=1}^{\infty} L_p[-n, n]$ is dense in $L_p(\mathbb{R})$; then use 1 (e).]

Let $C_0(\mathbb{R}) = \left\{h \in C(\mathbb{R}) : \lim_{x \to \pm\infty} h(x) = 0\right\}$. You may have seen in PM351 that $C_0(\mathbb{R})$ is a closed subspace of $C_b(\mathbb{R})$, the continuous bounded functions on $\mathbb{R}$. [If you haven't, I suggest you try to prove it for yourself.] In particular, $C_0(\mathbb{R})$ is a Banach space.

(b) Show that $C_c(\mathbb{R})$ is dense in $C_0(\mathbb{R})$.

(c) Is $C_0(\mathbb{R}) \subset L_p(\mathbb{R})$ for any $1 \leq p < \infty$? Explain with a proof or counter-example. How about when $p = \infty$?

3. Let $(f_n)_{n=1}^{\infty} \subset L^+[-1, 1]$ (non-negative integrable functions on $[-1, 1]$) satisfy the conditions

- $\lim_{n\to\infty} \int_{[-1,1]} f_n = 1$, and
- for each $\delta \in (0, 1)$, $\lim_{n\to\infty} \left(\int_{[-1,-\delta]} f_n + \int_{[\delta,1]} f_n\right) = 0$

(a) Show that if $\varphi \in C[-1, 1]$, then

$$\lim_{n\to\infty} \int_{[-1,1]} \varphi f_n = \varphi(0).$$

(b) Show, by way of example, that it is possible to have $\lim_{n\to\infty} f_n = 0$ a.e.

Part (a) says that "f_n *converges in distribution* to point mass at 0". We see from (b) that convergence in distribution may not be related to (a.e.) convergence.