Open Mapping Theorem Let $\mathcal{X}, \mathcal{Y}$ be Banach spaces, $T: \mathcal{X} \to \mathcal{Y}$ be bounded and linear. If T is surjective, then T is open, i.e. if $U \subset \mathcal{X}$ is open, $T(U) \subset \mathcal{Y}$ is open.

Corollary: (Inverse Mapping Theorem) Let $\mathcal{X}, \mathcal{Y}$ be Banach spaces, $T: \mathcal{X} \to \mathcal{Y}$ be linear and bounded. If T is bijective, then $T^{-1}: \mathcal{Y} \to \mathcal{X}$ is bounded.

Proofs: PM753.

Corollary: There exist sequences $(c_k)_{k=-\infty}^{\infty}$ in $c_0(\mathbb{Z})$ which are not associated to Fourier series, i.e. there is no f in $L_1(\mathbb{T})$ for which $c_k = c_k(f)$ for all k.

Proof: Let $T: L_1(\mathbb{T}) \to c_0(\mathbb{Z})$ be given by

$$Tf = \left(c_k(f)\right)_{k=-\infty}^{\infty}.$$

Then T is linear, bounded with $|||T||| \le 1$ (i.e. $\|Tf\|_\infty = \sup_{k \in \mathbb{Z}} |c_k(f)| \le \|f\|_1$), and range is in $c_0(\mathbb{Z})$ by R-L Lemma. Also, T is injective (see Corollary to Abstract Summability Kernel Theorem). If T were bijective, then we would have bounded $T^{-1}: c_0(\mathbb{Z}) \to L_1(\mathbb{Z})$. However, let

$$d_n = (\ldots 0, \underbrace{1}_{-n}, 1, \ldots, \underbrace{1}_{0}, \ldots, \underbrace{1}_{n}, 0, \ldots) \in c_0(\mathbb{Z})$$

so $\|d_n\|_\infty = 1$. Then $T^{-1}(d_n) = D_n$, the Dirichlet kernel of order n. But then

$$|||T^{-1}||| \ge \sup_{n \in \mathbb{N}} \|T^{-1}(d_n)\|_1 = \sup_{n \in \mathbb{N}} \underbrace{\|D_n\|_1}_{L_n} = \infty$$

which contradicts IVT, above. $\square$