Connectivity of random regular graphs generated by the pegging algorithm

Pu Gao *
Department of Combinatorics and Optimization
University of Waterloo
p3gao@math.uwaterloo.ca

Abstract

We study the connectivity of random d-regular graphs which are recursively generated by an algorithm motivated by a peer to peer network. We show that these graphs are asymptotically almost surely d-connected for any even constant $d \geq 4$.

1 Introduction

The properties (degree distribution, diameter, connectivity, short cycle distribution, hamiltonicity, etc) of random graphs in the classical Erdős-Rényi model [1] were widely studied and applied to various areas. Since a lot of networks, like the World Wide Web, have very different properties from the graphs in the Erdős-Rényi model, other models were introduced to simulate them. The preferential attachment model were introduced first by Yule [13], which produces random networks whose degree distributions obey the power law. Many authors [4, 6, 8] then applied this model or its variations to analyse scale-free networks. The most commonly studied scale-free networks include the World Wide Web and some social networks. Bourassa and Holt [5, 11] recently introduced a peer-to-peer network, called the SWAN network, whose underlying topology is a random regular graph. In the SWAN network, clients arrive and leave randomly. Their experimental results showed that this network has high connectivity and low latency amongst other things.

To model the SWAN network, Cooper, Dyer and Greenhill [7] defined a Markov chain on d-regular graphs with randomised size. The moves of the Markov chain are by the "clothespinning" (for arriving clients) or its reverse. The author and Wormald [9] studied random regular graphs generated by the "pegging algorithm", which repeats the pegging operation in each step, the same operation as clothespinning in [7]. We showed that the joint short cycle distribution of graphs generated this way is asymptotically independent Poisson, and also [9, 10] estimated the rate at which the joint distribution converges to its limit. This result supports the conjecture proposed in [9] of the contiguity between the random regular graphs in the pegging model and those of the uniform distribution. If the conjecture holds, then immediately we can derive asymptotical properties of the random d-regular graphs in the pegging model, such as being d-connected, containing a Hamilton

*Dept CO, 200 University Ave W, N2L 3G1, Waterloo, ON, Canada

cycle, and with diameter asymptotically almost surely (a.a.s.) $O(\log n)$, etc. The difficulty involved in proving the conjecture was discussed in [9]. Hence it is interesting to check other properties of these random regular graphs.

In this paper, we study the connectivity of graphs generated from the pegging algorithm. This is indicative of the connectivity properties of the SWAN network under long-term growth. It is well known [3, 12] that the random d-regular graphs are a.a.s. d-connected in the uniform model for any fixed constant $d \geq 3$. We show that the random d-regular graphs generated by the pegging algorithm, for any arbitrary even integer $d \geq 4$, are a.a.s. d-connected.

2 Main results

The general pegging operation is defined in [9]. The pegging algorithm starts from an initial d-regular graph, and repeats pegging operations at each step. Here we first give the definition of the pegging operation for even degrees. The odd degree case is a natural generalization of the even degree case, and its analysis is also analogous.

Pegging Operation for Even d
Input: a d-regular graph G, where d is even.
Choose a set E_1 of $d/2$ pairwise non-adjacent edges in $E(G)$ u.a.r.
Let $\{u_1, u_2, \ldots, u_d\}$ denote the vertices incident with edges in E_1.
Set $V(H) := V(G) \cup \{u\}$, where u is a new vertex.
Set $E(H) := (E(G) \setminus E_1) \cup \{uu_1, uu_2, uu_3, \ldots, uu_d\}$.
Output: H.

The newly introduced vertex u is called the *peg vertex*, and we say that the edges deleted are *pegged*. Figure 1 illustrates the pegging operation with $d = 4$.

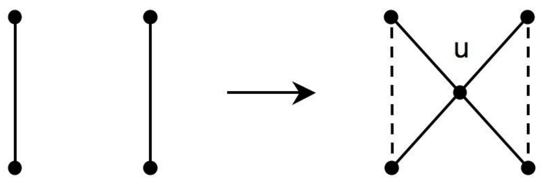

Figure 1: *Pegging operation when $d = 4$*

Let G_0 be the initial d-regular graph, and G_t be the resulting graph after t pegging operations are repeatedly applied. We retain the notation $\mathcal{P}(G_0, d)$ introduced in [9] for the random process $(G_0, G_1, G_2, \ldots)$. Let n_t and m_t be the number of vertices and edges respectively in G_t. Then $n_t = n_0 + t$ when d is even, whereas $m_t = dn_t/2$.

The joint short cycle distribution was studied in detail in [9]. Two of the results shown in [9] are useful to this paper. The first is a theorem for the joint distribution of the number of short cycles.

Theorem 2.1 ([9]) *For $k \geq 3$, define*

$$\mu_k = \frac{3^k - 9}{2k}. \tag{2.1}$$

Let G_0 and $k \geq 3$ be fixed, and let $X_{t,k}$ be the number of k-cycles in G_t. Then in $\mathcal{P}(G_0)$,

$$\mathbf{E}X_{t,k} = \mu_k + O\left(n_t^{-1}\right).$$

Moreover, the joint distribution of $X_{t,3}, \ldots, X_{t,k}$ is asymptotically that of independent Poisson variables with means $\mu_3, \ldots, \mu_k$.

The *excess* of a graph is the number of its edges minus the number of its vertices. Let $\Psi(i, r)$ be defined as the set of graphs with i vertices, minimum degree at least 2, and excess r. Define $W_{t,i,r}$ to be the number of subgraphs of G_t in $\Psi(i, r)$. The second useful result in [9] is a lemma for the expected number of subgraphs with given excess. Note that in the following lemma the constant implicit in $O()$ depends on i.

Lemma 2.1 ([9]) *For fixed $i > 0$ and $r \geq 0$,*

$$\mathbf{E}W_{t,i,r} = O(n_t^{-r}).$$

The above theorem and lemma are used in this paper to estimate the counts of some local structures. Let a *k-cut* be a vertex cut of size k, and a *k-edge-cut* be an edge cut of size k. The vertex cuts in G_t are closely related to the edge cuts. We say an edge cut A is generated by a vertex cut S in a graph G if A joins S and some component of $G - S$. Figure 2 is an example for the simple case of a 3-cut when $d = 4$. These are edge cuts of size 6, 5 and 7 generated by the 3-cuts in Figure 2. In fact, when $d = 4$, a 3-cut generates at least one edge cut of size at most 6. Similarly, a 2-cut generates some edge cut of size at most 4, and a 1-cut generates some edge cut of size at most 2. So the study of edge cuts of size at most 6 will be helpful for the discussion of vertex cuts of size at most 3.

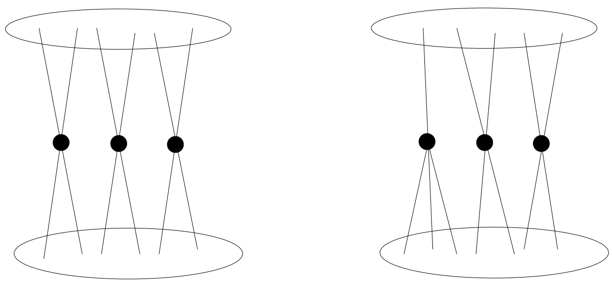

Figure 2: *a 3-cut which generates an edge-cut of size 6 or smaller*

We call an edge cut A *proper*, if no proper subset of A is an edge cut of G. Unless otherwise specified, all edge cuts discussed in this paper are proper edge cuts. The graph $G - S$ has two components if S is a proper edge cut. In this paper, we call these two components the *side-components* of the edge cut S.

Definition 2.1 *We call an edge cut A of a graph G trivial if it is of the form $A = E(S, \bar{S})$ for some S that induces a tree in G, where $\bar{S} = V(G) - S$. The notation $E(S, \bar{S})$ denotes the set of edges with one end in S and the other end in $\bar{S}$.*

For any S and A specified as above with $|S| = l$, there are $l-1$ edges in S. Since G_t is d-regular, $|A| = dl - 2(l-1)$, and hence a trivial edge cut is always of size $dl - 2(l-1)$ for some integer $l \geq 1$. For any vertex $v \in G_t$, there are $O(1)$ trees of size l that contain v. So the number of induced trees of size l is at most $O(n_t)$, which gives an upper bound on the number of trivial edge cuts. The expected number of cycles of length at most l is $O(1)$ by Theorem 2.1, and so there are $\Theta(n_t)$ $(l-1)$-paths a.a.s. Thus the number of trivial edge cuts is a.a.s. $\Theta(n_t)$.

Among all edge cuts other than trivial ones, we define the *semi-trivial* edge cut, which acts as a transition from trivial edge cuts to the rest.

Definition 2.2 *An edge cut is called semi-trivial if it is of the form $A = E(S, \bar{S})$ for some S that induces a connected unicyclic subgraph. Edge cuts that are neither trivial nor semi-trivial are called non-trivial.*

By definition, one of the side-components of a semi-trivial edge cut is connected and contains one cycle. If the side-component is of size l, then it contains l edges, and therefore the semi-trivial edge cut is of size $dl - 2l$. By Theorem 2.1, the expected number of semi-trivial edge cuts of size k, for any fixed k, is $\Theta(1)$.

Simply by checking the neighbours of a trivial or semi-trivial edge cut, we will prove the following lemma, indicating that, to determine the vertex connectivity, it is sufficient to study the non-trivial edge cuts.

Lemma 2.2 *For any graph in $\mathcal{P}(G_0, d)$, with $d \geq 3$, the edge cuts generated by vertex cuts of size at most $d-1$ are a.a.s. non-trivial.*

We first study the connectivity of random d-regular graphs when d is even. In this way we derive the following theorem for even degrees. However, we claim the same result holds for any odd degree $d \geq 3$ as well by analogous analysis.

Theorem 2.2 *Let $G_t \in \mathcal{P}(G_0, d)$ for any even $d \geq 4$, then G_t is a.a.s. d-connected.*

The proof of the theorem actually gives that the probability of $G_t \in \mathcal{P}(G_0, d)$ being not d-connected is $O\left(n_t^{-1/(M+1)}\right)$, where $M = d(d-1)/2$.

3 Proofs

The following lemma was shown in [9] and will be used in some calculations in the later part of this paper. The proof is elementary.

Lemma 3.1 *Let $(a_n)_{n\geq 1}$ be a sequence of nonnegative real numbers, and let $c > 0$, and $p \neq c+1$, be constants. If*

$$a_{n+1} = \left(1 - \frac{c}{n} + O(n^{-2})\right) a_n + O(n^{-p}),$$

then $a_n = O(n^{\delta})$ for all $n \geq 1$, where $\delta = \max\{-c, -p+1\}$.

Proof of Lemma 2.2. Assume that S is a vertex cut of size at most $d-1$, and S generates a trivial edge cut A, whose smaller side-component is S_1, where $|S_1| = l$ is bounded. Then the subgraph of G_t induced by $S_1 \cup S$ contains at most $l+d-1$ vertices, and at least $(l-1)+|A| = dl-l+1$ edges, since $|A| = dl-2(l-1)$. Hence the excess of this subgraph is at least $dl-l+1-(l+d-1) = (d-2)(l-1)$. Since $d \geq 3$, the excess can be 0 only if $l = 1$, which means that the vertex cut is the set of neighbours of some vertex, which is of size d. This contradicts $|S| \leq d-1$. Hence the subgraph has excess at least 1. Similarly, if A is semi-trivial, and $|S_1| = l$ is bounded, then $|A| = dl-2l$. Hence the subgraph induced by $S \cup S_1$ contains at most $l+d-1$ vertices, and at least $l+|A| = dl-l$ edges. So the excess is at least $dl-l-(l+d-1) = (d-2)(l-1)-1$. Since S_1 contains a cycle, $l \geq 3$. So the excess of the subgraph is at least 1. By Lemma 2.1, this occurs with probability $O(n_t^{-1})$. So the edge cuts generated by vertex cuts of size at most $d-1$ are a.a.s. neither trivial nor semi-trivial. ∎

We consider the random d-regular graph generated by the pegging algorithm, when $d \geq 4$ is even. The graph G_t contains $n_t = n_0 + t$ vertices, and $m_t = dn_t/2$ edges. Let N_t be the number of ways to choose $d/2$ non-adjacent edges. Then N_t is asymptotically $\binom{m_t}{d/2}$. Hence the number of ways to do a pegging operation at step t is asymptotically N_t. Let $Y_{t,k}$ be the number of non-trivial edge cuts of size k in G_t. Correspondingly, let $Y^*_{t,k}$, $\tilde{Y}_{t,k}$ be the number of trivial and semi-trivial edge cuts of size k. So $Y^*_{t,k} = \Theta(n_t)$ if $k = dl-2(l-1)$ for some integer $l \geq 1$, and $Y^*_{t,k} = 0$ for other values of k. By Theorem 2.1 there are a.a.s. $O(1)$ semi-trivial edge cuts of size k for any fixed integer k, i.e. $\tilde{Y}_{t,k} = O(1+n^{-1})$. By Lemma 2.2 we know that the study of the behavior of $Y_{t,k}$ is enough, but in some sense it relates to $\tilde{Y}_{t,k}$ as we will see later in this paper.

We study the random process $(Y_{t,k})_{t\geq 0}$ for any fixed k, or more precisely, we check the expected changes of the value of $Y_{t,k}$ in a single step. Let A be a k-edge-cut in G_t. We say that A is *destroyed* either if some edge in A is pegged or if A is no longer an edge cut in G_{t+1}. Therefore to destroy A simply requires that either at least one edge in A is pegged, or two edges from different side-components of A are pegged. We say that an edge cut A' in G_{t+1} is a *new edge cut created from* A if A is destroyed and A' contains at least one new edge created at step t. Note that whenever A is destroyed, there are always new edge cuts being created at the same time. The number and the size of new edge cuts depend on the way that the former edge cut is destroyed. For any given k-edge-cut A, there are three ways to destroy it, according to the relative positions of the two edges that are pegged. Let e_1 and e_2 be the two pegged edges, and v the peg vertex. Of course the d new edges added form a trivial d-edge-cut themselves, but we do not count this case since $Y_{t,k}$ counts only the non-trivial edge cuts.

Type 1(i): A contains e_1 and another $k-1$ edges as shown in Figure 3. The other $d/2-1$ edges pegged other than e_1 are all in the same side-component of A. Figure 3 is an example of $d=4$. In this case, a new k-edge-cut and a new $(k+d-2)$-edge cut are created.

Type 1(ii): A contains only one edge that is pegged, and the other $d/2-1$ edges pegged are not contained in the same side-component of A. In this case, a new $(k+i)$-edge-cut and a new $(k+d-2-i)$-edge-cut are created for some $2 \leq i \leq d-4$.

Type 2: A contains e_1, e_2 and another $k-2$ edges. Figure 4 is an example as $d=4$. The probability for A being destroyed this way is $O(n_t^{-2})$.

Type 3(i): None of the edges in A are pegged, and one of the pegged edges lies in one side-component of A, while the rest lie on the other side-component. See Figure 6 as an example with $d=4$. In this case, at most one new $(k+2)$-edge-cut and one new $(k+d-2)$-edge-cut are created.

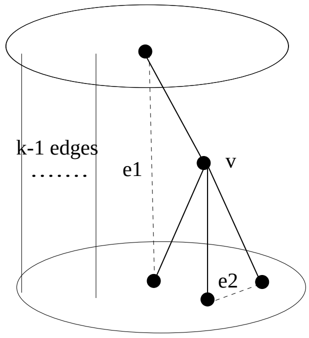

Figure 3: *only* e_1 *contained in the edge cut*

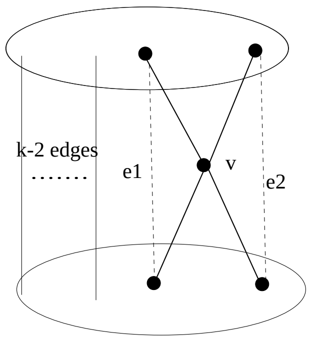

Figure 4: e_1 *and* e_2 *are both contained in the edge-cut*

A slight difference from Type 1 and 2 is that there might be other edge-cuts created besides the above two new edge-cuts, when the edge pegged is a bridge of some side-component. For example, let's consider $d = 4$. Let S_1 and S_2 be the two side-components. If e_1 is contained in a cycle of S_1 and e_2 is contained in a cycle of S_2, then 2 new $(k+2)$-edge cuts are created. This is illustrated in Figure 6. Otherwise, assume e_1 is a bridge of S_1. Then we have created a new $(i+1)$-edge cut and a new $(j+1)$-edge cut and a $(k+2)$-edge cut, where $i+j = k$. This is shown in the right hand side of Figure 5. Note that this implies the existence of an $(i + 1)$-edge cut and a $(j + 1)$-edge cut in G_t, and we will count the new $(i+1)$-edge cut and $(j+1)$-edge cut when the existing $(i+1)$-edge cut and $(j + 1)$-edge cut are destroyed with Type 1. Without over counting, we only count the creation of the new $(k+2)$-edge cuts for the destruction of A. For the same reason, if both e_1 and e_2 are bridges in S_1 and S_2, then no creation of new edge-cuts is counted for the destruction of A. In conclusion, at most two $(k + 2)$-edge-cuts are created for the Type 3(i) destruction of A.

Type 3(ii): A contains none of edges being pegged, and both side-components of A contains at least two pegged edges. In this case, a new $(k + i)$-edge cut and a new $(k + d - i)$-edge cut are created, for some $4 \leq i \leq d - 4$.

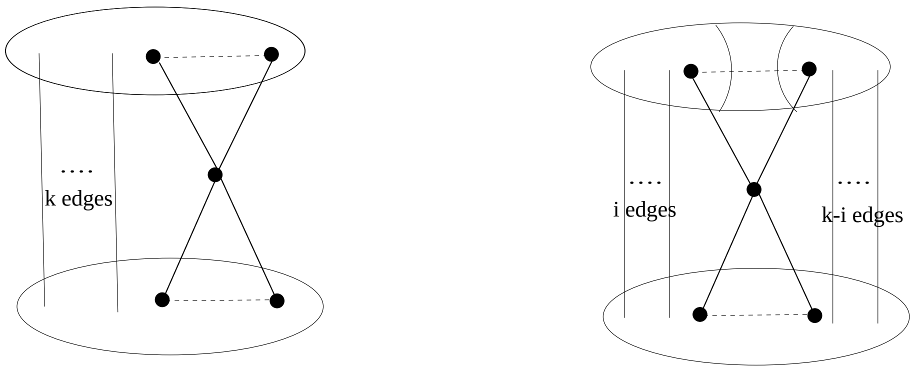

Figure 5: *neither e_1 nor e_2 is contained in the edge-cut*

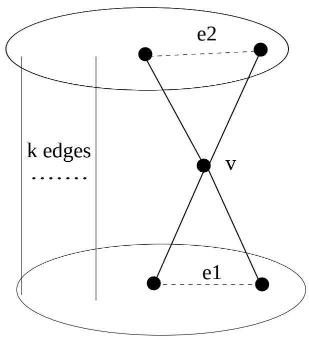

Figure 6: *neither e_1 nor e_2 is contained in the edge-cut*

Given any constant integer $M > 0$, define $\hat{C}(M,t)$ be the set of all non-trivial edge cuts with size at most M in graph G_t, and for any $t \geq 0$, and let $Y_{M,t} = |\hat{C}(M,t)|$. Hence $Y_{M,t} = \sum_{i \leq M} Y_{t,i}$. We can partition all edge cuts in $\hat{C}(M,t)$ into three types.

- Edge cuts which are created from destruction of some edge cut in $\hat{C}(M,t-1)$.
- Edge cuts that are in $\hat{C}(M,t-1)$ and remain from G_{t-1} to G_t.
- Edge cuts created from some semi-trivial edge cut in G_{t-1}.

The following lemma shows that $\hat{C}(M,t)$ is essentially empty.

Lemma 3.2 *Let $G_t \in \mathcal{P}(G_0, d)$, $M > 0$ be any given integer, then as $t \to \infty$,*

$$\mathbf{E}(|\hat{C}(M,t)|) = o(1).$$

To prove this lemma we check the expected changes in $\mathbf{E}(|\hat{C}(M,t)|)$ going from G_t to G_{t+1}. The contribution to changes comes from edge cuts of the first and the third types. The truth of this lemma is that the sizes of the edge cuts created are always at least that of the destroyed one, and the expected number of non-trivial edge cuts coming from semi-trivial edge cuts is very small. The outline of the proof is as follows.

- S1. In each step, the destruction of any edge cut does not created any edge cut smaller than the one destroyed, and the number of new edge cuts created is bounded.
- S2. In each step, the destruction of any edge cut creates at most one new edge cut that is of the same size of the one destroyed.
- S3. There is a significant probability ($\Theta(n_t^{-1})$), that all new edge cuts created are of strictly larger size than that of the destroyed one.
- S4. The probability of creating a non-trivial edge cut of size at most M from some semi-trivial edge cut is $O(n_t^{-2})$.

We are going to prove the statements $S1$-$S4$, and then show how these statements lead to the lemma.

Proof of Lemma 3.2. Let $\hat{Y}_{t,k}$ be the number of k-edge-cuts in $\hat{C}(M,t)$, and define the weight of a k-edge-cut to be $1/k!$. Hence the weight of $\hat{C}(M,t)$ is

$$W_t = \sum_{k=1}^{M} \frac{1}{k!} \hat{Y}_{t,k}.$$

We estimate the expected change from W_t to W_{t+1}. We first check the change caused by destruction of edge cuts in $\hat{C}(M,t)$. For a given k-edge cut A, we analyse the different ways that it is destroyed.

Type 1(i): A new k-edge cut and a new $(k+d-2)$-edge cut are created, so the weight change is $1/k! + 1/(k+d-2)! - 1/k! \leq 1/(k+2)!$. There are k ways to choose an edge in A, and at most $\binom{m_t}{d/2-1}$ ways to choose the rest $d/2-1$ edges, which lie in the same side-component of A. So the probability for this to occur is at most

$$\frac{k\binom{m_t}{d/2-1}}{N_t} \sim \frac{k}{n_t}.$$

The expected increase of weight is at most

$$\frac{1}{(k+2)!}\frac{k\hat{Y}_{t,k}}{n_t}.$$

Type 1(ii): A new $(k+i)$-edge cut and a new $(k+d-2-i)$-edge cut are created. The weight change is $1/(k+i)! + 1/(k+d-2-i)! - 1/k! \leq 2/(k+2)! - 1/k! < 0$.

Type 2: Some new $(k+i)$-edge cuts are created, with $0 \leq i \leq d-4$, and thus the contribution to the weight change is at most $1/k!$. The probability of this to occur is $O(n_t^{-2})$ as shown before. So the expected increase of weight is bounded by

$$\frac{1}{k!}\frac{\hat{Y}_{t,k}}{n_t^2}.$$

Type 3(i): A new $(k+2)$-edge cut and a new $(k+d-2)$-edge cut are created, and the weight change is $1/(k+2)! + 1/(k+d-2)! - 1/k! \leq 2/(k+2)! - 1/k!$. The probability of this occurrence depends on the size of each side-component. The probability is smaller as the sizes of the two side-components are more imbalanced. The worst case is that there is only bounded number of edges in one side-component, and $m_t - O(1)$ number of edges in the other side-component. There are asymptotically $\binom{m_t}{d/2-1}$ ways to choose $d/2-1$ edges from the larger side-component. Hence the probability for this to occur is at least

$$\frac{\binom{m_t}{d/2-1}}{N_t} \sim \frac{1}{n_t}.$$

The expected decrease of the weight is at least

$$\left(\frac{2}{(k+2)!} - \frac{1}{k!}\right)\frac{\hat{Y}_{t,k}}{n_t}.$$

Type 3(ii): A new $(k+i)$-edge cut and a new $(k+d-i)$-edge cut are created, and the weight change is $1/(k+i)! + 1/(k+d-i)! - 1/k! \leq 2/(k+2)! - 1/k! < 0$.

Another cause of the change of W_t is the expected number of non-trivial edge cuts created from semi-trivial edge cuts. Note that there are only $O(1+n_t^{-1})$ semi-trivial edge cuts of size at most M in G_t, since the number of cycles of size at most M is $O(1+n_t^{-1})$. Let A be a semi-trivial edge cut. All the ways of creating a non-trivial edge cut from A are listed as follows.

(1) Destroy A of Type 1, with e_1, e_2 being chosen such that they are adjacent to some common edge. Hence the common edge together with two of the new added edges will form a new triangle, which creates a non-trivial edge cut. There are only $O(1)$ ways to choose e_1 and e_2. So the probability of this to occur is $O(n_t^{-2})$.

(2) Peg at least two edges in A, i.e. destroy A of Type 2, which occurs with probability $O(n_t^{-2})$.

(3) Destroy A of Type 3, with e_1 and e_2 both adjacent to some edge in A, hence that edge together with two of the new edges form a new triangle and a new non-trivial edge cut appears. The number of choices of e_1 and e_2 is bounded and hence the probability for this to occur is $O(n_t^{-2})$.

There are only $O(1)$ choices of A. Hence the expected change in this case is $O(n_t^{-2})$. Thus we have

$$\begin{aligned}\mathbf{E}(W_{t+1} - W_t \mid \hat{C}(M,t)) \quad &\leq \quad \sum_{k=1}^{M-2} \frac{1}{(k+2)!}\frac{k\hat{Y}_{t,k}}{n_t} + \sum_{k=1}^{M} O\left(\frac{\hat{Y}_{t,k}}{n_t^2}\right) + \sum_{k=1}^{M-2}\left(\frac{2}{(k+2)!} - \frac{1}{k!}\right)\frac{\hat{Y}_{t,k}}{n_t} \\ &\quad - \sum_{k=M-1}^{M} \frac{1}{k!}\frac{\hat{Y}_{t,k}}{n_t} + O(n_t^{-2}).\end{aligned}$$

The first term comes from Type 1(i), second term comes from Type 2, and the third and fourth terms from Type 3(i). The contributions from Type 1(ii) and Type 3(ii) are ignored since they are negative. The fifth term comes from non-trivial edge cuts created from semi-trivial ones. So we have

$$
\begin{aligned}
&\mathbf{E}(W_{t+1} - W_t \mid \hat{C}(M,t)) \\
&\quad \le \sum_{k=1}^{M-2} -\left(\frac{1}{k!} - \frac{2}{(k+2)!} - \frac{k}{(k+2)!}\right)\frac{\hat{Y}_{t,k}}{n_t} - \sum_{k=M-1}^{M} \frac{1}{k!}\frac{\hat{Y}_{t,k}}{n_t} + \sum_{k=1}^{M} O\left(\frac{1+\hat{Y}_{t,k}}{n_t^2}\right) \\
&\quad \le \sum_{k=1}^{M-2} -\frac{1}{(k+1)!}\frac{\hat{Y}_{t,k}}{n_t} - \sum_{k=M-1}^{M} \frac{1}{k!}\frac{\hat{Y}_{t,k}}{n_t} + \sum_{k=1}^{M} O\left(\frac{1+\hat{Y}_{t,k}}{n_t^2}\right) \\
&\quad \le -\frac{1/(M+1)}{n_t} W_t + O(n_t^{-2}).
\end{aligned}
$$

Taking the expectation of both side of the above inequality gives

$$
\mathbf{E}\left(W_{t+1}\right) \le \left(1 - \frac{1/(M+1)}{n_t}\right)\mathbf{E}(W_t) + \omega(n_t),
$$

where $\omega(n_t)$ is some function of n_t such that $\omega(n_t) = O(n_t^{-2})$. Define $(a_t)_{t\ge 0}$ to be $a_0 = W_0$, and

$$
a_{t+1} = \left(1 - \frac{1/(M+1)}{n_t}\right) a_t + \omega(n_t), \quad \text{for all } t \ge 0.
$$

Assume $\mathbf{E}(W_t) \le a_t$ for some $t \ge 0$, then

$$
\mathbf{E}\left(W_{t+1}\right) \le \left(1 - \frac{1/(M+1)}{n_t}\right)\mathbf{E}(W_t) + \omega(n_t) \le \left(1 - \frac{1/(M+1)}{n_t}\right) a_t + \omega(n_t) = a_{t+1}.
$$

Hence $\mathbf{E}(W_t) \le a_t$ for all $t \ge 0$. By Lemma 3.1 $a_t = O\left(n_t^{-1/(M+1)}\right)$, and therefore $\mathbf{E}(W_t) = O\left(n_t^{-1/(M+1)}\right)$. Hence Lemma 3.2 follows. ∎

Proof of Theorem 2.2. Clearly any vertex cut of size at most $d-1$ generate an edge cut of size at most $d(d-1)/2$. By putting $M = d(d-1)/2$, the theorem follows directly from Lemma 2.2 and Lemma 3.2. More precisely, by Markov inequality, we have

$$
\mathbf{P}(G_t \text{ is not } d\text{-connected}) = \mathbf{P}(W_t \ge 1) + O(n_t^{-1}) = O\left(n_t^{-1/(M+1)}\right). \quad ∎
$$

The definition of a pegging operation for regular graphs of even degree d does not adapt directly to the case of odd d, but we may make a similar definition, illustrated in Figure 7 for $d = 3$. The general definition for any odd integer $d \ge 3$ is as follows.

Pegging Operation for Odd d

Input: a d-regular graph G, where d is odd.

1. Let $c := \lfloor d/2 \rfloor$ and choose a set $E_1 = \{u_1u_2, u_3u_4, \ldots, u_{2c-1}u_{2c}\}$ of c pariwise non-adjacent edges in $E(G)$ u.a.r., and another set $E_2 = \{u_{2c+1}u_{2c+2}, \ldots, u_{4c-1}u_{4c}\}$ of c pairwise non-adjacent edges in $E(G) \setminus E_1$ u.a.r.
2. $G := (G \setminus (E_1 \cup E_2)) \cup \{u, v\} \cup E_3 \cup \{uv\}$, where u and v are new vertices added to $V(G)$, and $E_3 = \{uu_1, \ldots, uu_{2c}, vu_{2c+1}, \ldots, vu_{4c}\}$.
3. Output: G.

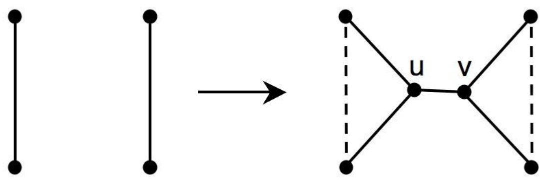

Figure 7: *Pegging operation when $d = 3$*

We can now follow the same routine to prove the connectivity result when $d \geq 3$ is odd. In each step, two new vertices and $d - 1$ new edges are added. The only difference from the even degree case is that, for any given k-edge cut that is destroyed, there can be up to four new edge cuts created instead of two. So there are more complicate transitions to obtaining a new k-edge cut for any k. It is straightforward but tedious to check the statements $S1$-$S4$ stated before the proof of Lemma 3.2. The author checked the case $d = 3$. So we omit the details and claim that $G_t \in \mathcal{P}(G_0, d)$ are d-connected for any arbitrary integer $d \geq 3$.

4 Acknowlegement

The author would like to thank N. Wormald who introduced us to this problem and also gave valuable suggestions on simplifying some of the proofs.

References

[1] B. Bollobás, (2001), Random Graphs (2nd ed.), Cambridge University Press.

[2] Albert-László Barabási, Scale-Free Networks, Scientific American, (May 2003), 288:60-69.

[3] B. Bollobás, Random graphs, Combinatorics (Swansea, 1981), pp. 80–102, London Math. Soc. Lecture Note Ser., 52, Cambridge Univ. Press, Cambridge-New York, 1981.

[4] Albert-Lszl Barabsi and Réka Albert, Emergence of scaling in random networks, Science (October 15, 1999), 286:509-512.

[5] V. Bourassa and F. Holt, "SWAN: Small-world wide area networks", in Proceeding of International Conference on Advances in Infrastructures (SSGRR 2003w), L'Aquila, Italy, 2003, paper# 64.

[6] B. Bollobás, O. Riordan, The diameter of a scale-free random graph, Combinatorica 24 (2004), no. 1, 5–34.

[7] C. Cooper, M. Dyer and C. Greenhill, "Sampling regular graphs and a peer-to-peer network", Proceedings of the sixteenth annual ACM-SIAM Symposium on Discrete Algorithms (2005), 980–988.

[8] F. Chung and L. Lu, "Complex graphs and networks", CBMS Regional Conference Series in Mathematics, 107. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 2006. viii+264 pp.

[9] P. Gao, N. Wormald, Short cycle distribution in random regular graphs recursively generated by pegging, submitted.

[10] P. Gao, N. Wormald, Rate of convergence of the short cycle distribution in random regular graphs generated by pegging, submitted.

[11] F.B. Holt, V. Bourassa, A.M. Bosnjakovic, J. Popovic, "Swan - highly reliable and efficient networks of true peers," in CRC Handbook on Theoretical and Algorithmic Aspects of Sensor, Ad Hoc Wireless, and Peer-to-Peer Networks (J. Wu, ed.), CRC Press, Boca Raton, Florida, 2005, pp. 799–824.

[12] N. Wormald, The asymptotic connectivity of labelled regular graphs, J. Combin. Theory Ser. B 31 (1981), no. 2, 156–167.

[13] G. Yule, A Mathematical Theory of Evolution, based on the Conclusions of Dr. J. C. Willis, F.R.S. Philosophical Transactions of the Royal Society of London, Ser. B (1925), 213: 2187