Pure Math 450, Final Exam Topics

Final Exam: Wednesday, April 11, 4:00-6:30PM, in RCH 301.

Measure Theory ($\leq 1/3$ of the exam)

- measurable sets, special sets (G_δ, Cantor etc.)
- measurable functions; properties: sums, products, compositions, approximation by simple measurable functions *(proof)*
- Lebesgue integral
- Monotone Convergence Theorem, Fatou's Lemma, Lebesgue Dominated Convergence Theorem
- L_p-spaces
  - completeness, separability *(proofs)*
  - Hölder's Inequality, Minkowski's Inequality
  - bounded linear operators and functionals *(proofs)*; see table below

space X	functional	symbol	Lipschitz const.
$L_1[a,b]$	$\Gamma_\varphi(f) = \int_a^b f\varphi$	$\varphi \in L_\infty[a,b]$ (or in $C[a,b]$)	$\lVert\Gamma_\varphi\rVert_* = \lVert\varphi\rVert_\infty$
$L_p[a,b]$	$\Gamma_g(f) = \int_a^b fg$	$g \in L_q[a,b]$	$\lVert\Gamma_g\rVert_* = \lVert g\rVert_q$
$C[a,b]$	$\Gamma_g(f) = \int_a^b fg$	$g \in L_1[a,b]$	$\lVert\Gamma_g\rVert_* = \lVert g\rVert_1$

Always, $\frac{1}{q} + \frac{1}{p} = 1$ and $f \in X$.

  - containment results: $C[a,b] \subsetneq L_p[a,b] \subsetneq L_r[a,b]$ if $1 \leq r < p \leq \infty$ *(proofs)*
    - $\|f\|_p \leq (b-a)^{1/p} \|f\|_\infty$, if $f \in L_\infty[a,b]$; $C[a,b] \subset L_\infty[a,b]$ and norms agree
    - $\|f\|_r \leq (b-a)^{\frac{p-r}{pr}} \|f\|_p$, if $f \in L_p[a,b]$
  - $C[a,b]$ is dense in $L_p[a,b]$, $1 \leq p < \infty$; but closed in $L_\infty[a,b]$ *(proofs)*

Fourier Analysis ($\geq 2/3$ of the exam)

- Fourier series, Fourier coefficients
- sums $s_n(f)$ for f in $L_1(\mathbb{T})$, sums $s_n(f,x)$ for f in $L(\mathbb{T})$ and x in $\mathbb{R}$
- convolutions, convolution operators $C(f)(g) = f * g$ where $f \in C(\mathbb{T})$, Lipschitz constants $\|C(f)\|_{\mathcal{B}}$ for $\mathcal{B} = L_1(\mathbb{T}), C(\mathbb{T})$; compare with $\|C(D_n)\|_{L_2(\mathbb{T})}$ (Riesz-Fischer).
- Nonconvergence of Fourier series in $L_1(\mathbb{T})$ and in $C(\mathbb{T})$ (even pointwise) *(outline of proof)*
  - Dirichlet kernel, D_n; Lebesgue constants, L_n
  - Banach-Steinhaus Theorem
  - norms of convolution operators on $L_1(\mathbb{T})$ and on $C(\mathbb{T})$

- Cesaro means, $\sigma_n(f)$ for f in $L_1(\mathbb{T})$
  - summability kernels, the Abstract Summability Kernel Theorem *(proof)*
  - Fejér kernel, K_n; Fejér's Theorem *(proof)*
  - uniqueness of Fourier coefficients *(proof)*
- Riemann-Lebesgue Lemma *(proof)*
  - Fourier transform, $T : L_1(\mathbb{T}) \to A(\mathbb{Z}) \subset c_0(\mathbb{Z})$
  - Open Mapping Theorem, why $A(\mathbb{Z}) \subsetneq c_0(\mathbb{Z})$
- $f \in L(\mathbb{T})$, $\int_{-\pi}^{\pi} \left|\frac{f(t)}{t}\right| dt < \infty \Rightarrow \lim_{n\to\infty} s_n(f,0) = 0$ *(proof)*
  - Localization Principle *(proof)*
  - Dini's Theorem (& Lipschitz version) *(proof)*
- Inner product and Hilbert spaces
  - Cauchy-Schwarz Inequality, Pythagoreas' Theorem
  - Linear Approximation Lemma *(proof)*
  - Orthonormal Basis Theorem: Bessel's (In)equality, Parseval's Identity *(proof)*
  - Hilbertian Fourier analysis, Riesz-Fischer Theorem, Plancherel Theorem *(proofs)*
- Fourier algebra $A(\mathbb{T})$
  - is an algebra of continuous functions *(proof)*
  - $\mathcal{D}(\mathbb{T}) \subset A(\mathbb{T})$ *(proof)*

Modes of convergence of Fourier series				
space containing f	Convergent to f in norm?		Convegence at a point x?	
	$s_n(f)$	$\sigma_n(f)$	$s_n(f,x)$	$\sigma_n(f,x)$
$L_1(\mathbb{T})$ ($L(\mathbb{T})$)	no$^{(\dagger)}$	yes$^{(*)}$	no, in general$^{(\dagger)}$; yes, if f is differentiable at $x^{(**)}$	yes, to $\omega_f(x)$ if it exists$^{(\bullet)}$; *[almost everywhere to $f(x)$]*$^{(***)}$
$C(\mathbb{T})$	no$^{(\dagger)}$	yes$^{(*\text{ or }\bullet)}$	no, in general$^{(\dagger)}$; yes, if f is differentiable at $x^{(**)}$	yes to $f(x)^{(\bullet)}$
$L_2(\mathbb{T})$	yes$^{(\ddagger)}$	yes$^{(*)}$	*[almost everywhere]*$^{(\bullet\bullet)}$	*[almost everywhere]*$^{(***)}$
$A(\mathbb{T})^{(\dagger\dagger)}$	yes	yes	yes	yes

$(\dagger)$	Banach-Steinhaus and growth of Lebesgue constants	$(*)$	Abstract summability kernel theorem
$(**)$	Dini's Theorem	$(\bullet)$	Fejér's Theorem
$(\ddagger)$	Orthonormal Basis Theorem	$(\dagger\dagger)$	assignment # 6
$(\bullet\bullet)$	theorem of L. Carlson *[beyond the scope of the course]*	$(***)$	Fejér-Lebesgue Theorem *[will not be on the exam]*

Containment relations: $A(\mathbb{T}) \subsetneq C(\mathbb{T}) \subsetneq L_2(\mathbb{T}) \subsetneq L_1(\mathbb{T})$.

- Hardy's Tauberian Theorem; application to bounded piecewise differentiable functions with integrable derivative *(proof)*
- Gibbs phenomenon *(proof of Lemma)*

Homework Assignment questions

Questions may not be given literally from assignments: they may be simplified; or aspects of them will be explicitly assumed and you will be asked to deduce others.

A4. Q1 all (may use $p = 1$ or 2); Q3 all

A5. Q1 (you will most likely be asked to take these results on faith); Q2 all; Q3 all; Q4

A6. Q1 all (may offer functions with much simpler calculations, maybe $F(t) = \frac{1}{2} - \frac{t}{2\pi}$ from Gibbs Lemma, or a step function like $\chi_{[-\pi/2,\pi/2]}$ or $-\chi_{[-\pi/2,0]} + \chi_{[0,\pi/2]}$); Q2 all; Q3 all