Distribution of the number of spanning regular subgraphs in random graphs

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Abstract

In this paper, we examine the moments of the number of d-factors in $\mathcal{G}(n,p)$ for all p and d satisfying $d^3 = o(p^2 n)$. We also determine the limiting distribution of the number of d-factors inside this range with further restriction that $(1-p)\sqrt{dn} \to \infty$ as $n \to \infty$.

1 Introduction

Studies of subgraphs in random graph spaces are one of the areas of major interest in random graph theory. In the fundamental work [4] by Erdős and Rényi, various types of subgraphs, for instance, trees and cycles of certain sizes, and general subgraphs of certain sizes or densities, are studied. A general approach of determining the limiting distribution of subgraphs of fixed sizes in the binomial model $\mathcal{G}(n,p)$ is investigated in [14, 15] by Ruciński, whereas the distributions of certain counts of some types of subgraphs, also with fixed sizes, in the random d-regular graph space $\mathcal{G}(n,d)$ are studied by Z. Gao and Wormald [5].

However, studying the distribution of subgraphs with larger size, in particular, the spanning subgraphs, is much more difficult and there is no general approach. The commonly studied spanning subgraphs include Hamilton cycles and spanning d-regular graphs (d-factors). In the binomial model $\mathcal{G}(n,p)$ where $p = \Omega(\ln n/n)$, even though the expected number of Hamilton cycles (or perfect matchings) grows fast as a function of n, it was unknown for a long time whether a Hamilton cycle exists or not until the breakthrough by Pósa [13]. The result was further strengthened by Komlós and Szemerédi [10] and Bollobás [2, pp. 239, Theorem 15]. The existence of a d-factor in $\mathcal{G}(n,p)$ was investigated by Shamir and Upfal [18]. However, the distributions of the number of Hamilton cycles and d-factors are not of usual types and have strong tails. These random variables have large deviation (comparable to their expectation) whereas they are positive asymptotically almost surely by the existence results for sufficiently large p.

The first breakthrough in determining the limiting distribution of spanning subgraphs is credited to Robinson and Wormald's work [16, 17] on showing that almost all regular graphs are Hamiltonian, using the small subgraph conditioning method. However, the distribution of the Hamilton cycle counts in the random d-regular graph model $\mathcal{G}(n,d)$ was not clearly revealed in that paper. Its

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limiting distribution was first sorted out by Janson [7], which according to the author, is implicitly hidden in the proof of Robinson and Wormald in [16, 17]. The limiting distribution is very unusual. It can be expressed as the distribution of a variable whose logarithm is a linear combination of infinitely many independent Poisson variables. Janson [8] also determined the log-normal distribution of the number of spanning trees, Hamilton cycles and 1-factors in $\mathcal{G}(n, p)$ by conditioning on the number of edges. The proof used the decomposition and projection methods [9].

In this paper, we study the moments and the limiting distribution of $X_{n,d}$, the number of d-factors in $\mathcal{G}(n, p)$. We prove that the distribution is asymptotically log-normal for a wide range of p and d, via a completely different approach than in [8]. Indeed, we will apply the *switching method* which allows us to estimate sharply the k-th moment of $X_{n,d}$ for all k not growing too fast with n. As we will discuss in Section 2, the k-th moment of the scaled variable $X_{n,d}/\mathbf{E}X_{n,d}$ grows so fast as a function of k that the distribution cannot be uniquely determined by its (finite) moments. However, the moment estimates coupled with the Z. Gao-Wormald theorem [6] enable us to prove the limiting normality of $X_{n,d}$ in the particular case when $d^2(1-p) = o(1)$, where the standard deviation is relatively small compared to $\mathbf{E}X_{n,d}$. To prove log-normality of $X_{n,d}$ in the other case where $\liminf_{n\to\infty} d^2(1-p) > 0$, we condition on the number of edges Y_n in $\mathcal{G}(n, p)$ and establish the concentration of $X_{n,d}$ in the random graph $\mathcal{G}(n, m)$ by the method of the second moment, and then use the limiting normality of Y_n.

Let $g(n, \mathbf{d})$ denote the number of graphs on n vertices with degree sequence $\mathbf{d}$. In particular, let $g(n, d)$ denote the number of d-regular graphs on n vertices. The following result is known from McKay [11].

Theorem 1.1 *Assume $d = o(n^{1/3})$. Then*

$$g(n, d) \sim \frac{(dn)!}{2^{dn/2}(dn/2)!(d!)^n} \cdot \exp\left(-\frac{d^2-1}{4}\right).$$

The restriction of d was further relaxed by McKay and Wormald in [12]. Clearly $\mathbf{E}X_{n,d} = g(n, d)p^{dn/2}$. However, calculating the second moment of $X_{n,d}$ is not trivial. We need to count the number of pairs of d-regular graphs (M_1, M_2), both on the same vertex set $[n]$, according to how the edge sets in M_1 and M_2 intersect. Brute-force counting is difficult for general d and high moments. In this paper, we use the switching method, which is surprisingly powerful in calculating the moments. This method was first introduced by McKay [11] to evaluate $g(n, d)$. In this paper the switchings are used in a different way but the basic ideas are the same.

2 Main results

Recall that $\mathcal{G}(n, p)$ denotes the probability space of random graphs on n vertices, in which each edge between a pair of vertices occurs independently with probability p. Recall also that $g(n, d)$ denotes the number of d-regular graphs on n vertices and $X_{n,d}$ denotes the number of d-factors in $G \in \mathcal{G}(n, p)$. Here both $p = p(n)$ and $d = d(n)$ are sequences indexed by n and we drop n from the notation when there is no confusion. Define

$$\mu_{n,d} = g(n, d)p^{dn/2}, \quad \beta_{n,d} = \exp\left(\frac{d^2(1-p)}{2p}\right). \tag{2.1}$$

In this paper, all the asymptotic notations refer to $n \to \infty$. We say $f(n,k) \sim g(n,k)$ uniformly for all k (in certain range) if there exists a sequence, (a_n) with $a_n \geq 0$ and $a_n \to 0$ as $n \to \infty$ such that $|f(n,k)/g(n,k) - 1| \leq a_n$ for all k and n. We define uniform convergence in a similar way. For a random variable Y, we let $\mathbf{E}Y = \mathbf{E}_p(Y)$ denote the expectation of Y in $\mathcal{G}(n,p)$. We normally drop p from the notation but we use it if we need to emphasise the value of p. We will prove the following.

Theorem 2.1 *Assume $d^3 = o(p^2 n)$. Then in $\mathcal{G}(n,p)$,*

$$\mathbf{E}(X_{n,d}^2) \sim \beta_{n,d}\mathbf{E}(X_{n,d})^2.$$

Moreover, for any $0 < \alpha < 7/8$,

$$\mathbf{E}(X_{n,d}^k) \sim \beta_{n,d}^{\binom{k}{2}}\mathbf{E}(X_{n,d})^k = \left(\frac{\mu_{n,d}}{\sqrt{\beta_{n,d}}}\right)^k \beta_{n,d}^{k^2/2},$$

uniformly for all positive integers k such that $k^4 d^3 = o(p^2 n)$ and $(7/4 - 2\alpha)\ln n + k\ln p \to \infty$ as $n \to \infty$.

We note here that the above theorem is likely to be true without the condition $(7/4-2\alpha)\ln n + k\ln p \to \infty$, and it is possible to weaken or remove this condition by a more careful analysis in the proof of Lemma 3.7 in Section 3. Since the current version is already strong enough to prove our main results on the d-factor distributions, we do not make a further effort to extend the theorem for a larger range of k. The proof of the theorem immediately yields the following proposition.

Proposition 2.2 *For any integer k such that $k^4 d^3 = o(n)$, the number of ordered k-tuples of pairwise disjoint d-factors in K_n is asymptotically $g(n,d)^k \exp\left(-\binom{k}{2}d^2/2\right)$.*

Let $W_{n,d} = X_{n,d}\sqrt{\beta_{n,d}}/\mu_{n,d}$. Theorem 2.1 implies that for any fixed integer $k \geq 0$, $\mathbf{E}W_{n,d}^k \to \beta_{n,d}^{k^2/2}$ as $n \to \infty$. Consider the case that $\beta = \lim_{n\to\infty}\beta_{n,d}$ exists and $\beta > 1$. Let W denote the random variable with log-normal distribution with parameters $\mu = 0$ and $\sigma^2 = \ln\beta$, i.e. $\ln W$ is distributed as $\mathcal{N}(0, \ln\beta)$, the normal distribution with expectation 0 and variance $\ln\beta$. It is easy to check that $\mathbf{E}W^k = \beta^{k^2/2}$ for every fixed integer $k \geq 0$. However, since the log-normal distribution is not uniquely determined by its moments, we can not claim that $W_{n,d} \to W$ in distribution, not even the existence of the limit. Indeed, we will determine the limiting distribution of $W_{n,d}$ (as below in Theorem 2.5), using a different approach than calculating the moments. For more details on the *moment problem*, readers can refer to [1, Theorem 30.1,30.2].

However, for a certain range of p where $\lim_{n\to\infty}\beta_{n,d} = 1$, we are able to determine the limiting distribution of $X_{n,d}$ using Theorem 2.1.

Theorem 2.3 *Assume that $d^2(1-p) = o(1)$ and $(1-p)\sqrt{dn} \to \infty$ as $n \to \infty$. Let*

$$\begin{aligned}\lambda_{n,d} &= \mu_{n,d}/\sqrt{\beta_{n,d}}\\ \sigma_{n,d} &= \lambda_{n,d} d\sqrt{(1-p)/2p}.\end{aligned}$$

Then $(X_{n,d} - \lambda_{n,d})/\sigma_{n,d}$ converges in distribution to the standard normal as $n \to \infty$.

Contrary to the large deviation of $X_{n,d}$ in $\mathcal{G}(n,p)$, this random variable is concentrated around its expectation in $\mathcal{G}(n,m)$.

Theorem 2.4 *Consider the random graph $\mathcal{G}(n,m)$. Let $N = \binom{n}{2}$ and $p = m/N$. Define*

$$\tilde{\mu}_{n,d} = g(n,d)\binom{N-dn/2}{m-dn/2}/\binom{N}{m}.$$

Assume $d^3 = o(p^2 n)$. Then in $\mathcal{G}(n,m)$, we have $\mathbf{E}X_{n,d} = \tilde{\mu}_{n,d}$ and

$$X_{n,d}/\tilde{\mu}_{n,d} \xrightarrow{p} 1,$$

as $n \to \infty$.

Using Theorem 2.4, we can prove the following main theorem which considers the case that the assumption $d^2(1-p) = o(1)$ in Theorem 2.3 is not satisfied.

Theorem 2.5 *Consider the random graph $\mathcal{G}(n,p)$. Assume $d^3 = o(p^2 n)$ and $\liminf_{n\to\infty} d^2(n)(1-p(n)) > 0$. Define*

$$\begin{aligned} \lambda_{n,d} &= \mu_{n,d}/\sqrt{\beta_{n,d}}, \\ s_{n,d} &= \ln \beta_{n,d} = d^2(1-p)/2p. \end{aligned}$$

Then

$$\frac{\ln\left(X_{n,d}/\lambda_{n,d}\right)}{\sqrt{s_{n,d}}} \xrightarrow{d} \mathcal{N}(0,1), \quad \textit{as } n \to \infty.$$

3 Proofs

We first prove Theorem 2.3 assuming Theorem 2.1. We use the following theorem from Z. Gao and Wormald [6] to show normality.

Theorem 3.1 *Assume $(\mu_n)_{n\geq 1}$ and $(\sigma_n)_{n\geq 1}$ are sequences of reals such that $\sigma_n/\mu_n \to 0$ as $n \to \infty$. Suppose that $f(n) \geq 0$ is a sequence of functions with $f(n) \to 0$ as $n \to \infty$ and $(X_n)_{n\geq 1}$ is a sequence of random variables such that for some constants $c' > c > 0$, for every n and for all integers k in the range $\mathcal{R}_n := \{k \in \mathbf{Z} :\ c\mu_n/\sigma_n \leq k \leq c'\mu_n/\sigma_n\}$,*

$$\mathbf{E}X_n^k = (1 + f(n,k))\mu_n^k \exp\left(\frac{k^2\sigma_n^2}{2\mu_n^2}\right),$$

for some $f(n,k)$ with $|f(n,k)| \leq f(n)$ for all $k \in \mathcal{R}_n$ and for all n. Then $(X_n - \mu_n)/\sigma_n$ converges in distribution to the standard normal as $n \to \infty$.

Proof of Theorem 2.3. Apply Theorem 3.1 with $X_n = X_{n,d}$, $\mu_n = \lambda_{n,d}$ and $\sigma_n = \sigma_{n,d}$. In order to show that $(X_{n,d} - \lambda_{n,d})/\sigma_{n,d}$ converges to the standard normal, we only need to check that all the hypotheses of Theorem 3.1 hold. Let $q = 1 - p$. Since $d^2q = o(1)$ and $q\sqrt{dn} \to \infty$ as $n \to \infty$, we have $\beta_{n,d} \to 1$. We also have $\sigma_{n,d}/\lambda_{n,d} = d\sqrt{q/2p} = O(\sqrt{d^2q}) = o(1)$. Lastly, since

$$\frac{\lambda_{n,d}}{\sigma_{n,d}} = (d\sqrt{q/2p})^{-1} = O((d\sqrt{q})^{-1}),$$

the theorem is proved by showing that uniformly for all integers $k \le (d\sqrt{q})^{-1}$,

$$\mathbf{E}X_{n,d}^k \sim \lambda_{n,d}^k \exp(k^2\sigma_{n,d}^2/2\lambda_{n,d}^2). \tag{3.1}$$

Since $d^2q = o(1)$ and $q\sqrt{dn} \to \infty$ as $n \to \infty$, we have $p = \Omega(1)$ and $\sqrt{d^3/n} = o(d^2q) = o(1)$, which implies $d^3 = o(p^2n)$. Also, for all $k \le (d\sqrt{q})^{-1}$, we have $k^4d^3 = o(p^2n)$ and

$$\ln n + k\ln(1-q) = \ln n - O(kq) = \ln n - O(\sqrt{q}/d) \to \infty,$$

as $n \to \infty$. Hence Theorem 2.1 with $\alpha = 3/8$ implies that (3.1) holds for all $k \le (d\sqrt{q})^{-1}$. ∎

In order to prove Theorem 2.1, we need to compute the expected number of k-tuples $(M_1, \ldots, M_k)$, where each M_i is a d-factor of $\mathcal{G}(n,p)$. To illustrate the idea of the proof, we first compute the second moment ($k = 2$).

Let $F_d(\ell)$ denote the class of ordered pairs of graphs (M_1, M_2), where M_1 and M_2 are both d-regular graphs on n vertices on the same vertex set $[n]$ and share exactly ℓ common edges. Hence the number of edges in each graph in $F_d(\ell)$ is $dn - \ell$. Let $f_d(\ell) = |F_d(\ell)|$. We first show the following lemma.

Lemma 3.2 *Let $\alpha = 1/10$. For all $1 \le \ell \le (1-\alpha)dn/2$,*

$$\frac{f_d(\ell)}{f_d(\ell-1)} = \frac{d^2}{2\ell}(1 + O(\ell/dn + d/n)).$$

Proof. We define two types of switchings.

s-switching: Take an edge x that is contained in both M_1 and M_2. Label the end vertices of x as u_2 and u_2'. Then take an edge y such that $y \in M_1 \setminus M_2$ and label the end vertices of y as u_1 and u_1'. Then take an edge $z \in M_2 \setminus M_1$ and label its end vertices as u_3 and u_3'. An s-switching replaces x and y by $\{u_1, u_2\}$ and $\{u_1', u_2'\}$ in M_1 and replaces x and z by $\{u_2, u_3\}$ and $\{u_2', u_3'\}$ in M_2. An s-switching is applicable on the chosen triple $\{x, y, z\}$ with the given labeling, only if

(a) all six vertices u_i and u_i' for $i = 1, 2, 3$ are distinct;

(b) all of $\{u_1, u_2\}, \{u_1', u_2'\}, \{u_2, u_3\}, \{u_2', u_3'\}$ are not in $M_1 \cup M_2$.

inverse s-switching: Choose a pair of directed 2-paths (u_1, u_2, u_3) and (u_1', u_2', u_3') such that $\{u_1, u_2\}, \{u_1', u_2'\} \in M_1 \setminus M_2$ and $\{u_2, u_3\}, \{u_2', u_3'\} \in M_2 \setminus M_1$. The inverse s-switching replaces $\{u_1, u_2\}$ and $\{u_1', u_2'\}$ by $\{u_1, u_1'\}$ and $\{u_2, u_2'\}$ in M_1 and replaces $\{u_2, u_3\}$ and $\{u_2', u_3'\}$ by $\{u_2, u_2'\}$ and $\{u_3, u_3'\}$ in M_2. The s-switching is applicable on the chosen pair of directed paths only if

(a') all vertices u_1, u_2, u_3, u_1', u_2', u_3' are distinct;

(b') none of $\{u_1, u_1'\}$, $\{u_2, u_2'\}$ and $\{u_3, u_3'\}$ are contained in $M_1 \cup M_2$.

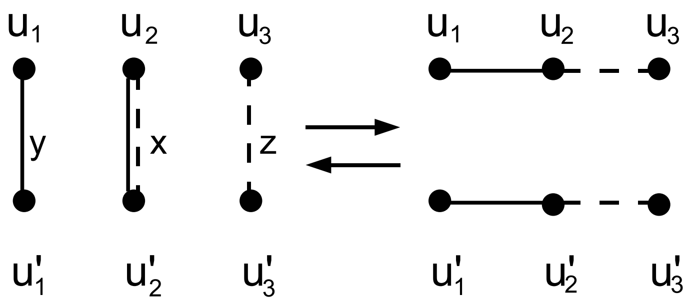

Figure 1: *s-switching and its inverse*

An example of the s-switching and its inverse is shown in Figure 1 where the solid lines denote edges in M_1 and the dashed lines denote edges in M_2.

For any $g \in F_d(\ell)$, an s-switching converts g into a graph in $F_d(\ell-1)$. For each such g, let $N(g)$ denote the number of ways to choose the three edges x, y and z and to label their end vertices so that an s-switching can be applied. There are ℓ ways to choose x and two ways to label its end vertices. For any chosen x, the number of ways to choose y (or z) is $dn/2 - \ell - O(d^2)$, where the term $\ell + O(d^2)$ counts all edges in $M_1 \cap M_2$ and all choices of y such that x and y are adjacent or u_1, u_2 are adjacent or u_1', u_2' are adjacent. For each chosen y (or z), there are two ways to label its end vertices.

$$N(g) = 8\ell\Big((dn/2 - \ell + O(d^2))^2 + O(d^2 n)\Big) = 2\ell d^2 n^2 (1 + O(\ell/dn + d/n)),$$

where the error term $O(d^2 n)$ acounts for the case when y and z are adjacent. On the other hand, for any $g' \in F_d(\ell-1)$, an inverse s-switching converts g' into a graph in $F_d(\ell)$. Let $N'(g')$ denote the number of ways to choose the two directed 2-paths so that an inverse s-switching can be applied. Every vertex is incident with d edges in M_1 and another d edges in M_2, and so the number of directed 2-paths (u_1, u_2, u_3) with $\{u_1, u_2\} \in M_1$ and $\{u_2, u_3\} \in M_2$ is approximately $d^2 n$. The only miscount occurs when the vertex u_2 is incident with an edge in $M_1 \cap M_2$. Hence $N'(g') = (d^2 n)^2 + O(d^5 n + d^3 n\ell)$, where the error term $O(d^5 n)$ accounts for all miscounts that violate constraints (a') and (b') while the error term $O(d^3 n\ell)$ accounts for the case that one of the two paths contains an edge in $M_1 \cap M_2$. Clearly, $\sum_{g \in F_d(\ell)} N(g) = \sum_{g' \in F_d(\ell-1)} N'(g')$. Moreover, for any $\ell \le (1-\alpha)dn/2$, $dn/2 - \ell = \Omega(dn)$. So,

$$\frac{f_d(\ell)}{f_d(\ell-1)} = \frac{(d^2 n)^2 + O(d^4 n + d^3 n\ell)}{8\ell\Big((dn/2 - \ell + O(d^2))^2 + O(d^2 n)\Big)} = \frac{d^2}{2\ell}(1 + O(\ell/dn + d/n)). \quad \blacksquare$$

Lemma 3.3 *Let $t > 0$ be an integer. For any $d = o(n^{1/3})$ and any degree sequence $\mathbf{d}$ such that $dn - \sum_{i=1}^n d_i = t$, $g(n, \mathbf{d}) \le g(n,d) n^{-t} \exp(O(dn))$.*

Proof. The asymptotic value of $g(n,d)$ was given in Theorem 1.1, whereas the asymptotic value of $g(n, \mathbf{d})$ was also given in [11]. But here we only use a crude upper bound as follows. For any degree sequence $\mathbf{d}$ with $M = \sum_{i=1}^n d_i$ being even,

$$g(n, \mathbf{d}) \le \frac{M!}{2^{M/2}(M/2)! \prod_{i=1}^n d_i!}.$$

For any real $x > 0$, define $x! = \prod_{i=0}^{\lceil x\rceil -1}(x-i)$. Since $\prod_{i=1}^n d_i! \ge ((M/n)!)^n$, we have

$$g(n, \mathbf{d}) \le \frac{M!}{2^{M/2}(M/2)!((M/n)!)^n}.$$

Thus,

$$\frac{g(n,\mathbf{d})}{g(n,d)} \le \frac{(dn-t)!}{(dn)!}\left(\frac{d!}{(d-t/n)!}\right)^n \exp(O(dn)) = n^{-t}\exp(O(dn)). \quad \blacksquare$$

Lemma 3.4 *Let $\alpha = 1/10$ and let $t_n = \sum_{\ell > (1-\alpha)dn/2} f(\ell)$. Then $\ln(t_n/g(n,d)^2) \le -(1-\alpha)dn\ln n + O(dn)$.*

Proof. We count the number of (M_1, M_2) such that both M_1 and M_2 are d-regular graphs on $[n]$ and they share at least $(1-\alpha)dn/2$ edges. There are $g(n,d)$ ways to choose M_1. Given any M_1, there are at most $2^{dn/2}$ ways to choose a set T of shared edges. Given any such T, the number of ways to choose M_2 such that $T \subseteq M_2$ is at most $g(n,d)n^{-2|T|}\exp(O(dn))$ by Lemma 3.3. Therefore,

$$t_n \le g(n,d)^2 2^{dn/2} n^{-(1-\alpha)dn}\exp(O(dn)) = g(n,d)^2 n^{-(1-\alpha)dn}\exp(O(dn)). \quad \blacksquare$$

Lemma 3.5 *Assume $d^3 = o(p^2 n)$. Then $\mathbf{E}_p(X_{n,d}^2) \sim \beta_{n,d}\mu_{n,d}^2$.*

Proof. As explained before, every graph in $F_d(\ell)$ contains $dn-\ell$ edges. So

$$\mathbf{E}_p(X_{n,d}^2) = \sum_{\ell=0}^{dn/2} p^{dn-\ell} f_d(\ell) = p^{dn}\sum_{\ell=0}^{dn/2}\frac{f_d(\ell)}{p^\ell}.$$

By Lemma 3.2, for any $1 \le \ell \le (9/10)dn/2$,

$$\frac{p^{-\ell}f_d(\ell)}{p^{-(\ell-1)}f_d(\ell-1)} = \frac{d^2}{2p\ell}(1 + O(\ell/dn + d/n)).$$

Hence, for any function $w(n)$ that tends to infinity arbitrarily slowly as n tends to infinity, we have

$$\frac{p^{-\ell}f_d(\ell)}{p^{-(\ell-1)}f_d(\ell-1)} = o(1),$$

for all $d^2 w(n)/p \le \ell \le (9/10)dn/2$. On the other hand,

$$p^{dn}\sum_{\ell=(9/10)dn/2}^{dn/2}\frac{f_d(\ell)}{p^\ell} \le g(n,d)^2 n^{-(9/10)dn}\exp(O(dn)),$$

by Lemma 3.4. Thus,

$$\begin{aligned}
\mathbf{E}_p(X_{n,d}^2) &\sim p^{dn}\sum_{\ell=0}^{\lfloor d^2w(n)/p\rfloor}\frac{f_d(\ell)}{p^\ell}+O(g(n,d)^2n^{-(9/10)dn}\exp(O(dn)))\\
&= p^{dn}f_d(0)\sum_{i=0}^{\lfloor d^2w(n)/p\rfloor}\frac{d^{2i}}{(2p)^i i!}(1+O(i(i/dn+d/n)))+O(g(n,d)^2n^{-(9/10)dn}\exp(O(dn)))\\
&= p^{dn}f_d(0)\exp(d^2/2p)\left(1+O\left(\frac{d^4w(n)^2}{p^2dn}+\frac{d^3w(n)}{pn}\right)\right)+O(g(n,d)^2n^{-(9/10)dn}\exp(O(dn))).
\end{aligned}$$

Since $d^3=o(p^2n)$, we have

$$\mathbf{E}_p(X_{n,d}^2)\sim p^{dn}f_d(0)\exp(d^2/2p)+O(g(n,d)^2n^{-(9/10)dn}\exp(O(dn))). \tag{3.2}$$

In particular, (3.2) holds for $p=1$. Thus,

$$\mathbf{E}_1(X_{n,d}^2)\sim f_d(0)\exp(d^2/2)+O(g(n,d)^2n^{-(9/10)dn}\exp(O(dn)))=f_d(0)\exp(d^2/2)+o(g(n,d)^2).$$

However, $\mathbf{E}_1(X_{n,d}^2)=X_{n,d}^2=g(n,d)^2$. Hence,

$$f_d(0)\sim g(n,d)^2\exp(-d^2/2).$$

Combining with (3.2), we have

$$\mathbf{E}_p(X_{n,d}^2)\sim p^{dn}g(n,d)^2\exp(-d^2/2+d^2/2p)+O(g(n,d)^2n^{-(9/10)dn}\exp(O(dn)))\sim\beta_{n,d}\mu_{n,d}^2,$$

since $n^{-(9/10)dn}\exp(O(dn))=o(p^{dn})$. ∎

It follows right away that when $d^2(1-p)=o(1)$, the number of d-factors in $\mathcal{G}(n,p)$ is concentrated around its expectation.

Corollary 3.6 *Assume $d^2(1-p)=o(1)$. Then for any $\epsilon>0$,*

$$\mathbf{P}_{\mathcal{G}(n,p)}(|X_{n,d}-\mu_{n,d}|\ge\epsilon\mu_{n,d})=o(1).$$

Proof. By the definition of $\beta_{n,d}$ in (2.1), $\beta_{n,d}=1+o(1)$ when $d^2(1-p)=o(1)$. Hence the corollary follows by Lemma 3.5 and Chebyshev's inequality. ∎

Now we extend the proof of Lemma 3.2 and 3.5 to show Theorem 2.1. We count the number of k-tuples $\mathbf{M}=(M_1,\ldots,M_k)$, where each M_i is a d-regular graph on the same vertex set $[n]$. Let $H=(h_{i,j})_{i,j\le k}$ be a matrix and let F_H denote the set of $\mathbf{M}$ such that the number of edges in M_i that are shared by exactly j of the k graphs from $\mathbf{M}$ is $h_{i,j}$. For each $1\le i\le k$, we let $x_i=\sum_{j=2}^k h_{i,j}$, denoting the number of edges in M_i that are shared by some other M_j with $j\ne i$. Let $\mathbf{Z}_+^\ell$ denote the set of ℓ-dimensional vectors with each component a nonnegative integer. Let $J=\{\mathbf{j}=(j_2,\ldots,j_k)\in\mathbf{Z}_+^{k-1}:\ \sum_{i=2}^k ij_i\le kdn/2\}$. Given $\mathbf{j}=(j_2,\ldots,j_k)\in J$, define $j_1=kdn/2-\sum_{i=2}^k ij_i$. Let $F_d(\mathbf{j})$ denote the set of $\mathbf{M}$ such that in $\cup_{i=1}^k M_i$, there are exactly j_i edges that are contained in exactly i of the k d-regular graphs in $\mathbf{M}$. Let $f_d(\mathbf{j})=|F_d(\mathbf{j})|$. Given $\mathbf{j}\in J$, let $\mathcal{H}(\mathbf{j})=\{H: ij_i=\sum_{\ell=1}^k h_{\ell,i},\ \forall\ 1\le i\le k\}$. Then, $F_d(\mathbf{j})=\cup_{H\in\mathcal{H}(\mathbf{j})}F_H$.

Similarly as in the proof where $k = 2$, it is hard to analyse the switchings when there is some x_i such that x_i is close to $dn/2$. Therefore, we will restrict our analysis of switchings on a subset of $F_d(\mathbf{j})$ instead. Let $0 < \alpha < 1$ be an arbitrary constant. For any $\mathbf{j} \in J$, let $\mathcal{H}^\alpha(\mathbf{j})$ be the subset of $\mathcal{H}(\mathbf{j})$ such that $x_i \le (1-\alpha)dn/2$ for all $1 \le i \le k$. Let $F_d^\alpha(\mathbf{j}) = \cup_{H \in \mathcal{H}^\alpha(\mathbf{j})} F_H$. Analogous to Lemma 3.4, we bound $\sum_{\mathbf{j}\in J} |F_d(\mathbf{j}) \setminus F_d^\alpha(\mathbf{j})|$. Recall that $\mu_{n,d} = g(n,d)p^{dn/2}$.

Lemma 3.7 *Let $0 < \alpha < 1$ be an arbitrary constant and let*

$$t_n = \sum_{\mathbf{j}\in J} |F_d(\mathbf{j}) \setminus F_d^\alpha(\mathbf{j})|.$$

Then

$$\ln(t_n/\mu_{n,d}^k) \le -\frac{dn}{2}\Big((7/4 - 2\alpha)\ln n + k \ln p\Big) + O(dn).$$

Proof. By the definition of t_n,

$$t_n = \sum_{\mathbf{j}\in J} \sum_{H \in \mathcal{H}(\mathbf{j}) \setminus \mathcal{H}^\alpha(\mathbf{j})} |F_H|.$$

By the definition of $\mathcal{H}^\alpha(\mathbf{j})$, for any $H \in \mathcal{H}(\mathbf{j}) \setminus \mathcal{H}^\alpha(\mathbf{j})$, there exists i with $x_i > (1-\alpha)dn/2$. We first bound the number of all $\mathbf{M}$ with at least $(1-\alpha)dn/2$ edges of M_i being shared by other M_ℓ, for a fixed i. There are $g(n,d)$ ways to choose M_i. Colour the edges of M_i with k colours so that if an edge is in M_j for $j \ge 1$ but not in any M_ℓ with $\ell < j$, it receives colour j. Colour all edges that are only contained in M_i with colour 1. Given M_i, there are $k^{dn/2}$ ways to k-colour all edges in M_i. Let T be the set of edges with colour not equal to 1. For any given T with $|T| > (1-\alpha)dn/2$, the number of ways to choose the other $k-1$ M_ℓ is at most $g(n,d)^{k-1} n^{-(1-\alpha)dn} \exp(O(dn))$ by Lemma 3.3. Thus, for any fixed i, the number $\mathbf{M}$ with at least $(1-\alpha)dn/2$ edges of M_i being shared by other M_ℓ is at most $g(n,d)^k k^{dn/2} n^{-(1-\alpha)dn} \exp(O(dn))$. There are k choices for i and the number of choices of $\mathbf{j}$ is at most $(dn/2)^k$. Therefore,

$$t_n \le k(dn/2)^k g(n,d)^k k^{dn/2} n^{-(1-\alpha)dn} \exp(O(dn)) = g(n,d)^k k^{dn/2} n^{-(1-\alpha)dn} \exp(O(dn)).$$

Thus,

$$\ln(t_n/\mu_{n,d}^k) \le \frac{dn}{2}\ln k - (1-\alpha)dn \ln n - \frac{kdn}{2}\ln p + O(dn).$$

Since $k^4 d^3 = o(p^2 n)$, we have $k = o(n^{1/4})$. Thus, $\ln k \le \ln n/4$. Then the conclusion of the lemma follows. ∎

Proof of Theorem 2.1. The case $k = 1$ is trivially true and the case $k = 2$ is true by Lemma 3.5. Now consider an integer $k \ge 3$ such that $k^4 d^3 = o(p^2 n)$ and $(7/4 - 2\alpha)\ln n + k \ln p \to \infty$ as $n \to \infty$.

We extend the s-switching into the s_i-*switching* for any $2 \le i \le k$ as follows. Choose any edge x that is contained in exactly i of the k d-regular graphs in $\mathbf{M}$. Label the two end vertices of x as u and u'. Assume the i d-regular graphs that contain x are $M_{\ell_1}, \ldots, M_{\ell_i}$. Then pick another i edges $y_1, \ldots, y_i$, such that for each $1 \le j \le i$, y_j is contained in M_{ℓ_j} but not other regular graphs in $\mathbf{M}$. Label the end vertices of every y_j as u_j and u'_j. An s_i-switching replaces x and y_j by $\{u, u_j\}$ and $\{u', u'_j\}$ in M_{ℓ_j} for every $1 \le j \le i$. The s_i-switching is applicable only if

(a) all vertices u, u', u_j, u'_j, for all $1 \le j \le i$ are distinct;

(b) none of $\{u, u_j\}$, $\{u', u'_j\}$ for all $1 \le j \le i$ are contained in any of the regular graphs in $\mathbf{M}$.

Similarly we can extend the inverse s-switching to *inverse s_i-switching*. We choose a star centered at u where for $j = 1, \ldots, i$, uu_j is contained in M_{ℓ_j} but not other regular graphs in $\mathbf{M}$. Similarly, we choose another star centered at u' where for $j = 1, \ldots, i$, $u'u'_j$ is contained in M_{ℓ_j} but not other regular graphs in $\mathbf{M}$. The inverse s_i-switching replaces the edges in the two stars by $\{u, u'\}$ and $\{u_j, u'_j\}$ for all $1 \le j \le i$. The inverse s_i-switching is applicable only if

(a') all vertices u, u', u_j, u'_j for $1 \le j \le i$ are distinct;

(b') Neither $\{u, u'\}$ nor any of $\{u_j, u'_j\}$ for $1 \le j \le i$ is contained in any of the regular graphs in $\mathbf{M}$.

Figure 2 gives an example that illustrates the s_3-switching and its inverse.

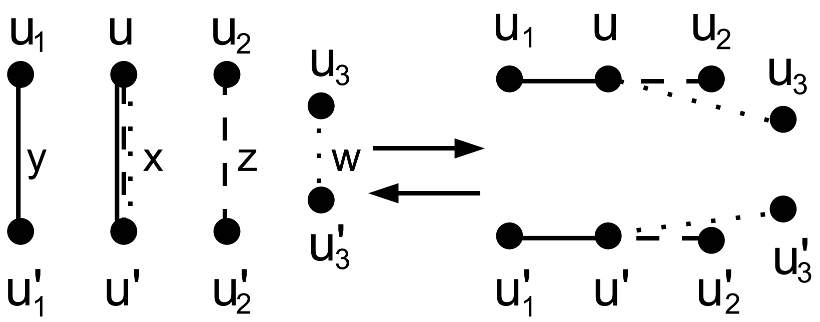

Figure 2: *s_3-switching and its inverse*

Fix any $i \ge 2$, let $\mathbf{j} = (j_2, \ldots, j_i, 0, \ldots, 0)$ and let $\mathbf{j}' = (j_2, \ldots, j_i - 1, 0, \ldots, 0)$, where $j_i \ge 1$. Clearly an s_i-switching converts an element $g \in F_d(\mathbf{j})$ to an element $g' \in F_d(\mathbf{j}')$. We define $N(g)$ and $N'(g')$ similarly as in the proof of Lemma 3.2. An analogous argument gives that for any $g \in F_d(\mathbf{j})$,

$$\begin{aligned} j_i(dn/2)^i 2^{i+1} \ge N(g) &\ge j_i(dn/2)^i 2^{i+1} - j_i i (dn/2)^{i-1} 2^{i+1} \sum_{2\le \ell \le i} j_\ell + O\left(j_i (dn/2)^{i-1} 2^i \cdot kd^2\right) \\ &= 2 j_i d^i n^i \left(1 - 2i \sum_{2\le \ell\le i} j_\ell/dn + O\left(kd/n\right)\right), \end{aligned} \tag{3.3}$$

where the second inequality holds because the number of ways to choose the i edges $y_1, \ldots, y_i$ such that at least one of them is contained in more than one regular graph is at most $i(dn/2)^{i-1}2^{i+1}\sum_{2\le\ell\le i} j_\ell$. On the other hand, for any $g \in F_d^\alpha(\mathbf{j}) \subseteq F_d(\mathbf{j})$,

$$N(g) \ge j_i 2^{i+1}\Big(dn/2 - (1-\alpha)dn/2 + O(kd^2)\Big)^i = \Omega(j_i (dn)^i), \tag{3.4}$$

since in every M_ℓ, the number of edges that are shared by other regular graphs of $\mathbf{M}$ is at most $(1-\alpha)dn/2$ by the definition of $F_d^\alpha(\mathbf{j})$.We also have the following observation.

Observation: For any $0<\beta_1\le\beta_2<1$, and any F_1 such that $F_d^{\beta_1}(\mathbf{j}')\subseteq F_1\subseteq F_d^{\beta_2}(\mathbf{j}')$, let F_2 be the set of $\mathbf{M}$ that can be obtained from F_1 by an inverse s_i-switching. Then $F_d^{\beta_1}(\mathbf{j})\subseteq F_2\subseteq F_d^{\beta_2+2/dn}(\mathbf{j})$.

Next, we estimate $N'(g')$. For any $g'\in F_d(\mathbf{j}')$, each vertex has degree at most kd since every vertex is incident with d edges from every $M_j\in\mathbf{M}$. There are $\binom{k}{i}$ ways to choose a set of i distinct regular graphs $M_{\ell_1},\dots,M_{\ell_i}$ from $\mathbf{M}$ and given each choice, there are approximately $(d^in)^2$ ways to choose the two stars. Thus,

$$\begin{aligned}N'(g') &= \binom{k}{i}\left((d^in)^2+O\left(kd^{2i+1}n+d^in\cdot d^{i-1}\sum_{2\le\ell\le i}j_\ell\right)\right)\\ &= \binom{k}{i}(d^in)^2\left(1+O\left(kd/n+\sum_{2\le\ell\le i}j_\ell/dn\right)\right),\end{aligned}$$

where the error term $O(kd^{2i+1}n)$ accounts for the case when condition (a') or (b') is violated whereas the error term $O(d^in\cdot d^{i-1}\sum_{\ell\le i}j_\ell)$ accounts for the case when one of the two stars contains an edge that is commonly shared by at least two of the regular graphs in $\mathbf{M}$. Then it follows that for any $i\ge 2$ and for any $\mathbf{j}\in J$ with $\mathbf{j}=(j_2,\dots,j_i,0,\dots,0)$, $j_i\ge 1$, and $F_d^\alpha(\mathbf{j})$ not empty, by (3.3) and (3.4),

$$\frac{|F'|}{|F_d^\alpha(j_2,\dots,j_i-1,0,\dots,0)|}=\frac{\binom{k}{i}d^i}{2j_in^{i-2}}\left(1+O\left(kd/n+i\sum_{2\le\ell\le i}j_\ell/dn\right)\right).$$

where F' is the set of $\mathbf{M}$ that can be obtained from $|F_d^\alpha(j_2,\dots,j_i-1,0,\dots,0)|$ by an inverse s_i-switching. By our observation, $F'\supseteq F_d^\alpha(j_2,\dots,j_i,0,\dots,0)$. Thus, for any $i\ge 3$ and $j_i\ge 1$,

$$\frac{|F_d^\alpha(j_2,\dots,j_i,0,\dots,0)|p^{-(i-1)j_i}}{|F_d^\alpha(j_2,\dots,j_i-1,0,\dots,0)|p^{-(i-1)(j_i-1)}}\le\frac{k^id^i}{2j_in^{i-2}p^{i-1}}\left(1+O\left(kd/n+i\sum_{2\le\ell\le i}j_\ell/dn\right)\right)=o(1), \tag{3.5}$$

and for any $j_2\ge dn/\ln n$,

$$\frac{|F_d^\alpha(j_2,0,\dots,0)|p^{-j_2}}{|F_d^\alpha(j_2-1,0,\dots,0)|p^{-(j_2-1)}}\le\frac{k^2d^2}{2j_2p}\left(1+O\left(kd/n+j_2/dn\right)\right)=o(1). \tag{3.6}$$

Given $\mathbf{j}\in J$, every graph in $F_d(\mathbf{j})$ contains $kdn/2-\sum_{i=2}^k(i-1)j_i\ge dn/2$ edges. So

$$\mathbf{E}_p(X_{n,d}^k)=\sum_{\mathbf{j}\in J}|F_d(\mathbf{j})|p^{kdn/2-\sum_{i=2}^k(i-1)j_i}=\sum_{\mathbf{j}\in J}|F_d^\alpha(\mathbf{j})|p^{kdn/2-\sum_{i=2}^k(i-1)j_i}+O(\gamma_n), \tag{3.7}$$

where $\gamma_n=\sum_{\mathbf{j}\in J}|F_d(\mathbf{j})\setminus F_d^\alpha(\mathbf{j})|p^{dn/2}$. By (3.5) and (3.6),

$$\begin{aligned}\mathbf{E}_p(X_{n,d}^k) &\sim \sum_{j_2=0}^{dn/2}|F_d^\alpha(j_2,0,\dots,0)|p^{kdn/2-j_2}+O(\gamma_n)\\ &\sim \sum_{j_2=0}^{dn/\ln n}|F_d^\alpha(j_2,0,\dots,0)|p^{kdn/2-j_2}+O(\gamma_n). \end{aligned}\tag{3.8}$$

Next, we estimate $|F_d^\alpha(j_2, 0, \dots, 0)|$. Define a sequence $(S_\ell)_{\ell \geq 0}$ as follows. Let $S_0 = F_d^\alpha(0, \dots, 0)$. For any $\ell \geq 1$, let S_ℓ be defined as the set of $\mathbf{M}$ that can be obtained from $S_{\ell-1}$ by an inverse s_2-switching. Then by our observation above, for each ℓ, $F_d^\alpha(\ell, 0, \dots, 0) \subseteq S_\ell \subseteq F_d^{\alpha+2\ell/dn}(\ell, 0, \dots, 0)$. Let $s(\ell) = |S_\ell|$ for all $\ell \geq 0$. Since $0 < \alpha < 7/8$, for any $\ell \leq dn/\ln n$, $0 < \alpha + 2\ell/dn < 7/8$. By (3.3) and (3.4), for any $\ell \leq dn/\ln n$ and any $g \in S_\ell$,

$$\frac{1}{N(g)} = \frac{1}{2j_2 d^2 n^2} \left(1 + O\left(j_2/dn + kd/n\right)\right).$$

Then for any $j_2 \leq dn/\ln n$,

$$\frac{s(j_2)p^{-j_2}}{s(j_2-1)p^{-(j_2-1)}} = \frac{\binom{k}{2}d^2}{2j_2 p} \left(1 + O\left(kd/n + j_2/dn\right)\right).$$

By noting that $S_{j_2} \supseteq F_d^\alpha(j_2, 0, \dots, 0)$ for all $j_2 \geq 0$ and by (3.8), we have

$$\mathbf{E}_p(X_{n,d}^k) \sim \sum_{j_2=0}^{dn/\ln n} s(j_2) p^{kdn/2 - j_2} + O(\gamma_n).$$

Since $(7/4 - 2\alpha)\ln n + k \ln p \to \infty$ as $n \to \infty$ by our assumption, by Lemma 3.7, $\gamma_n = o(\mu_{n,d}^k)$. Let $M = k^2 d^2/p$. Then since $k^4 d^3 = o(p^2 n)$, we have $Mkd = o(n)$ and $M^2 = o(dn)$. Hence, for any arbitrarily slowly growing function $w(n)$, uniformly for all k in this range,

$$\begin{aligned}
\mathbf{E}_p(X_{n,d}^k) &\sim \sum_{j_2=0}^{dn/\ln n} s(j_2) p^{kdn/2 - j_2} + o(\mu_{n,d}^k) \sim p^{kdn/2} \sum_{j_2=0}^{Mw(n)} p^{-j_2} s(j_2) + o(\mu_{n,d}^k) \\
&= p^{kdn/2} s(0) \exp\left(\binom{k}{2} d^2/2p\right)\left(1 + O\left(Mw(n)\left(\frac{kd}{n} + \frac{Mw(n)}{dn}\right)\right)\right) + o(\mu_{n,d}^k) \\
&\sim p^{kdn/2} s(0) \exp\left(\binom{k}{2} d^2/2p\right) + o(\mu_{n,d}^k).
\end{aligned}$$

In particular, taking $p = 1$ in the above formula yields

$$g(n,d)^k = \mathbf{E}_1(X_{n,d}^k) \sim s(0) \exp\left(\binom{k}{2} d^2/2\right). \tag{3.9}$$

Thus,

$$\mathbf{E}_p(X_{n,d}^k) \sim \mu_{n,d}^k \exp\left(\binom{k}{2}(d^2/2p - d^2/2)\right) = \mu_{n,d}^k \beta_{n,d}^{\binom{k}{2}}. \quad \blacksquare$$

Proof of Proposition 2.2. It follows directly from (3.9). ∎

For any real x and any integer $\ell \geq 0$, define the ℓ-th falling factorial $[x]_\ell$ to be $\prod_{i=0}^{\ell-1}(x - i)$.

Lemma 3.8 *Let $N = \binom{n}{2}$ and let $p = m(n)/N$, where $0 < m(n) < N$. Then for any integer $\ell = \ell(n) \geq 0$ such that $\limsup_{n\to\infty} \ell(n)/m(n) < 1$,*

$$\binom{N-\ell}{m-\ell} \Big/ \binom{N}{m} = p^\ell \exp\left(-\frac{1-p}{pN}\frac{\ell^2 - \ell}{2} + O(\ell^3/m^2)\right).$$

Moreover, if $\ell = \Omega(\sqrt{m})$, then

$$\binom{N-\ell}{m-\ell}\Big/\binom{N}{m} = p^\ell \exp\left(-\frac{1-p}{pN}\frac{\ell^2}{2} + O(\ell^3/m^2)\right).$$

Proof.

$$\begin{aligned}\binom{N-\ell}{m-\ell}\Big/\binom{N}{m} &= \frac{[m]_\ell}{[N]_\ell} = \prod_{i=0}^{\ell-1}\frac{m-i}{N-i} \\ &= \prod_{i=0}^{\ell-1}\frac{m}{N}\exp\left(-\frac{i}{m}+\frac{i}{N}+O(i^2/m^2)\right) \quad (\text{since } \limsup_{n\to\infty}\ell(n)/m(n) < 1) \\ &= p^\ell \exp\left(-\frac{1-p}{pN}\frac{\ell^2-\ell}{2} + O(\ell^3/m^2)\right).\end{aligned}$$

If we have further that $\ell = \Omega(\sqrt{m})$, then $\ell/pN = O(\ell^3/m^2)$. ∎

Proof of Theorem 2.4. The probability space considered in this proof is $\mathcal{G}(n,m)$. Let $N = \binom{n}{2}$ and let $p = m/N$. We Apply Lemma 3.8 with $\ell = dn/2$. Since $d^3 = o(p^2 n)$, we have $\ell = \Omega(\sqrt{m})$. Thus,

$$\mathbf{E}X_{n,d} = g(n,d)\binom{N-dn/2}{m-dn/2}\Big/\binom{N}{m} = g(n,d)p^{dn/2}\exp\left(-\frac{1-p}{pN}\frac{d^2n^2}{8} + O(d^3n^3/m^2)\right).$$

Following the notations above Lemma 3.2, we have

$$\mathbf{E}X_{n,d}^2 = \sum_{\ell=0}^{dn/2} f_d(\ell)\binom{N-(dn-\ell)}{m-(dn-\ell)}\Big/\binom{N}{m}.$$

Let

$$g_d(\ell) = f_d(\ell)\binom{N-(dn-\ell)}{m-(dn-\ell)}\Big/\binom{N}{m}.$$

Since

$$\binom{N-(dn-\ell)}{m-(dn-\ell)}\Big/\binom{N-(dn-\ell+1)}{m-(dn-\ell+1)} = \frac{N-dn+\ell}{m-dn+\ell} = \frac{1}{p}(1+O(dn/m)),$$

by Lemma 3.2, for any $\ell \le (9/10)dn/2$,

$$\frac{g_d(\ell)}{g_d(\ell-1)} = \frac{d^2}{2p\ell}\left(1+O(\ell/dn + d/n + dn/m)\right).$$

We first bound $\gamma_n = \sum_{\ell>(9/10)dn/2} g_d(\ell)$. Since for any $\ell \le dn/2$, $\binom{N-(dn-\ell)}{m-(dn-\ell)} \le \binom{N-dn/2}{m-dn/2}$, by Lemma 3.4,

$$\gamma_n \le g(n,d)^2 n^{-(9/10)dn}\exp(O(dn))p^{dn/2}\exp\left(-\frac{1-p}{pN}\frac{d^2n^2}{8} + O(d^3n^3/m^2)\right). \tag{3.10}$$

Following the same approach as in Lemma 3.5, we obtain

$$\begin{aligned}
\mathbf{E}X_{n,d}^2 &\sim \gamma_n + p^{dn} f_d(0) \exp\left(-\frac{1-p}{pN}\frac{(dn)^2}{2} + O(d^3/p^2n)\right) \sum_{i=0}^{d^2 w(n)/p} \frac{d^{2i}}{(2p)^i i!}\left(1 + O\left(\frac{i^2}{dn} + \frac{id}{n} + \frac{idn}{m}\right)\right) \\
&\sim \gamma_n + p^{dn} f_d(0) \exp\left(-\frac{1-p}{pN}\frac{(dn)^2}{2} + \frac{d^2}{2p}\right) \sim \gamma_n + p^{dn} g(n,d)^2 \exp\left(-\frac{d^2(1-p)}{p} + \frac{d^2}{2p} - \frac{d^2}{2}\right) \\
&= \gamma_n + p^{dn} g(n,d)^2 \exp(-d^2(1-p)/2p),
\end{aligned}$$

as $d^3 = o(p^2 n)$. By (3.10), $\gamma_n = o(p^{dn} g(n,d)^2 \exp(-d^2(1-p)/2p))$. Thus,

$$\mathbf{E}X_{n,d}^2 \sim (\mathbf{E}X_{n,d})^2,$$

since

$$(\mathbf{E}X_{n,d})^2 \sim p^{dn} g(n,d)^2 \exp\left(-\frac{1-p}{pN}\frac{d^2n^2}{4}\right) = p^{dn} g(n,d)^2 \exp\left(-\frac{d^2(1-p)}{2p}\right).$$

By Theorem 1.1 and the assumption $d^3 = o(p^2 n)$, we have $\mathbf{E}X_{n,d} \to \infty$ as $n \to \infty$. Then by Chebychev's inequality, for any $\epsilon > 0$,

$$\mathbf{P}(|X_{n,d}/\mathbf{E}X_{n,d} - 1| > \epsilon) \to 0, \quad \text{as } n \to \infty.$$

Hence,

$$X_{n,d}/\mathbf{E}X_{n,d} \xrightarrow{p} 1,$$

as $n \to \infty$. ∎

Proof of Theorem 2.5. Let Y_n denote the number of edges in $\mathcal{G}(n,p)$, then $Y_n \sim Bin(N,p)$. Hence we have

$$Y_n - pN = O_p(\sqrt{p(1-p)N}), \tag{3.11}$$

where $f(n) = O_p(g(n))$ for some $g(n) \geq 0$ means $\mathbf{P}(|f(n)| > Kg(n)) \to 0$ as $K \to \infty$ and $n \to \infty$. Similarly we use the notation $f(n) = o_p(g(n))$ meaning that for every $\epsilon > 0$, $\mathbf{P}(|f(n)| > \epsilon g(n)) \to 0$ as $n \to \infty$. Since $d^3 = o(p^2 n)$, we have $d^3 = o_p(Y_n^2 n/N^2)$ and thus $d^3 n^3/Y_n^2 = o_p(1)$. By conditioning on Y_n and applying Theorem 2.4 and Lemma 3.8, we have

$$\ln X_{n,d}(\mathcal{G}(n,p)) - \ln g(n,d) - \frac{dn}{2}\ln(Y_n/N) + \frac{1 - Y_n/N}{Y_n}\frac{d^2n^2}{8} \xrightarrow{p} 0. \tag{3.12}$$

By (3.11),

$$\begin{aligned}
\frac{1 - Y_n/N}{Y_n}\frac{d^2n^2}{8} &= \frac{d^2(1-p)}{4p}\left(1 + O_p\left(\sqrt{\frac{p}{(1-p)N}} + \sqrt{\frac{1-p}{pN}}\right) + O(n^{-1})\right) \\
&= \frac{d^2(1-p)}{4p} + o_p(1), \qquad (3.13)
\end{aligned}$$

where the equality above holds because $d^3 = o(p^2 n)$. We also have

$$\ln(Y_n/N) = \ln p(1 + Y_n^* \sqrt{(1-p)/pN}) = \ln p + \sqrt{(1-p)/pN}\, Y_n^* + O_p((1-p)/pN), \tag{3.14}$$

where

$$Y_n^* = \frac{Y_n - pN}{\sqrt{p(1-p)N}}$$

is the normalised variable of Y_n. Recall from (2.1) that $\mu_{n,d} = g(n,d)p^{dn/2}$ and $\ln \beta_{n,d} = d^2(1-p)/2p$ and from the statement of Theorem 2.5 that $\lambda_{n,d} = \mu_{n,d}/\sqrt{\beta_{n,d}}$ and $s_{n,d} = \ln \beta_{n,d} = d^2(1-p)/2p$. Combining with (3.12)–(3.14), we have

$$\ln(X_{n,d}/\lambda_{n,d}) - \sqrt{s_{n,d}}(1+O(n^{-1}))Y_n^* \xrightarrow{p} 0. \tag{3.15}$$

Since $\liminf_{n\to\infty} d^2(n)(1-p(n)) > 0$, we have $s_{n,d} = \Omega(1)$. Thus (3.15) immediately yields

$$\frac{\ln(X_{n,d}/\lambda_{n,d})}{\sqrt{s_{n,d}}} = Y_n^* + o_p(1).$$

Since $Y_n^* \xrightarrow{d} \mathcal{N}(0,1)$, the theorem follows. ∎

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