Induced subgraphs in sparse random graphs with given degree sequence

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Abstract

Let $\mathcal{G}_{n,d}$ denote the uniformly random d-regular graph on n vertices. For any $S \subset [n]$, we obtain estimates of the probability that the subgraph of $\mathcal{G}_{n,d}$ induced by S is a given graph H. The estimate gives an asymptotic formula for any $d = o(n^{1/3})$, provided that H does not contain almost all the edges of the random graph. The result is further extended to the probability space of random graphs with a given degree sequence.

1 Introduction

Properties of subgraphs and induced subgraphs in random graph models have been investigated by various authors. Ruciński [12, 14] studied the distribution of the count of small subgraphs in the standard random graph model $\mathcal{G}_{n,p}$, and conditions under which the distribution converges to the normal distribution. He also studied properties of induced subgraphs in [13].

Techniques for analysing the standard random graph model $\mathcal{G}_{n,p}$ often do not apply in the random regular graph model $\mathcal{G}_{n,d}$. We take the vertex set of the graph to be $[n]$ in both these models. For $S \subseteq [n]$, let G_S denote the subgraph of G induced by S. For a graph H with vertex set S, computing the probabilities $\mathbf{P}(G_S \supseteq H)$ and $\mathbf{P}(G_S = H)$ in $\mathcal{G}_{n,p}$ is trivial, but computing them in $\mathcal{G}_{n,d}$ is not easy, especially when the degree $d \to \infty$ as $n \to \infty$. McKay [8] estimated lower and upper bounds of $\mathbf{P}(G_S \supseteq H)$ in $\mathcal{G}_{n,d}$ when the degree sequence of H and d satisfy certain conditions. These bounds are useful in estimating the asymptotic value of $\mathbf{P}(G_S \supseteq H)$ when H is small or d is not too large. Z. Gao and the third author [6] proved

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that the distribution of the number of small subgraphs with certain restrictions (such as d not growing too quickly) converges to the normal distribution in $\mathcal{G}_{n,d}$. No such results on induced subgraphs have been derived, although the main results of [8] could be used as a basis for obtaining results on induced subgraphs. However, this would require severe restrictions on the size of the subgraphs, and seems unlikely to apply to subgraphs with more than $n^{2/3}$ vertices for any d.

On the other hand, for very dense regular graphs, Krivelevich, Sudakov and Wormald [7] computed $\mathbf{P}(G_S = H)$ in $\mathcal{G}_{n,d}$ when n is odd, $d = (n-1)/2$ and $|V(H)| = o(\sqrt{n})$. McKay [11] has recently given a stronger result, for more general degree sequences and provided H has less than $n^{1+\epsilon}$ edges for some $\epsilon > 0$.

An asymptotic formula of the probability that $G_S = H$ or $G_S \supseteq H$ in a random bipartite graph with a specified degree sequence has been derived by Bender [2] when the maximum degree is bounded. The result was extended further by Bollobás and McKay [4] and by McKay [9] when the maximum degree goes to infinity slowly as n goes to infinity. Greenhill and McKay [5] recently derived an asymptotic formula for the case when the random bipartite graph is sufficiently dense and H is sparse enough.

For a vector $\mathbf{d} = (d_1, \ldots, d_n)$ of nonnegative integers, let $M = M(\mathbf{d}) = \sum_{i=1}^{n} d_i$ and let $\mathcal{G}_{\mathbf{d}}$ denote the class of graphs with degree sequence $\mathbf{d}$ and the uniform distribution (so $\mathcal{G}_{\mathbf{d}}$ is a generalisation of $\mathcal{G}_{n,d}$). In this paper, we compute the probability that $G_S = H$ in $\mathcal{G}_{\mathbf{d}}$ when $d_{\max} = o((M - 2m(H))^{1/4})$, where $m(H)$ denotes the number of edges in H and $d_{\max} = \max\{d_1, \ldots, d_n\}$. The power of this result is that there is no major restriction on the size or density of H. In Section 2, as a direct application of our main result, we compute the probability that a given set of vertices in $\mathcal{G}_{n,d}$ is an independent set. Our results will also be useful as a basic tool for studying the properties of induced subgraphs in the binomial random graph $\mathcal{G}(n,p)$, such as the subgraph induced by the vertices of even degree, or odd degree.

A graph G is called a *B-graph with vertex bipartition* (L, R) if $V(G) = L \cup R$, and L is an independent set of G. If the graph is not necessarily simple, i.e. loops and multiple edges are allowed, we call it a *B-multigraph* instead. An edge in a B-graph or B-multigraph is called a *mixed edge* if it has one end vertex in L and one in R, and a *pure edge* if both its end vertices are in R. Given a nonnegative integer vector $\mathbf{d}$, let $\mathcal{G}(L, R, \mathbf{d})$ be the set of B-graphs with bipartition L and R and degree sequence $\mathbf{d}$ and let $g(L, R, \mathbf{d}) = |\mathcal{G}(L, R, \mathbf{d})|$. By convention, $g(L, R, \mathbf{d}) = 0$ if $\mathbf{d}$ is not nonnegative.

Given a sequence $\mathbf{d}$, let $g(\mathbf{d})$ denote the number of graphs on vertex set $[n]$ with degree sequence $\mathbf{d}$. Given $S = [s] \subset [n]$, let H be a given graph on vertex set S with degree sequence $(k_i)_{1 \le i \le s}$. Let $\mathbf{d}'$ be the integer vector defined by $d'_i = d_i - k_i$ for $i \in S$ and $d'_i = d_i$ for $i \in [n] \backslash S$. Then the number of graphs with degree sequence $\mathbf{d}$ and with $G_S = H$ is $g(S, [n] \setminus S, \mathbf{d}')$, and so the probability that $G_S = H$ in $\mathcal{G}_{n,\mathbf{d}}$ equals $g(S, [n] \setminus S, \mathbf{d}')/g(\mathbf{d})$. So the study of induced subgraphs leads directly to the question of counting B-graphs.

Given a sequence $\mathbf{d} = (d_1, \ldots, d_n)$, let $d_{\max} = \max\{d_i, i \in [n]\}$ and let $M_2(\mathbf{d}) = \sum_{i=1}^{n} d_i(d_i - 1)$. Define $\mu(\mathbf{d})$ to be $M_2(\mathbf{d})/2M(\mathbf{d})$. The following theorem by McKay [9] gives an asymptotic formula for $g(\mathbf{d})$ when $d_{\max}^4 = o(M(\mathbf{d}))$. (The restriction on $d_{\max}$ was relaxed further by McKay and Wormald in [10], but to do so requires a few extra terms in the exponential factor of the asymptotic formula, and is not needed for the purpose of this paper.)

Theorem 1.1 (McKay) *Let* $\mathbf{d} = (d_1, \ldots, d_n)$ *with* $\sum_{i=1}^n d_i$ *even and* $d_{\max} = o(M(\mathbf{d})^{1/4})$. *The number of graphs with degree sequence* $\mathbf{d}$ *is uniformly*

$$\frac{M(\mathbf{d})!}{2^{M(\mathbf{d})/2}(M(\mathbf{d})/2)!\prod_{i=1}^n d_i!} \cdot \exp\left(-\mu(\mathbf{d}) - \mu(\mathbf{d})^2 + O(d_{\max}^4/M(\mathbf{d}))\right)$$

as $n \to \infty$.

By "uniformly" in the above theorem we mean the constant implicit in $O(.)$ is the same for all choices of $\mathbf{d}$ as a function of n, for a given function implicit in the $o(.)$ term. A special case of Theorem 1.1 gives that the number of d-regular graphs on n vertices is asymptotically

$$\frac{(dn)!}{2^{dn/2}(dn/2)!(d!)^n} \cdot \exp\left(-\frac{d^2-1}{4}\right),$$

when $d = o(n^{1/3})$.

Our main result is an asymptotic formula for $g(L, R, \mathbf{d})$, to an accuracy matching McKay's formula in Theorem 1.1. This is given in Section 2, together with its direct applications to estimating $\mathbf{P}(G_S = H)$ in $\mathcal{G}_{\mathbf{d}}$, and some special cases are also given there. The proofs use the switching method, first introduced by McKay [9], with refinements by McKay and Wormald [10], and suitably modified for our purposes here. In Section 3 we use switchings to estimate the ratios between probabilities defined by the counts of loops and various types of multiple edges. In Section 4 we again use switchings to evaluate some variables appearing in those estimates, and in Section 5 we use these to prove the main theorem.

2 Main results

Given a sequence $\mathbf{d} = (d_1, \ldots, d_n)$, our main goal in this paper is to estimate $g(L, R, \mathbf{d})$. We first define some notation. For any positive integer n, let $[n]$ denote the set $\{1, 2, \ldots, n\}$. For any $S \subset L \cup R$, define

$$M_1(\mathbf{d}, S) = \sum_{i \in S} d_i, \quad M_2(\mathbf{d}, S) = \sum_{i \in S} d_i(d_i - 1),$$

$$\mu_0(\mathbf{d}, L, R) = \frac{(M_1(\mathbf{d}, R) - M_1(\mathbf{d}, L))M_2(\mathbf{d}, R)}{2M_1(\mathbf{d}, R)^2}, \tag{2.1}$$

$$\mu_1(\mathbf{d}, L, R) = \frac{M_2(\mathbf{d}, R)M_2(\mathbf{d}, L)}{2M_1(\mathbf{d}, R)^2}, \tag{2.2}$$

$$\mu_2(\mathbf{d}, L, R) = \mu_0(\mathbf{d}, L, R)^2. \tag{2.3}$$

We drop the notations L and R from $\mu_i(\mathbf{d}, L, R)$ for $i = 0, 1, 2$ when the context is clear. Note also that if $M_1(\mathbf{d}, R) < M_1(\mathbf{d}, L)$, then $g(L, R, \mathbf{d})$ is trivially 0, so we may assume that

$$M_1(\mathbf{d}, R) \geq M_1(\mathbf{d}, L). \tag{2.4}$$

The following theorem, proved in Section 5, gives an asymptotic formula for $g(L, R, \mathbf{d})$.

Theorem 2.1 *Let* $\mathbf{d} = (d_1, \ldots, d_n)$ *with* $\sum_{i=1}^n d_i$ *even,* $d_{\max} = o(M(\mathbf{d})^{1/4})$ *and* $M_1(\mathbf{d}, R) \geq M_1(\mathbf{d}, L)$*. Then uniformly over all* L *and* $\mathbf{d}$ *as* $n \to \infty$*,*

$$g(L, R, \mathbf{d}) \;=\; \frac{M_1(\mathbf{d}, R)!e^{-\mu_0(\mathbf{d})-\mu_1(\mathbf{d})-\mu_2(\mathbf{d})}}{2^{(M_1(\mathbf{d},R)-M_1(\mathbf{d},L))/2}((M_1(\mathbf{d}, R) - M_1(\mathbf{d}, L))/2)! \prod_{i=1}^n d_i!}\left(1 + O\left(\frac{d_{\max}^4}{M(\mathbf{d})}\right)\right).$$

Applying Theorems 2.1 and 1.1 we directly get the following. Here $d'_{\max}$ denotes $\max\{d'_1, \ldots, d'_n\}$.

Corollary 2.2 *Let* $\mathbf{d} = (d_1, \ldots, d_n)$ *with* $\sum_{i=1}^n d_i$ *even and* $d_{\max} = o(M(\mathbf{d})^{1/4})$*. Let* $S = [s] \subset [n]$*, let* H *be a graph on vertex set* S *with degree sequence* $\mathbf{k} = (k_1, \ldots, k_s)$*, let* $h = \sum_{i=1}^h k_i$ *and let* $\mathbf{d}' = (d'_1, \ldots, d'_n)$ *with* $d'_i = d_i - k_i$ *for* $i \in S$ *and* $d'_i = d_i$ *for* $i \notin S$*. If* $d'_i < 0$ *for some* $i \in [n]$ *or* $M_1(\mathbf{d}', [n] \setminus S) < M_1(\mathbf{d}', S)$*, then* $\mathbf{P}_{\mathcal{G}_{\mathbf{d}}}(S, H) = 0$*. Otherwise, if* $d'_{\max} = o(M(\mathbf{d}')^{1/4})$*, then uniformly*

$$\mathbf{P}_{\mathcal{G}_{\mathbf{d}}}(S, H) \;=\; \exp\left(-\mu_0(\mathbf{d}') - \mu_1(\mathbf{d}') - \mu_2(\mathbf{d}') + \mu(\mathbf{d}) + \mu(\mathbf{d})^2 + O\left(\frac{{d'}_{\max}^4}{M(\mathbf{d}')} + \frac{d_{\max}^4}{M(\mathbf{d})}\right)\right)$$
$$\times \frac{M_1(\mathbf{d}', [n] \setminus S)! 2^{M_1(\mathbf{d}',S)+h/2}(M(\mathbf{d})/2)!}{((M_1(\mathbf{d}', [n] \setminus S) - M_1(\mathbf{d}', S))/2)! M(\mathbf{d})!} \prod_{i=1}^s [d_i]_{k_i}.$$

where $\mu_i(\mathbf{d}') = \mu_i(\mathbf{d}', S, [n] \setminus S)$ *for* $i = 0$*,* 1 *and* 2*.*

Proof. Recall that $g(\mathbf{d})$ denote the number of graphs on vertex set $[n]$ with degree sequence $\mathbf{d}$. We have

$$\mathbf{P}_{\mathcal{G}_{\mathbf{d}}}(S, H) = \frac{g(S, [n] \setminus S, \mathbf{d}')}{g(\mathbf{d})}.$$

The corollary now follows from the formulae for $g(S, [n] \setminus S, \mathbf{d}')$ in Theorem 2.1 and $g(\mathbf{d})$ in Theorem 1.1. ∎

Let $\mathbf{P}_{\mathcal{G}_{n,d}}(S, H)$ denote the probability that $G_S = H$ for a random d-regular graph G.

Corollary 2.3 *Given* $0 < s < n$*, let* $S = [s] \subset [n]$*, let* H *be a graph on vertex set* S *with degree sequence* $\mathbf{k} = (k_1, \ldots, k_s)$ *with* $k_i \leq d$ *for all* $1 \leq i \leq s$*, and put* $h = \sum_{i=1}^h k_i$*. Assume* $d = o((n - s)^{1/3})$*. Then*

$$\mathbf{P}_{\mathcal{G}_{n,d}}(S, H) \;=\; \exp\left(-\mu_0(\mathbf{d}') - \mu_1(\mathbf{d}') - \mu_2(\mathbf{d}') + \frac{d^2 - 1}{4} + O(d^4/(dn - h))\right)$$
$$\times \frac{(dn - ds)!(dn/2)! 2^{ds-h/2}}{((dn - 2ds + h)/2)!(dn)!} \prod_{i=1}^s [d]_{k_i}\,,$$

where $d'_i = d - k_i$ *for* $i \in S$ *and* $d'_i = d$ *for* $i \notin S$*, and* μ_i *is defined as in Corollary 2.2.*

Proof. We apply Corollary 2.2. By the definition of $\mu(\mathbf{d})$, we immediately get that $\mu(\mathbf{d}) + \mu(\mathbf{d})^2 = (d^2 - 1)/4$ when $\mathbf{d}$ is a constant sequence with each term d. We also have $M(\mathbf{d}) = dn$, $M(\mathbf{d}') = dn - h$, $M_1(\mathbf{d}', S) = ds - h$, $M_1(\mathbf{d}', [n] \setminus S) = dn - ds$, and $d'_{\max} \leq d$. Moreover,

$$\frac{(d'_{\max})^4}{M(\mathbf{d}')} = \frac{d^4}{dn - h} = \frac{d^3}{n - h/d} \leq \frac{d^3}{n - s} = o(1),$$

since $h \leq ds$ and $d = o((n - s)^{1/3})$. ∎

The formula in Corollary 2.3 easily simplifies if the graph H is not too large.

Corollary 2.4 *Let S, H, $\mathbf{k}$ and h be defined as in Corollary 2.3. If $s \geq 1$, $d = o(n^{1/3})$, $s^2 d = o(n)$ and $d^2 s = o(n)$, then*

$$\mathbf{P}_{\mathcal{G}_{n,d}}(S, H) = \big(1 + O((d^3 + s^2 d + d^2 s)/n)\big)(dn)^{-h/2} \prod_{i=1}^{s} [d]_{k_i}.$$

Proof. Since $d^2 s = o(n)$, we have $h = O(ds) = o(n)$ and hence $d^4/(dn - h) = O(d^3/n)$. Similarly,

$$M_1(\mathbf{d}', n\backslash[S]) = dn + O(ds), \quad M_i(\mathbf{d}', S) = O(d^i s)\ (i = 1, 2), \quad M_2(\mathbf{d}', n\backslash[S]) = d(d-1)(n - O(s))$$

and hence from (2.1)–(2.3),

$$\mu_0(\mathbf{d}') = \frac{d-1}{2} + O(ds/n), \quad \mu_1(\mathbf{d}') = O\left(d^2 s/n\right), \quad \mu_2(\mathbf{d}') = \frac{(d-1)^2}{4} + O(d^2 s/n).$$

Thus $\mu_0(\mathbf{d}') + \mu_1(\mathbf{d}') + \mu_2(\mathbf{d}') = (d^2 - 1)/4 + O(d^2 s/n)$.

The corollary now follows upon applying Stirling's formula in the form $n! = \sqrt{2\pi n}(n/e)^n(1 + O(n^{-1}))$ to obtain

$$\begin{aligned}\frac{(dn - ds)!(dn/2)!2^{ds-h/2}}{((dn - 2ds + h)/2)!(dn)!} &= \left(1 + O\left(\frac{s}{n}\right)\right)\left(\frac{dn}{e}\right)^{-h/2} \frac{(1 - s/n)^{dn-ds}}{(1 - 2s/n + h/dn)^{(dn-2ds+h)/2}} \\ &= (dn)^{-h/2}\left(1 + O\left(\frac{s^2 d}{n}\right)\right). \quad \blacksquare\end{aligned}$$

Another interesting special case is when H is empty.

Corollary 2.5 *Assume $d = o(n^{1/3})$. Then for any $S \subset [n]$ with $s = |S| < n/2$,*

$$\mathbf{P}_{\mathcal{G}_{n,d}}(S \text{ is independent}) = \big(1 + O(d^3/n)\big) \exp\left(f(d, \delta)\right) \frac{(dn - ds)!(dn/2)!2^{ds}}{((dn - 2ds)/2)!(dn)!},$$

where $\delta = \delta(n) = s/n$, and

$$f(d, \delta) = -\frac{\delta(d-1)(\delta d - 2 + \delta)}{4(1-\delta)^2}.$$

Proof. This is a simple application of Corollary 2.3 with $h = 0$, noting that

$$\mu_0 = \frac{(d-1)(n-2s)}{2(n-s)}, \ \mu_1 = \frac{(d-1)^2 s}{2(n-s)}, \ \mu_2 = \frac{(d-1)^2(n-2s)^2}{4(n-s)^2}. \quad \blacksquare$$

Note that if $d(n - 2s) \to \infty$, then the probability that S is independent under the conditions in Corollary 2.5 can be further simplified using Stirling's formula to

$$\big(1 + O(d^3/n) + O(1/(dn - 2ds))\big)\sqrt{\frac{1-\delta}{1-2\delta}} \left(\frac{(1-\delta)^{1-\delta}}{(1-2\delta)^{(1-2\delta)/2}}\right)^{dn} \exp\left(f(d, \delta)\right).$$

3 The main switchings

We can use the pairing model to generate B-graphs with the vertex partition $L \cup R$ and the degree sequence $\mathbf{d} = \{d_1, \ldots, d_n\}$. Consider n buckets representing the n vertices. Let bucket i contain d_i points. Take a random pairing of these points. We say a pairing is *restricted* if no pair has both ends in buckets representing vertices in L. Let $\mathcal{M}(L, R, \mathbf{d})$ be the class of all restricted pairings. Every such pairing corresponds to a B-multigraph by contracting all points in each bucket to form a vertex. In the rest of the paper, a bucket in a pairing is also called a vertex. Recall that an edge is pure if both of its end vertices are in R. A pair in a pairing is called a *mixed (pure) pair* if it corresponds to a mixed (pure) edge in the corresponding B-multigraph. Thus, in a restricted pairing, each pair is either mixed or pure. Note that any simple B-graph corresponds to $\prod_{i=1}^{n} d_i!$ restricted pairings in $\mathcal{M}(L, R, \mathbf{d})$. Hence, all simple B-graphs occur with the same probability in the pairing model.

The main goal of this section is to compute the probability that a B-multigraph generated by the pairing model is simple. We say that $\{\{u_1, u_1'\}, \{u_2, u_2'\}, \{u_3, u_3'\}\}$ is a triple pair if u_1, u_2, u_3 are in one vertex and u_1', u_2', u_3' are in another vertex. We call the two vertices involved the *end vertices* of the triple pair. If the end vertices are in L and R respectively, the triple pair is called a *mixed triple pair*, and otherwise it is *pure*. Given a random restricted pairing, let T_1 and T_2 be the number of mixed and pure triple pairs respectively. In this section, there is only one degree sequence $\mathbf{d}$ referred to, so we drop the notation $\mathbf{d}$ from $M(\mathbf{d})$ and $M_i(\mathbf{d}, L)$, $M_i(\mathbf{d}, R)$, $\mu_i(\mathbf{d})$ for simplicity. Since $M_1(R) \geq M_1(L)$ by assumption (2.4), we have $M_1(R) \geq M/2$.

We often use the fact that for any positive even integer m, if $U(m)$ denotes the number of pairings of m points, then

$$U(m) = \prod_{i=0}^{m/2-1} (m - 2i - 1) = \frac{m!}{2^{m/2}(m/2)!}.$$

Lemma 3.1 $\mathbf{E}(T_1) = O(d_{\max}^4/M)$ *and* $\mathbf{E}(T_2) = O(d_{\max}^4/M)$.

Proof. For any two vertices $i \in L$ and $j \in R$, we compute the probability that there is a triple pair with end vertices i and j. There are $\binom{d_i}{3}$ ways to choose three points from the vertex i and $\binom{d_j}{3}$ ways to choose three points from the vertex j. There are 6 ways to match the six chosen points to form a triple pair. The probability for the three particular pairs to occur is

$$\frac{[M_1(R) - 3]_{M_1(L)-3} U(M_1(R) - M_1(L))}{[M_1(R)]_{M_1(L)} U(M_1(R) - M_1(L))} \sim M_1(R)^{-3}$$

(noting that $M_1(R) \geq M_1(L)$ implies $M_1(R) \to \infty$). This is because the number of ways to match the remaining $M_1(R) - 3$ points in L to points in R, except for the three chosen points in the vertex j, is $[M_1(R) - 3]_{M_1(L)-3}$, and the number of matchings of the remaining $M_1(R) - M_1(L)$ points in R is $U(M_1(R) - M_1(L))$, whilst the total number of restricted pairings is $[M_1(R)]_{M_1(L)}U(M_1(R)-M_1(L))$. Hence we have

$$\begin{aligned}\mathbf{E}(T_1) &\sim \sum_{i\in L}\sum_{j\in R} 6\binom{d_i}{3}\binom{d_j}{3}M_1(R)^{-3} = O\left(\left(\sum_{i\in L} d_i^3\right)\left(\sum_{j\in R} d_j^3\right)\right)M^{-3}\\ &= O\left(\frac{d_{\max}^4 M_1(L)M_1(R)}{M^3}\right) = O\left(\frac{d_{\max}^4}{M}\right),\end{aligned}$$

where the second equality uses $M/2 \le M_1(R) \le M$.

A similar argument gives

$$\begin{aligned}\mathbf{E}(T_2) &\sim \sum_{i,j\in R, i<j} 6\binom{d_i}{3}\binom{d_j}{3}M_1(R)^{-3} = O\left(\left(\sum_{i\in R} d_i^3\right)\left(\sum_{j\in R} d_j^3\right)\right)M^{-3}\\ &= O\left(\frac{d_{\max}^4 M_1(R)^2}{M^3}\right) = O\left(\frac{d_{\max}^4}{M}\right). \quad \blacksquare\end{aligned}$$

A pair $\{u, u'\}$ is called a *loop* if u and u' are contained in the same vertex and two pairs $\{u_1, u_1'\}, \{u_2, u_2'\}$ are called a *double pair* if u_1, u_2 are in one vertex and u_1', u_2' are in another vertex. We call two loops that contain points from a common vertex a *double loop*. Let I be the number of double loops. The proof of the following is a simple modification of the proof of the previous lemma, so is omitted.

Lemma 3.2 *$\mathbf{E}(I) = O(d_{\max}^3/M)$.* $\blacksquare$

Lemmas 3.1 and 3.2 show that a.a.s. there are no triple pairs or double loops in a random restricted pairing, under the assumption $d_{\max}^4 = o(M(\mathbf{d}))$. So we only need to consider single loops (i.e. without another loop at the same vertex) and double pairs. In a restricted pairing, there are two types of double pairs. One is that u_1, u_2 are contained in a vertex in L and u_1', u_2' are contained in a vertex in R. The other is that all of u_1, u_2, u_1' and u_2' are contained in vertices in R. We call the former type *mixed* and the latter type *pure*.

Let B_0, B_1 and B_2 be the numbers of single loops, mixed double pairs and pure double pairs respectively. We first compute the expected value of B_i for $i = 0, 1, 2$. Recall from (2.1)–(2.3) that

$$\mu_0 = \frac{(M_1(R)-M_1(L))M_2(R)}{2M_1(R)^2}, \quad \mu_1 = \frac{M_2(R)M_2(L)}{2M_1(R)^2}, \quad \mu_2 = \mu_0^2.$$

Lemma 3.3 *For $i = 0, 1, 2$ we have $\mathbf{E}B_i = O(\mu_i)$. If $d_{\max} = o(M^{1/3})$ and $M_1(R) - M_1(L) \to \infty$, then, more precisely, $\mathbf{E}B_i \sim \mu_i$ for $i = 0$ and 1, and $\mathbf{E}B_2 = (1 + o(1))\mu_2 + o(1)$.*

Proof. Using small modifications of the proof of Lemma 3.1, we immediately get

$$\begin{aligned}\mathbf{E}B_0 &= \sum_{i\in R}\binom{d_i}{2}\frac{[M_1(R)-2]_{M_1(L)}U(M_1(R)-M_1(L)-2)}{[M_1(R)]_{M_1(L)}U(M_1(R)-M_1(L))}\\ &= \sum_{i\in R}\frac{[d_i]_2}{2}\frac{O\left(M_1(R)-M_1(L)\right)}{M_1(R)^2} = O(\mu_0);\end{aligned}$$

$$
\begin{aligned}
\mathbf{E}B_1 &= \sum_{i\in L}\sum_{j\in R} 2\binom{d_i}{2}\binom{d_j}{2}\frac{[M_1(R)-2]_{M_1(L)-2}U(M_1(R)-M_1(L))}{[M_1(R)]_{M_1(L)}U(M_1(R)-M_1(L))} \\
&\sim \frac{M_2(L)M_2(R)}{2}M_1(R)^{-2} = \mu_1;
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{E}B_2 &= \sum_{i,j\in R, i<j} 2\binom{d_i}{2}\binom{d_j}{2}\frac{[M_1(R)-4]_{M_1(L)}U(M_1(R)-M_1(L)-4)}{[M_1(R)]_{M_1(L)}U(M_1(R)-M_1(L))} \\
&= \frac{1}{2}\sum_{i\in R}\sum_{j\in R} 2\binom{d_i}{2}\binom{d_j}{2}\frac{[M_1(R)-4]_{M_1(L)}U(M_1(R)-M_1(L)-4)}{[M_1(R)]_{M_1(L)}U(M_1(R)-M_1(L))} \\
&\quad -\frac{1}{2}\sum_{i\in R} 2\binom{d_i}{2}\binom{d_i}{2}\frac{[M_1(R)-4]_{M_1(L)}U(M_1(R)-M_1(L)-4)}{[M_1(R)]_{M_1(L)}U(M_1(R)-M_1(L))} \qquad (3.1) \\
&= \frac{M_2(R)^2}{4}\frac{O((M_1(R)-M_1(L))^2)}{M_1(R)^4} - \alpha = O(\mu_2) - \alpha,
\end{aligned}
$$

where $\alpha = O(d_{\max}^3/M)$ is nonnegative. This gives the first part of the lemma.

If furthermore $d_{\max} = o(M^{1/3})$ and $M_1(R) - M_1(L) \to \infty$, then all the $O(.)$ terms in the displayed equations above can be replaced by $(1+o(1))(.)$. The lemma follows. ∎

Corollary 3.4 *If $d_{\max}^4 = o(M)$ and $M_2(R) = O(d_{\max}^3)$, then the probability that there exists a loop or a double pair is $O(d_{\max}^4/M)$.*

Proof. If $d_{\max}^4 = o(M)$ and $M_2(R) = O(d_{\max}^3)$, then $\mathbf{E}B_0 = O(M_2(R)/M_1(R)) = O(d_{\max}^3/M)$; $\mathbf{E}B_1 = O(M_2(L)d_{\max}^3/M^2) = O(d_{\max}^4/M)$ (since $M_2(L)/M_1(R) \le M_2(L)/M_1(L) \le d_{\max}$); $\mathbf{E}B_2 = O(d_{\max}^6/M^2) = o(d_{\max}^2/M)$. The result follows by the first moment principle. ∎

We will need to prescribe some upper bounds on the likely values of the random variables of interest. Define

$$\eta(L) = M_2(L)/M_1(L), \quad \eta(R) = M_2(R)/M_1(R)$$

and let

$$k_0 = \max\{\ln M, 8\eta(L), 8\eta(R)\}, \quad k_1 = k_2 = \max\{\ln M, 8\eta(L)^2, 8\eta(R)^2\}\ (i=1,2). \qquad (3.2)$$

Clearly $\eta(L) = O(d_{\max})$ and $\eta(R) = O(d_{\max})$.

Lemma 3.5 *If $d_{\max}^4 = o(M)$, then $\mathbf{P}\big(B_i \ge k_i\big) = O(M^{-1})$ for $i = 0, 1, 2$.*

Proof. For any $h = o(\sqrt{M})$, the probability that there exist h single loops is bounded above by the h-th factorial moment of B_0. Following the same pattern of proof as for Lemma 3.1, this is at most

$$\sum_{\substack{i_1,\ldots,i_h\in R\\ i_1<\cdots<i_h}} \left(\prod_{j=1}^{h}\binom{d_{i_j}}{2}\right)\frac{[M_1(R)-2h]_{M_1(L)}U(M_1(R)-M_1(L)-2h)}{[M_1(R)]_{M_1(L)}U(M_1(R)-M_1(L))}$$
$$\le \frac{M_2(R)^h}{2^h h!}\frac{[M_1(R)-2h]_{M_1(L)}U(M_1(R)-M_1(L)-2h)}{[M_1(R)]_{M_1(L)}U(M_1(R)-M_1(L))}$$
$$= \frac{M_2(R)^h}{2^h h!}\frac{\prod_{i=0}^{h-1}(M_1(R)-M_1(L)-2i)}{\prod_{i=0}^{2h-1}(M_1(R)-i)} \le \frac{M_2(R)^h}{2^h h!}\frac{(M_1(R)-M_1(L))^h}{(1+o(1))M_1(R)^{2h}}, \tag{3.3}$$

where the last inequality above holds because $M_1(R)=\Theta(M)$ and $h=o(\sqrt{M})$, which yields $\prod_{i=0}^{2h-1}(M_1(R)-i)\sim M_1(R)^{2h}$. Hence the probability that h loops exist is at most

$$(1+o(1))\frac{M_2(R)^h}{2^h h!}M_1(R)^{-h} \le (1+o(1))\left(\frac{eM_2(R)}{2hM_1(R)}\right)^h = (1+o(1))\left(\frac{e\eta(R)}{2h}\right)^h.$$

Similarly we have that for any $h=o(\sqrt{M})$, the probability that there exist h mixed double pairs is at most

$$\frac{1}{h!}\sum_{i_1,\ldots,i_h\in L, j_1,\ldots,j_h\in R}\left(\prod_{\ell=1}^{h}2\binom{d_{i_\ell}}{2}\binom{d_{j_\ell}}{2}\right)\frac{[M_1(R)-2h]_{M_1(L)-2h}U(M_1(R)-M_1(L))}{[M_1(R)]_{M_1(L)}U(M_1(R)-M_1(L))}$$
$$\le (1+o(1))\frac{M_2(L)^hM_2(R)^h}{2^h h!}M_1(R)^{-2h} \le (1+o(1))\left(\frac{e\eta(L)\eta(R)}{2h}\right)^h, \tag{3.4}$$

where the factor $1/h!$ accounts for the multiple counting caused by the $h!$ ways to order the h double pairs. Similarly, the probability that there exist h pure double pairs is at most

$$\frac{1}{h!}\sum_{i_1,\ldots,i_h\in R, j_1,\ldots,j_h\in R}\left(\prod_{\ell=1}^{h}2\binom{d_{i_\ell}}{2}\binom{d_{j_\ell}}{2}\right)\frac{[M_1(R)-4h]_{M_1(L)}U(M_1(R)-M_1(L)-4h)}{[M_1(R)]_{M_1(L)}U(M_1(R)-M_1(L))}$$
$$\le (1+o(1))\frac{M_2(R)^{2h}}{2^h h!}\frac{(M_1(R)-M_1(L))^{2h}}{M_1(R)^{4h}} \le (1+o(1))\left(\frac{e\eta(R)^2}{2h}\right)^h. \tag{3.5}$$

Note that $\eta(L)$ and $\eta(R)$ are both bounded above by $d_{\max}$. By the definition of k_i in (3.2), $k_i=O(\ln M+d_{\max}^2)$ for $i=0,1,2$. Since $d_{\max}^4=o(M)$, we therefore have $k_i=o(\sqrt{M})$. Hence

$$\begin{aligned}
\mathbf{P}\big(B_0\ge k_0\big) &= O\left(\left(\frac{e\eta(R)}{2k_0}\right)^{k_0}\right) = O\left(\left(\frac{e}{16}\right)^{\ln M}\right) = O(M^{-1}),\\
\mathbf{P}\big(B_1\ge k_1\big) &= O\left(\left(\frac{e}{2k_1}\cdot\eta(L)\cdot\eta(R)\right)^{k_1}\right) = O\left(\left(\frac{e}{16}\right)^{\ln M}\right) = O(M^{-1}),\\
\mathbf{P}\big(B_2\ge k_2\big) &= O\left(\left(\frac{e\eta(R)^2}{2k_2}\right)^{k_2}\right) = O\left(\left(\frac{e}{16}\right)^{\ln M}\right) = O(M^{-1}). \quad \blacksquare
\end{aligned}$$

Lemma 3.6 *Assuming $d_{\max}^4 = o(M)$,*

(i) if $M_2(R) = O(d_{\max}^5 + d_{\max}^3 \ln^2 M)$, then with probability $1 - O(d_{\max}^4/M)$, $B_0 \le d_{\max}$ and $B_i \le d_{\max}^2$ for $i = 1, 2$;

(ii) if $M_1(R) - M_1(L) = O(d_{\max}^4 + d_{\max}^2 \ln^2 M)$, then with probability $1 - O(d_{\max}^4/M)$, $B_0 \le d_{\max}$ and $B_2 \le d_{\max}^2$;

(iii) if $M_2(L) = O(d_{\max}^5 + d_{\max}^3 \ln^2 M)$, then with probability $1 - O(d_{\max}^4/M)$, $B_1 \le d_{\max}^2$.

Proof. These statements follow easily, after some simple estimations, from (3.3), (3.4) and (3.5). As the proofs are similar for the three cases, we only provide the proof of $B_0 \le d_{\max}$ with probability $1 - O(d_{\max}^4/M)$ in case (i).

Suppose $M_2(R) = O(d_{\max}^5 + d_{\max}^3 \ln^2 M)$. Then $\eta(R) = O((d_{\max}^5 + d_{\max}^3 \ln^2 M)/M)$. We only need to show that $\mathbf{E}([B_0]_h/h!) = O(d_{\max}^4/M)$ for $h = d_{\max} + 1$. By (3.3),

$$\begin{aligned}
\mathbf{E}([B_0]_{d_{\max}+1}/(d_{\max}+1)!) &= O\left(\left(\frac{C(d_{\max}^5 + d_{\max}^3 \ln^2 M)}{d_{\max} M}\right)^{d_{\max}+1}\right) \\
&= O\left(\left(C \cdot \frac{d_{\max}^4 + d_{\max}^2 \ln^2 M}{M}\right)^{d_{\max}+1}\right),
\end{aligned}$$

for some constant $C > 0$. Clearly, this is $O(d_{\max}^4/M)$ for any $d_{\max} \ge 1$. ∎

We now redefine the values k_i as follows. Let ζ_0, ζ_1 and ζ_2 be (large) constants specified later (at the beginning of Section 5). If $M_2(R) \le \zeta_0(d_{\max}^5 + d_{\max}^3 \ln^2 M)$, use $k_0 = d_{\max}$ and $k_i = d_{\max}^2$ for $i = 1$ and 2; if $M_1(R) - M_1(L) \le \zeta_1(d_{\max}^4 + d_{\max}^2 \ln^2 M)$, use $k_0 = d_{\max}$, and $k_2 = d_{\max}^2$; if $M_2(L) \le \zeta_2(d_{\max}^5 + d_{\max}^3 \ln^2 M)$, use $k_1 = d_{\max}^2$. With the modified values, we have the following immediately from the previous two results.

Corollary 3.7 *If $d_{\max}^4 = o(M)$, then $\mathbf{P}\big(B_i \ge k_i\big) = O(d_{\max}^4/M)$ for $i = 0, 1, 2$.*

Define $\mathcal{C}_{l_0,l_1,l_2}$ be the class of restricted pairings in $\mathcal{M}(L, R, \mathbf{d})$ that contains l_0 loops, l_1 mixed double pairs, l_2 pure double pairs and no double loops or triple pairs. Also, let $\mathbf{P}(\mathbf{d})$ be the probability that a random pairing $\mathcal{P} \in \mathcal{M}(L, R, \mathbf{d})$ corresponds to a simple B-graph.

The following corollary is obtained from Lemmas 3.1 and 3.2 and Corollary 3.7 by noting that the sum of $|\mathcal{C}_{l_0,l_1,l_2}|$ over *all* l_0, l_1, l_2 is the total number of pairings with $T_1 = T_2 = I = 0$.

Corollary 3.8

$$\frac{1}{\mathbf{P}(\mathbf{d})} = \big(1 + O(d_{\max}^4/M)\big) \sum_{l_0=0}^{k_0} \sum_{l_1=0}^{k_1} \sum_{l_2=0}^{k_2} \frac{|\mathcal{C}_{l_0,l_1,l_2}|}{|\mathcal{C}_{0,0,0}|}.$$

With this corollary in mind, in the rest of the paper when considering $|\mathcal{C}_{l_0,l_1,l_2}|$ we implicitly assume that $0 \le l_i \le k_i$ for $i = 0, 1$ and 2.

Given a restricted pairing $\mathcal{P}$, we say the ordered pair of pairs $((u_1, u_1'), (u_2, u_2'))$ forms a directed 2-path in $\mathcal{P}$ if u_1' and u_2 lie in the same vertex and the three vertices where u_1, u_1'

and u'_2 lie in respectively are all distinct. We then say that the two pairs (u_1, u'_1) and (u_2, u'_2) are adjacent. For instance, the ordered pair of pairs $((1,2),(3,4))$ forms a directed 2-path in the four examples in Figure 1. Note that a directed 2-path in a pairing corresponds to a directed 2-path in the corresponding B-multigraph. Let v denote the vertex that contains u'_1 and u_2. We say the directed 2-path $((u_1, u'_1), (u_2, u'_2))$ in $\mathcal{P}$ is *simple* if neither of $\{u_1, u'_1\}$ and $\{u_2, u'_2\}$ is contained in a double pair and there is no loop at v.

There are four types of directed 2-paths in which we are interested in this paper. These 2-paths will be used later to define our switching operations. Those with all vertices lying in R are of *type 1*. A directed 2-path $((a,b),(c,d))$ is of *type 2* if a lies in a vertex in L and the other points all lie in vertices in R, *type 3* if a and d are in vertices in L and the vertex containing b and c is in R, and *type 4* if a and d lie in vertices in R and the vertex containing b and c is in L.

Given a restricted pairing $\mathcal{P}$, let t be the number of pure pairs in $\mathcal{P}$. Then

$$t = (M_1(R) - M_1(L))/2. \tag{3.6}$$

Let $A_i(\mathcal{P})$ denote the number of simple directed 2-paths of type i in $\mathcal{P}$, for $i = 1,2,3,4$, and let

$$a_i(l_0, l_1, l_2) = \mathbf{E}(A_i(\mathcal{P}) \mid \mathcal{P} \in \mathcal{C}_{l_0,l_1,l_2}). \tag{3.7}$$

We drop $\mathcal{P}$ from the notation $A_i(\mathcal{P})$ if the context is clear. Clearly $A_4(\mathcal{P}) = \sum_{i\in L} d_i(d_i-1) - O(l_1 d_{\max}) = M_2(L) - O(l_1 d_{\max})$ for any $\mathcal{P} \in \mathcal{C}_{l_0,l_1,l_2}$ since the number of non-simple directed 2-paths of type 4 is bounded by $O(l_1 d_{\max})$.

The switching operations we are going to use are ideologically similar to the switching operations used by McKay and Wormald [10]. Although those switchings cannot be applied here because they do not preserve the property of the pairings being restricted, they can easily be adjusted and adapted to our current needs. The main twist is that there are a number of alternative switchings available to use, and we need to specify which ones should be used, and for what values of the parameters, to achieve the desired result. The following two switching operations are used to prove Lemma 3.9. These switchings are designed so that the number of loops decreases by exactly 1 after the operation is applied.

(a) L_1-*switching*: take a loop $\{2,3\}$ and two pure pairs $\{1,5\}$, $\{4,6\}$ such that the six points are located in the five distinct vertices as drawn in Figure 2. Replace the three pairs $\{2,3\}, \{1,5\}, \{4,6\}$ by $\{1,2\}, \{3,4\}, \{5,6\}$.

(b) L_2-*switching*: take a loop $\{2,3\}$ and two mixed pairs $\{1,5\}$, $\{4,6\}$ such that the six points are located in the five distinct vertices as drawn in Figure 3. Replace the three pairs $\{2,3\}, \{1,5\}, \{4,6\}$ by $\{1,2\}, \{3,4\}, \{5,6\}$.

For any switching operation that converts a pairing $\mathcal{P}_1$ to another pairing $\mathcal{P}_2$, we call the operation that converts $\mathcal{P}_2$ to $\mathcal{P}_1$ the inverse of that switching. Thus, the *inverse* L_1-*switching* can be defined as follows. Take a 2-directed path (not necessarily simple) $((1,2),(3,4))$ of type 1 and a pure pair $\{5,6\}$ such that the points 1, 2, 4, 5 and 6 lie in five distinct vertices. Replace $\{1,2\}$, $\{3,4\}$ and $\{5,6\}$ by $\{2,3\}$, $\{1,5\}$ and $\{4,6\}$. The inverse L_2-switching can be defined in the same way.

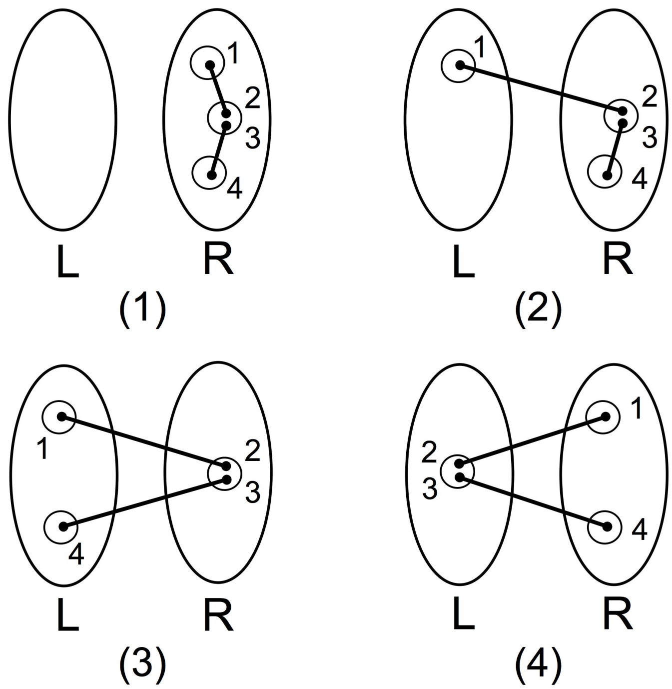

Figure 1: *four different types of 2-paths*

The following lemma estimates the ratio $|\mathcal{C}_{l_0,l_1,l_2}|/|\mathcal{C}_{l_0-1,l_1,l_2}|$ by counting ways to perform certain L_1-switchings and their inverses. We express the present results in terms of the numbers $a_i(l_0, l_1, l_2)$, defined in (3.7), whose estimation we postpone until later.

Lemma 3.9 *Assume $l_0 \geq 1$. Let $a_1 = a_1(l_0 - 1, l_1, l_2)$ and $a_3 = a_3(l_0 - 1, l_1, l_2)$. Then*

$$
\begin{aligned}
(i): \quad & \textit{If } t \geq 1, \\
& \frac{|\mathcal{C}_{l_0,l_1,l_2}|}{|\mathcal{C}_{l_0-1,l_1,l_2}|} = \frac{a_1}{4l_0 t}(1 + O((d_{\max}^2 + l_0 + l_2)/t)), \\
(ii): \quad & \textit{If } M_1(L) \geq 1 \textit{ and } t \geq 1, \\
& \frac{|\mathcal{C}_{l_0,l_1,l_2}|}{|\mathcal{C}_{l_0-1,l_1,l_2}|} = \frac{t a_3}{l_0 M_1(L)^2}(1 + O((d_{\max}^2 + l_1)/M_1(L) + (d_{\max}^2 + l_0 + l_2)/t)).
\end{aligned}
$$

Proof. Let $\mathcal{P} \in \mathcal{C}_{l_0,l_1,l_2}$ and we consider the number of L_1-switching operations that convert $\mathcal{P}$ to some $\mathcal{P}' \in \mathcal{C}_{l_0-1,l_1,l_2}$. For the purpose of counting, we label the points in the pairs that are under consideration as shown in Figure 2. So for any pair under consideration, we will incorporate in our counting the number of ways we can label the points in the pair. Let N denote the number of ways to choose the pairs and label the points in them so that an L_1-switching can be applied to these pairs, which converts $\mathcal{P}$ to some $\mathcal{P}' \in \mathcal{C}_{l_0-1,l_1,l_2}$ without creating any new loops or double pairs. This implies that the switching operations counted by N destroy only one loop and there is no simultaneous creation or destruction of other loops or double pairs.

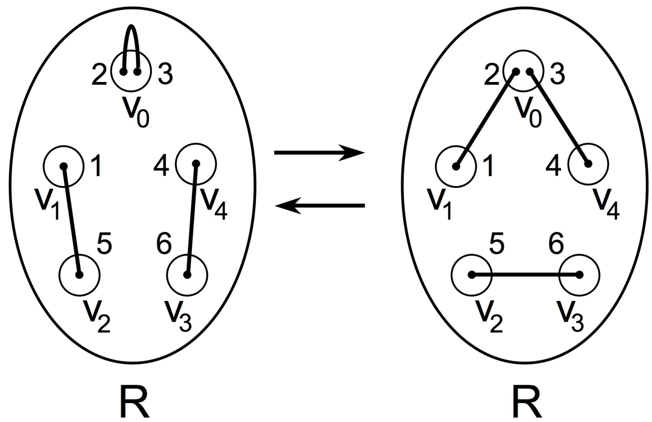

Figure 2: *L_1-switching and its inverse*

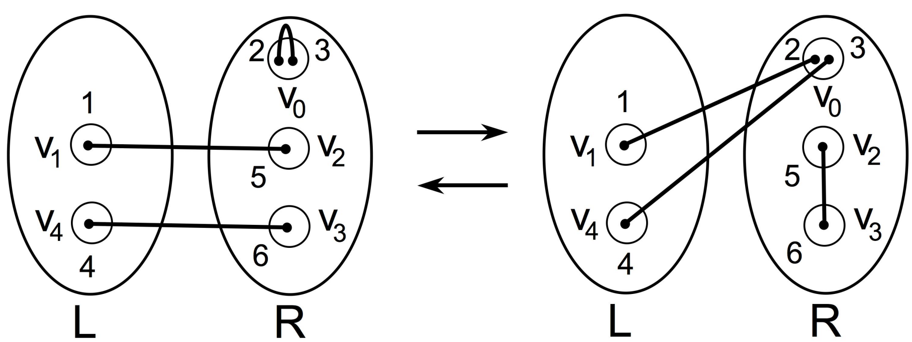

Figure 3: *L_2-switching and its inverse*

We first give a rough count of N, that includes some forbidden cases (due to creating double pairs, etc) and then estimate the error. There are l_0 ways to choose a loop e_0 and $t(t-1)$ ways to choose (e_1, e_2), an ordered pair of two distinct pure pairs. For any chosen loop e_0, there are two ways to distinguish the two end points to label the points 2 and 3 as shown in Figure 2. For each of the other pairs, there are two ways to label its two endpoints, as 1 and 5, or 4 and 6, as the case may be. Hence a rough estimation of N is $8l_0t(t-1)$, including the count of some forbidden choices of e_0, e_1 and e_2, which we estimate next. Let the vertices that contain points $2, 1, 5, 6, 4$ be denoted by v_0, v_1, v_2, v_3, v_4 respectively as shown in Figure 2. The only possible exclusions caused by invalid choices in the above are the following:

(a) the loop e_0 is adjacent to e_1 or e_2, or e_1 is adjacent to e_2, in which case, the L_1-switching is not applicable since the definition of the L_1-switching excludes cases where the edges are adjacent because it requires the end vertices to be distinct;

(b) there exists a pair between $\{v_0, v_1\}$, or $\{v_0, v_4\}$, or $\{v_2, v_3\}$ in $\mathcal{P}$, in which case there will be more double pairs created when the L_1-switching is applied;

(c) the pair e_1 or e_2 is a loop or is contained in a double pair, in which case a loop or double pair is simultaneously destroyed.

First we show that the number of exclusions from case (a) is $O(l_0 t d_{\max})$. The number of pairs of (e_0, e_1) is at most $l_0 t$. For any given e_0 and e_1, the number of ways to choose a pair e_2 such that e_2 is adjacent to e_0 or e_1 is at most $2d_{\max}$ since both e_0 and e_2 are adjacent to at most $d_{\max}$ pairs. Hence the number of triples of (e_0, e_1, e_2) such that e_2 is adjacent to either e_0 or e_1 is at most $2l_0 t d_{\max}$. By symmetry, the number of triples of (e_0, e_1, e_2) such that e_1 is adjacent to either e_0 or e_2 is also at most $2l_0 t d_{\max}$. Hence the number of exclusions from case (a) is $O(l_0 t d_{\max})$.

Next we show that the number of exclusions from case (b) is $O(l_0 t d_{\max}^2)$. As just explained, the number of pairs of (e_0, e_1) is at most $l_0 t$. For any given e_0 and e_1, the number of ways to choose a pair e_2 such that v_3 is adjacent to v_2 or v_4 is adjacent to v_0 is at most $2d_{\max}^2$, since both e_0 and e_1 have at most $d_{\max}^2$ edges that are of distance 2 away. Hence the number of triples (e_0, e_1, e_2) such that v_3 is adjacent to v_2 or v_4 is adjacent to v_0 is $O(l_0 t d_{\max}^2)$. By symmetry, the number of triples (e_0, e_1, e_2) such that v_3 is adjacent to v_2 or v_0 is adjacent to v_1 is $O(l_0 t d_{\max}^2)$. Hence the number of exclusions from case (b) is $O(l_0 t d_{\max}^2)$.

Now we show that the number of exclusions from case (c) is $O(l_0^2 t + l_0 t l_2)$. The number of ways to choose e_0, e_1, e_2 such that e_1 or e_2 is a loop is at most $2l_0^2 t$ and the number of ways to choose these three pairs such that e_1 or e_2 is contained in a double pair is at most $2 \cdot l_0 t \cdot 2l_2 = O(l_0 t l_2)$. Hence the number of exclusions from case (c) is $O(l_0^2 t + l_0 t l_2)$.

Thus, the number of exclusions in the calculation of N is $O(l_0 t d_{\max}^2 + l_0^2 t + l_0 t l_2)$. So $N = 8l_0 t^2 (1 + O(d_{\max}^2/t + (l_0 + l_2)/t))$.

Now choose an arbitrary pairing $\mathcal{P}' \in \mathcal{C}_{l_0-1,l_1,l_2}$. Let N' be the number of ways to choose the pairs and label points in them so that an inverse L_1-switching operation can be applied to these pairs such that $\mathcal{P}'$ is converted to some $\mathcal{P} \in \mathcal{C}_{l_0,l_1,l_2}$ without destroying any loops or double pairs. To apply this operation we need to choose e_0', e_1', e_2', such that (e_0', e_1') is a simple directed 2-path of type 1 and e_2' is a pure pair. We consider the directed 2-path (e_0', e_1') because it automatically gives a unique way of distinguishing vertices v_1, v_0 and v_4 and labelling points as 1, 2, 3 and 4 in Figure 2. There are $A_1(\mathcal{P}')$ simple directed 2-paths of type 1, and hence $A_1(\mathcal{P}')$ ways to choose the points as 1, 2, 3 and 4. The number of ways to choose a pure pair e_2' is t and so there are $2t$ ways to fix the vertices v_2, v_3 and the points $\{5, 6\}$. The only possible exclusions to the above choices are listed the following cases.

(a) There exists a pair between $\{v_1, v_2\}$ or $\{v_3, v_4\}$ in $\mathcal{P}'$, since then at least one double pair will be created if the inverse L_1-switching is applied.

(b) The pair e_2' is a loop, in which case the inverse L_1-switching is not applicable, or e_2' is contained in a double pair, in which case a double pair is destroyed after the application of the inverse L_1-switching.

(c) The pair e_2' is adjacent to the 2-path or is contained in the 2-path, in which case the inverse L_1-switching operation is not applicable.

The number of exclusions from case (a) is $O(A_1(\mathcal{P}')d_{\max}^2)$ and the numbers of exclusions from case (b) and (c) are $O(A_1(\mathcal{P}')l_0 + A_1(\mathcal{P}')l_2)$ and $O(A_1(\mathcal{P}')d_{\max})$ respectively.

Thus, the number of exclusions from case (a)–(d) is $O(A_1(\mathcal{P}')d_{\max}^2 + A_1(\mathcal{P}')l_0 + A_1(\mathcal{P}')l_2)$. So

$$\mathbf{E}(N') = \mathbf{E}\big(2A_1t(1+O(d_{\max}^2/t+(l_0+l_2)/t)) \mid \mathcal{P}' \in \mathcal{C}_{l_0-1,l_1,l_2}\big) = 2a_1t(1+O(d_{\max}^2/t+(l_0+l_2)/t)).$$

Next we count the pairs of $(\mathcal{P},\mathcal{P}')$ such that $\mathcal{P} \in \mathcal{C}_{l_0,l_1,l_2}$, $\mathcal{P}' \in \mathcal{C}_{l_0-1,l_1,l_2}$, and $\mathcal{P}'$ is obtained by applying an L_1-switching to $\mathcal{P}$, which destroys only one loop without creating any new loops or double pairs. (Since the parameters l_1 and l_2 are unchanged, no double pairs can be destroyed.) Then the number of such pairs of pairings equals both $|\mathcal{C}_{l_0,l_1,l_2}|\mathbf{E}(N)$ and $|\mathcal{C}_{l_0-1,l_1,l_2}|\mathbf{E}(N')$. Thus,

$$\frac{|\mathcal{C}_{l_0,l_1,l_2}|}{|\mathcal{C}_{l_0-1,l_1,l_2}|} = \frac{a_1}{4l_0t}(1+O(d_{\max}^2/t+(l_0+l_2)/t)).$$

This proves part (i) of Lemma 3.9. Analogously we can deduce the following by analysing the L_2-switching and its inverse.

$$\begin{aligned}\frac{|\mathcal{C}_{l_0,l_1,l_2}|}{|\mathcal{C}_{l_0-1,l_1,l_2}|} &= \frac{2ta_3 + O(d_{\max}^2 a_3) + O(l_0a_3 + l_2a_3)}{2l_0M_1(L)^2 + O(d_{\max}^2 M_1(L)l_0 + l_0M_1(L)l_1)} \\ &= \frac{ta_3}{l_0M_1(L)^2}(1+O(d_{\max}^2/M_1(L) + d_{\max}^2/t + (l_0+l_2)/t + l_1/M_1(L))).\end{aligned}$$

Then we obtain part (ii) of Lemma 3.9. ∎

We use the following two switching operations to prove Lemma 3.10. These switchings are designed to reduce the number of mixed double pairs by exactly one.

(a) D_1-*switching*: take a mixed double pair $\{\{3,4\},\{5,6\}\}$ and also two pure pairs $\{1,2\}$ and $\{7,8\}$ such that the eight points are located in the six distinct vertices as shown in Figure 4. Replace the four pairs by $\{1,3\},\{5,7\},\{2,4\},\{6,8\}$.

(b) D_2-*switching*: take a mixed double pair $\{\{3,4\},\{5,6\}\}$ and also two mixed pairs $\{1,2\}$ and $\{7,8\}$ such that the eight points are located in the six distinct vertices as shown in Figure 5. Replace the four pairs by $\{1,4\},\{6,7\},\{2,3\},\{5,8\}$.

The inverse switchings are defined analogously to the earlier ones. For instance, the inverse D_1-switching is defined as follows. Take a directed 2-path $((1,3),(5,7))$ of type 4 and a directed 2-path $((2,4),(6,8))$ of type 1 such that the eight points are located in six distinct vertices as shown in Figure 4. Replace these four pairs by $\{1,2\}$, $\{3,4\}$, $\{5,6\}$ and $\{7,8\}$.

Lemma 3.10 *Assume $l_1 \ge 1$. Let $a_1 = a_1(0,l_1-1,l_2)$ and $a_3 = a_3(0,l_1-1,l_2)$. Then*

(i): *If $t \ge 1$ and $M_2(L) \ge 1$,*

$$\frac{|\mathcal{C}_{0,l_1,l_2}|}{|\mathcal{C}_{0,l_1-1,l_2}|} = \frac{M_2(L)a_1}{8l_1t^2}(1+O((d_{\max}^3 + l_1d_{\max})/M_2(L) + (d_{\max}^2 + l_2)/t));$$

(ii): *If $M_1(L) \ge 1$ and $M_2(L) \ge 1$,*

$$\frac{|\mathcal{C}_{0,l_1,l_2}|}{|\mathcal{C}_{0,l_1-1,l_2}|} = \frac{a_3M_2(L)}{2l_1M_1(L)^2}(1+O((d_{\max}^2 + l_1)/M_1(L) + (d_{\max}^3 + l_1d_{\max})/M_2(L))).$$

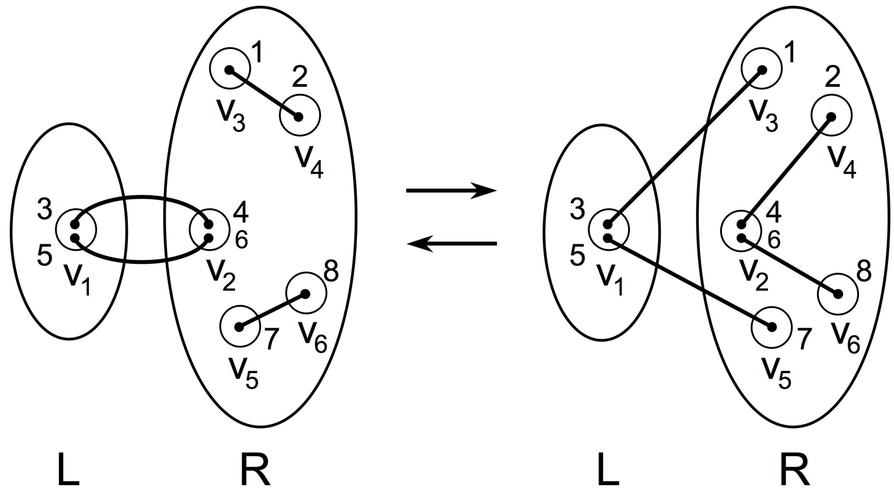

Figure 4: D_1*-switching and its inverse*

Proof. For a given pairing $\mathcal{P} \in \mathcal{C}_{0,l_1,l_2}$, let N be the number of ways to choose the pairs and label the points in them so that a D_1-switching can be applied to these pairs such that $\mathcal{P}$ is converted to some $\mathcal{P}' \in \mathcal{C}_{0,l_1-1,l_2}$ without creating any new loops and double pairs. In order to apply a D_1-switching operation, we need to choose a mixed double pair $\{e_1, e_2\}$ and an ordered pair of distinct pure pairs (e_3, e_4). The number of ways to choose $\{e_1, e_2\}$ is l_1 in $\mathcal{C}_{0,l_1,l_2}$ and hence the number of ways to label the points as $3, 4, 5, 6$ is $2l_1$. The number of ways to choose the ordered pair of pure pairs (e_3, e_4) is $t(t-1)$. For any chosen (e_3, e_4), there are 4 ways to label points as $1, 2, 7, 8$. Let the vertices that contain points $3, 4, 1, 2, 7, 8$ be $v_1, v_2, v_3, v_4, v_5, v_6$ as shown in Figure 4. Hence a rough count of N is $8l_1t(t-1)$ including the count of a few forbidden choices of e_1, e_2, e_3, e_4, which are listed as follows.

(a) The pair e_1 is adjacent to e_3 or e_4, or e_3 is adjacent to e_4, in which case the D_1-switching is not applicable.

(b) There exists a pair between $\{v_1, v_3\}$, or $\{v_2, v_4\}$, or $\{v_2, v_6\}$, or $\{v_1, v_5\}$ in $\mathcal{P}$, since another double pair will be created after the D_1-switching is applied.

(c) The pair e_3 or e_4 is contained in a double pair, since another double pair is destroyed after the D_1-switching is applied.

The numbers of forbidden choices of e_1, e_2, e_3, e_4 coming from case (a), (b) and (c) are $O(l_1 t d_{\max})$, $O(l_1 t d_{\max}^2)$ and $O(l_1 t l_2)$ respectively. So $N = 8l_1t^2(1 + O(d_{\max}^2/t + l_2/t))$.

For a given pairing $\mathcal{P}' \in \mathcal{C}_{0,l_1-1,l_2}$, let N' be the number of ways to choose the pairs and label the points in them so that an inverse D_1-switching operation can be applied to these pairs which converts $\mathcal{P}'$ to some $\mathcal{P} \in \mathcal{C}_{0,l_1,l_2}$ without destroying any loops or double pairs simultaneously. In order to apply such an operation, we need to choose two simple directed 2-paths, one of type 1 and the other of type 4. There are $A_1(\mathcal{P}')$ simple directed 2-paths of type 1, each of which gives a way of labelling points as $2, 4, 6, 8$, and there are $A_4(\mathcal{P}')$ simple directed 2-paths of type 4, each of which gives a way of labelling points as $1, 3, 5, 7$. Hence

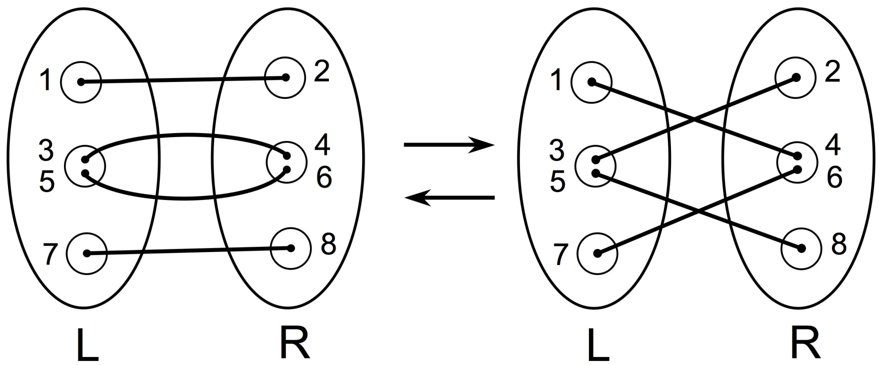

Figure 5: *D_2-switching and its inverse*

a rough count of N' is $A_1(\mathcal{P}')A_4(\mathcal{P}')$ including the counts of a few forbidden choices of such two 2-paths which are listed in the following two cases.

(a) If we have $v_i = v_j$, for $i \in \{3,5\}$ and $j \in \{2,4,6\}$, then the operation is not applicable.

(b) If there already exists a pair between $\{v_1, v_2\}$, or $\{v_3, v_4\}$, or $\{v_5, v_6\}$ in $\mathcal{P}'$, then more than one double pair will be created in this case if the inverse D_1-switching is applied.

Note that since these two directed 2-paths are both simple, there is no destruction of double pairs when the inverse D_1-switching is applied. The numbers of forbidden choices of the two directed 2-paths from case (a) and (b) are respectively $O(A_1(\mathcal{P}')d_{\max}^2)$ and $O(A_1(\mathcal{P}')d_{\max}^3)$. So $\mathbf{E}(N') = \mathbf{E}\big(A_1(\mathcal{P}')A_4(\mathcal{P}') \mid \mathcal{P}' \in \mathcal{C}_{0,l_1-1,l_2}\big) + O(a_1 d_{\max}^3) = a_1(M_2(L) - O(l_1 d_{\max}))(1 + O(d_{\max}^3/M_2(L)))$. Since $l_1 \geq 1$, we have $M_2(L) \geq 1$. Hence

$$\begin{aligned}\frac{|\mathcal{C}_{0,l_1,l_2}|}{|\mathcal{C}_{0,l_1-1,l_2}|} &= \frac{a_1 M_2(L)(1 + O(d_{\max}^3/M_2(L)) + O(l_1 d_{\max}/M_2(L)))}{8 l_1 t^2 (1 + O(d_{\max}^2/t) + O(l_2/t))} \\ &= \frac{a_1 M_2(L)}{8 l_1 t^2}(1 + O(d_{\max}^3/M_2(L) + d_{\max}^2/t + l_2/t + l_1 d_{\max}/M_2(L))),\end{aligned}$$

and this shows part (i) of Lemma 3.10. Similarly we can obtain part (ii) by analysing the D_2-switching and its inverse. ∎

The following two switching operations, designed to reduce the number of pure double pairs by exactly one, are used for the next lemma.

(a) *D_3-switching*: take a pure double pair $\{\{1,2\},\{3,4\}\}$ and also two pure pairs $\{5,6\}$ and $\{7,8\}$ such that the eight points are located in the six distinct vertices as shown in Figure 6. Replace the four pairs by $\{1,5\},\{2,6\},\{3,7\},\{4,8\}$.

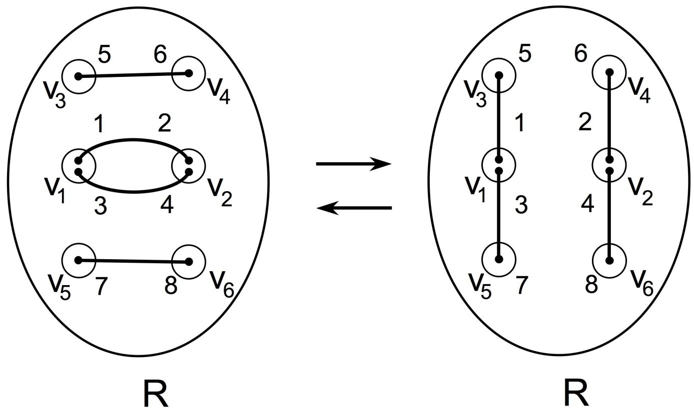

Figure 6: *D_3-switching and its inverse*

(a) *D_4-switching*: take a pure double pair $\{\{1,2\},\{3,4\}\}$ and also four mixed pairs $\{5,6\}$, $\{7,8\}$, $\{9,10\}$, $\{11,12\}$ such that the twelve points are located in the ten distinct vertices as shown in Figure 7. Replace the six pairs by $\{6,10\}$, $\{8,12\}$, $\{1,5\}$, $\{3,9\}$, $\{2,11\}$, $\{4,7\}$.

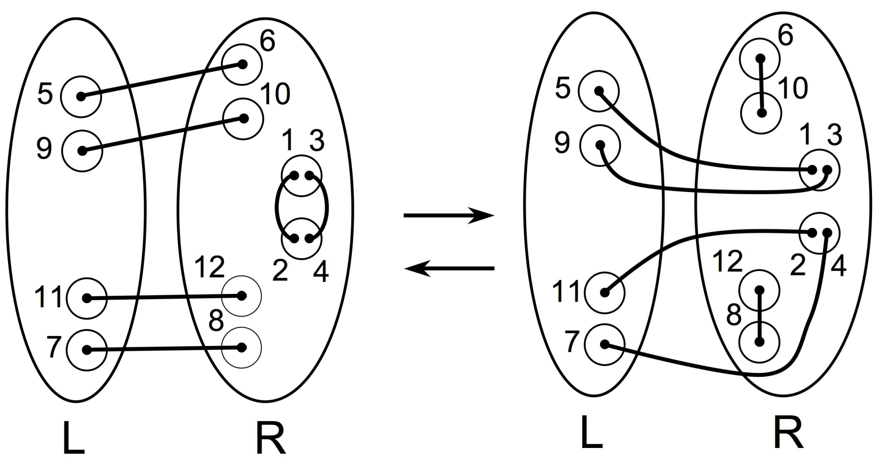

Figure 7: *D_4-switching and its inverse*

The inverse switchings are defined in the obvious way. For example, for the inverse of the D_3-switching, take two directed paths of type 1, $((5,1),(3,7))$ and $((6,2),(4,8))$, such that the eight points are located in six distinct vertices as shown in Figure 6. Replace these four pairs by $\{5,6\}$, $\{1,2\}$, $\{3,4\}$, $\{7,8\}$.

Define $b_i(l_0,l_1,l_2) = \mathbf{E}(A_i(\mathcal{P})^2 \mid \mathcal{P} \in \mathcal{C}_{l_0,l_1,l_2})$ for $i = 1$ and 3.

Lemma 3.11 *Assume $l_2 \geq 1$. For $i = 1, 3$, let $b_i = b_i(0, 0, l_2 - 1)$ for short. Then*

$$(i): \quad \text{If } t \geq 1 \text{ and } b_1 \geq 1,$$
$$\frac{|\mathcal{C}_{0,0,l_2}|}{|\mathcal{C}_{0,0,l_2-1}|} = \frac{b_1}{16 l_2 t^2}(1 + O((d_{\max}^2 + l_2)/t + d_{\max}^3 a_1/b_1)).$$
$$(ii): \quad \text{If } M_1(L) \geq 1,\ b_3 \geq 1 \text{ and } t \geq 1,$$
$$\frac{|\mathcal{C}_{0,0,l_2}|}{|\mathcal{C}_{0,0,l_2-1}|} = \frac{t^2 b_3}{l_2 M_1(L)^4}(1 + O(d_{\max}^3 a_3/b_3 + d_{\max}^2/M_1(L) + (d_{\max}^2 + l_2)/t)).$$

Proof. For a given pairing $\mathcal{P} \in \mathcal{C}_{0,0,l_2}$, let N be the number of ways to choose the pairs and label the points in them so that a D_3-switching operation can be applied, which converts $\mathcal{P}$ to some $\mathcal{P}' \in \mathcal{C}_{0,0,l_2-1}$ without creating any loops and double pairs simultaneously. In order to apply a D_3-switching operation, we need to choose a pure double pair $\{e_1, e_2\}$ and an ordered pair of distinct pure pairs (e_3, e_4). The number of ways to choose $\{e_1, e_2\}$ is l_2 in $\mathcal{C}_{0,0,l_2}$ and there are four ways to label the points as $1, 2, 3, 4$ for any chosen double pair. The number of ways to choose an ordered pair of two pure pairs (e_3, e_4) is $t(t-1)$ and hence the number of ways to label the points as $5, 6, 7, 8$ is $4t(t-1)$. Hence a rough count of N is $16 l_2 t(t-1)$ including the counts of forbidden choices of pairs $e_1, \ldots, e_4$ which we estimate next. Let the vertices that contain points $1, 2, 5, 6, 7, 8$ be $v_1, v_2, v_3, v_4, v_5, v_6$ as shown in Figure 6. The forbidden choices of the pairs $e_1, \ldots, e_4$ are listed in the following three cases.

(a) When e_1 is adjacent to e_3 or e_4 or when e_3 is adjacent to e_4, then the D_3-switching is not applicable.

(b) If there exists a pair between $\{v_1, v_3\}$, or $\{v_2, v_4\}$, or $\{v_1, v_5\}$, or $\{v_2, v_6\}$ in $\mathcal{P}$, then more double pairs will be created after the application of the switching operation.

(c) If e_3 or e_4 is contained in a double pair, then another double pair would be destroyed after the application of the switching operation.

The numbers of forbidden choices of $e_1, \ldots, e_4$ coming from (a),(b) and (c) are $O(l_2 t d_{\max})$, $O(l_2 t d_{\max}^2)$ and $O(l_2^2 t)$ respectively. So $N = 16 l_2 t^2 (1 + O(d_{\max}^2/t + l_2/t))$.

For any pairing $\mathcal{P}' \in \mathcal{C}_{0,0,l_2-1}$, let N' be the number of ways to choose the pairs and label the points in them so that an inverse D_3-switching can be applied to these pairs, which converts $\mathcal{P}'$ to some $\mathcal{P} \in \mathcal{C}_{0,0,l_2}$ without simultaneously destroying any loops or double pairs. In order to apply such an operation, we need to choose an ordered pair of distinct simple directed 2-paths of type 1. The number of ways to do that is $A_1(\mathcal{P}')(A_1(\mathcal{P}') - 1)$. So the number of ways to label the points $1, 2, \ldots, 8$ is $A_1(\mathcal{P}')(A_1(\mathcal{P}') - 1)$, which gives a rough count of N'. The forbidden choices of the two paths whose counts should be excluded from N' are listed in the following cases.

(a) The two paths share some common vertex or common pair. In this case the inverse D_3-switching is not applicable.

(b) There exists a pair between $\{v_1, v_2\}$ or $\{v_3, v_4\}$ or $\{v_5, v_6\}$ in $\mathcal{P}'$. In this case, more double pairs will be created after the inverse D_3-switching operation is applied.

The numbers of ways to choose the ordered pair of 2-paths in case (a) and (b) are $O(A_1(\mathcal{P}')d_{\max}^2)$ and $O(A_1(\mathcal{P}')d_{\max}^3)$ respectively. Thus, $\mathbf{E}(N') = b_1(1 + O(d_{\max}^3 a_1/b_1))$.

Hence

$$\frac{|\mathcal{C}_{0,0,l_2}|}{|\mathcal{C}_{0,0,l_2-1}|} = \frac{b_1}{16 l_2 t^2}(1 + O(d_{\max}^2/t + d_{\max}^3 a_1/b_1 + l_2/t)).$$

Similarly by analysing the D_4-switching and its inverse, we obtain Lemma 3.11(ii). ∎

4 More switchings to estimate a's and b's

The lemmas in the previous section give ratios of the sizes of 'adjacent' classes $\mathcal{C}_{i,j,k}$, but those estimates are in terms of $a_i(l_0, l_1, l_2)$ ($i = 1, 2, 3$) defined in (3.7), b_i ($i = 1, 3$) defined just before Lemma 3.11, and t defined in (3.6). In this section, we use further switchings to estimate the values of these variables. The following two switchings are used for a_i.

(a) S_1-*switching*: Take a mixed pair and label the points in it by $\{1, 2\}$ as shown in Figure 8. Take a simple directed 2-path that is vertex disjoint from the chosen mixed pair. Label the points by $3, 4, 5, 6$. Replace these three pairs by $\{2, 3\}$, $\{1, 4\}$ and $\{5, 6\}$.

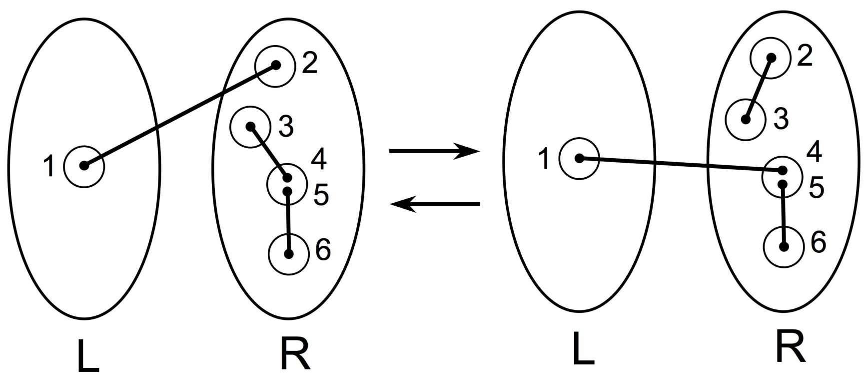

Figure 8: S_1-*switching and its inverse*

(b) S_2 *switching*: Take a pure pair $\{5, 6\}$ and a simple directed 2-path $((1, 2), (3, 4))$ such that the six points are located in five distinct vertices shown as in Figure 9. Replace these three pairs by $\{1, 2\}$, $\{3, 5\}$ and $\{4, 6\}$.

The inverse switchings are defined in the obvious way. Recall that we often abbreviate $A_i(\mathcal{P})$, defined just after (3.6), to A_i.

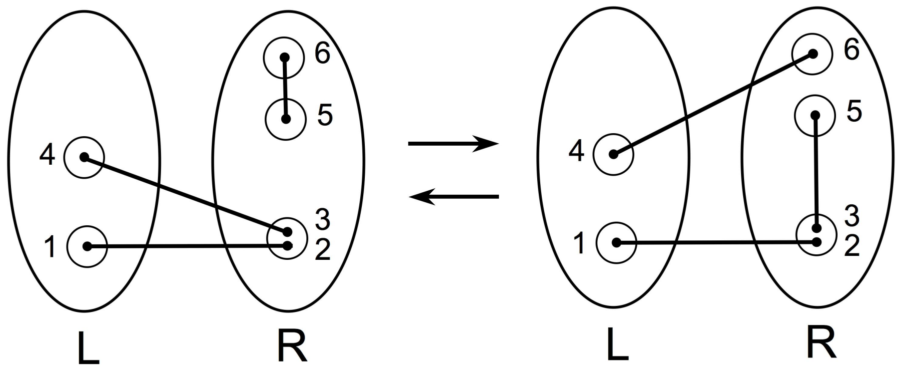

Figure 9: S_2*-switching and its inverse*

Lemma 4.1 *Given l_0, l_1 and l_2, let $\ell = l_0 + l_1 + l_2$. Suppose $d_{\max}^4 = o(M)$ and $\ell = o(M)$. Then*

$$
\begin{aligned}
&(i): \ \textit{if } M_1(L) \le M/4 \textit{ and } M_2(R) \ge 1,\\
&\quad a_1(l_0,l_1,l_2) = \frac{(M_1(R)-M_1(L))^2 M_2(R)}{M_1(R)^2}(1+O((d_{\max}^2+\ell)/t + (\ell d_{\max} + l_0 d_{\max}^2)/M_2(R)));\\
&(ii): \ \textit{if } M_1(L) > M/4 \textit{ and } M_2(R) \ge 1,\\
&\quad a_3(l_0,l_1,l_2) = \frac{M_1(L)^2 M_2(R)}{M_1(R)^2}(1+O((d_{\max}^2+\ell)/M_1(L) + (\ell d_{\max} + l_0 d_{\max}^2)/M_2(R))).
\end{aligned}
$$

Proof. Let $a_i = a_i(l_0,l_1,l_2)$ for $i = 1,2,3$. We use the S_1-switching to compute the ratio a_1/a_2 and the S_2-switching to compute the ratio a_3/a_2. We count the ordered pairs of pairings $(\mathcal{P},\mathcal{P}')$ such that both $\mathcal{P}$ and $\mathcal{P}'$ are from $\mathcal{C}_{l_0,l_1,l_2}$, and $\mathcal{P}'$ is obtained from $\mathcal{P}$ by applying an S_1-switching to $\mathcal{P}$ without any creation or destruction of loops or double pairs. Let N_1 denote the number of such ordered pairs of pairings.

We first prove part (i). Assume $M_1(L) \le M/4$. For any $\mathcal{P} \in \mathcal{C}_{l_0,l_1,l_2}$, the number of S_1-switching operations that can be applied to it is

$$
A_1 M_1(L) + O(A_1 d_{\max}^2 + A_1 l_1) = A_1 M_1(L)\Big(1 + O(d_{\max}^2/M_1(L) + l_1/M_1(L))\Big). \tag{4.1}
$$

For any $\mathcal{P} \in \mathcal{C}_{l_0,l_1,l_2}$, the number of inverse S_1-switching operations that can be applied to it is

$$
A_2 \cdot 2t + O(A_2 d_{\max}^2 + A_2(l_0+l_2)) = A_2 \cdot 2t\Big(1 + O(d_{\max}^2/t + (l_0+l_2)/t)\Big). \tag{4.2}
$$

The total number of S_1-switching operations that can be applied to pairings in $\mathcal{C}_{l_0,l_1,l_2}$ is

$$
\sum_{\mathcal{P}\in\mathcal{C}_{l_0,l_1,l_2}} A_1(\mathcal{P}) M_1(L)\Big(1+O((d_{\max}^2+l_1)/M_1(L))\Big) = a_1 M_1(L)\Big(1+O((d_{\max}^2+\ell)/M_1(L))\Big)|\mathcal{C}_{l_0,l_1,l_2}|,
$$

and the total number of inverse S_1-switching operations that can be applied to pairings in $\mathcal{C}_{l_0,l_1,l_2}$ is

$$\sum_{\mathcal{P}\in\mathcal{C}_{l_0,l_1,l_2}} A_2(\mathcal{P})\cdot 2t\Big(1+O(d_{\max}^2/t+(l_0+l_2)/t)\Big) = a_2\cdot 2t\Big(1+O(d_{\max}^2/t+\ell/t)\Big)|\mathcal{C}_{l_0,l_1,l_2}|.$$

These two numbers are both equal to N_1. Hence

$$\begin{aligned}\frac{a_2}{a_1} &= \frac{M_1(L)}{2t}(1+O(d_{\max}^2/t+d_{\max}^2/M_1(L)+\ell/M_1(L)+\ell/t)) \\ &= \frac{M_1(L)}{2t}\Big(1+O((d_{\max}^2+\ell)/t)\Big)+O((d_{\max}^2+\ell)/t).\end{aligned}\tag{4.3}$$

Similarly, by the S_2-switching and its inverse we get

$$\begin{aligned}\frac{a_3}{a_2} &= \frac{M_1(L)}{2t}(1+O(d_{\max}^2/t+d_{\max}^2/M_1(L)+\ell/M_1(L)+\ell/t)) \\ &= \frac{M_1(L)}{2t}\Big(1+O((d_{\max}^2+\ell)/t)\Big)+O((d_{\max}^2+\ell)/t).\end{aligned}\tag{4.4}$$

Hence

$$\begin{aligned}a_2 &= a_1\left(\frac{M_1(L)}{2t}\Big(1+O((d_{\max}^2+\ell)/t)\Big)+O((d_{\max}^2+\ell)/t)\right), \\ a_3 &= a_1\left(\frac{M_1(L)}{2t}\Big(1+O((d_{\max}^2+\ell)/t)\Big)+O((d_{\max}^2+\ell)/t)\right)^2.\end{aligned}$$

Recall that $d_{\max}^4 = o(M)$ and $\ell = o(M)$. Since $M_1(L)\le M/4$, we have $t=\Omega(M)$ and $M_1(L)/t\le 1$ (indeed we only need that $M_1(L)/t$ is bounded) and so

$$a_3 = a_1\left(\left(\frac{M_1(L)}{2t}\right)^2\Big(1+O((d_{\max}^2+\ell)/t)\Big)+O((d_{\max}^2+\ell)/t)\right).$$

Hence

$$\begin{aligned}a_1+2a_2+a_3 &= a_1\left(1+\left(2\frac{M_1(L)}{2t}+\left(\frac{M_1(L)}{2t}\right)^2\right)\Big(1+O((d_{\max}^2+\ell)/t)\Big)+O((d_{\max}^2+\ell)/t)\right) \\ &= a_1\left(\left(1+\frac{M_1(L)}{2t}\right)^2\Big(1+O((d_{\max}^2+\ell)/t)\Big)+O((d_{\max}^2+\ell)/t)\right) \\ &= a_1\left(1+\frac{M_1(L)}{2t}\right)^2(1+O((d_{\max}^2+\ell)/t)).\end{aligned}\tag{4.5}$$

For any pairing $\mathcal{P}\in\mathcal{C}_{l_0,l_1,l_2}$, the number of simple directed 2-paths in $\mathcal{P}$ is $\sum_{i\in L\cup R} d_i(d_i-1) - O(\ell d_{\max}+l_0 d_{\max}^2)$, since the number of non-simple directed 2-paths is bounded by

$O(l_0 d_{\max}^2 + l_1 d_{\max} + l_2 d_{\max}) = O(\ell d_{\max} + l_0 d_{\max}^2)$. On the other hand, the number of simple directed 2-paths in $\mathcal{P}$ is $A_1 + 2A_2 + A_3 + A_4$, since $2A_2$ counts the number of directed 2-paths of type 2 and the opposite direction. Then

$$A_1 + 2A_2 + A_3 + M_2(L) - O(l_1 d_{\max}) = \sum_{i \in L \cup R} d_i(d_i - 1) - O(\ell d_{\max} + l_0 d_{\max}^2).$$

Thus,

$$A_1 + 2A_2 + A_3 = M_2(R) + O(\ell d_{\max} + l_0 d_{\max}^2) = M_2(R)(1 + O((\ell d_{\max} + l_0 d_{\max}^2)/M_2(R))). \quad (4.6)$$

Combining this with (4.5), we have

$$a_1 = \frac{(M_1(R) - M_1(L))^2 M_2(R)}{M_1(R)^2}(1 + O(d_{\max}^2/t + \ell/t + (\ell d_{\max} + l_0 d_{\max}^2)/M_2(R))),$$

which proves part (i).

Next we show part (ii). Assume $M_1(L) > M/4$. We observe that (4.3) also gives

$$\frac{a_1}{a_2} = \frac{2t}{M_1(L)}\Big(1 + O((d_{\max}^2 + \ell)/M_1(L))\Big) + O((d_{\max}^2 + \ell)/M_1(L)),$$

and (4.4) gives

$$\frac{a_2}{a_3} = \frac{2t}{M_1(L)}\Big(1 + O((d_{\max}^2 + \ell)/M_1(L))\Big) + O((d_{\max}^2 + \ell)/M_1(L)).$$

Since $M_1(L) \geq M/4$, we have $t/M_1(L) < 1$ (so $t/M_1(L)$ is bounded). Calculation similar to part (i) yields

$$\begin{aligned} a_1 + 2a_2 + a_3 &= a_3\left(1 + \frac{2t}{M_1(L)}\right)^2 \Big(1 + O((d_{\max}^2 + \ell)/M_1(L))\Big) \\ &= M_2(R)(1 + O((\ell d_{\max} + l_0 d_{\max}^2)/M_2(R))). \end{aligned}$$

Hence

$$a_3 = \frac{M_1(L)^2 M_2(R)}{M_1(R)^2}(1 + O((d_{\max}^2 + \ell)/M_1(L) + (\ell d_{\max} + l_0 d_{\max}^2)/M_2(R))).$$

This proves part (ii) of the lemma. ∎

Recall the definition of $b_i(l_0, l_1, l_2)$ above Lemma 3.11. We next estimate these using simple modifications of the S_i-switchings for $i = 1, 3$. (Note: in this lemma, our abbreviation b_i contains no shift of index, whilst it did in Lemma 3.11.)

Lemma 4.2 *For $i = 1, 3$, let $a_i = a_i(l_0, l_1, l_2)$ and $b_i = b_i(l_0, l_1, l_2)$, and let $\ell = l_0 + l_1 + l_2$. Assume $M_2(R) \geq 1$, $d_{\max}^4 = o(M)$ and $\ell = o(M)$. Then*

(i): *if $M_1(L) \leq M/4$,* $b_1 = a_1^2(1 + O((d_{\max}^2 + \ell)/t + (\ell d_{\max} + l_0 d_{\max}^2 + d_{\max}^2)/M_2(R)))$;

(ii): *if $M_1(L) > M/4$,*
$b_3 = a_3^2(1 + O((d_{\max}^2 + \ell)/M_1(L) + (\ell d_{\max} + l_0 d_{\max}^2 + d_{\max}^2)/M_2(R)))$.

Proof. For $1 \le i \le 5$, let $X_i(\mathcal{P})$ denote the number of ordered pairs of vertex disjoint simple 2-paths in $\mathcal{P}$ where the first path has type j_i and the second has type h_i, with $(j_1, h_1) = (1,1)$, $(j_2, h_2) = (3,3)$, $(j_3, h_3) = (1,2)$, $(j_4, h_4) = (1,3)$, and $(j_5, h_5) = (2,3)$. We drop $\mathcal{P}$ from the notation $X_i(\mathcal{P})$ when the context is clear.

The S_3-switching, as illustrated in Figure 10, is a slight modification of the S_1-switching. To apply it, we need to choose a mixed pair and two simple 2-paths of type 1 such that they are pairwise disjoint. To apply its inverse, we need to choose a pure pair and two simple 2-paths of type 2 and 1 respectively such that they are pairwise disjoint. Compared with the S_1-switching, the S_3-switching requires an additional simple directed 2-path of type 1. However, the pairs in the extra 2-path remain after the S_3-switching is applied since they are vertex-disjoint from the pairs that are removed. The S_4-switching, as illustrated in Figure 11, is a similar modification of the S_2-switching.

We will first estimate $\mathbf{E}(X_i(\mathcal{P}) \mid \mathcal{P} \in \mathcal{C}_{l_0,l_1,l_2})$ for $i \in [5]$ and then use this to estimate b_1 and b_3. Following the analogous argument as in Lemma 4.1, we can estimate the ratio $\mathbf{E}(X_3(\mathcal{P}) \mid \mathcal{P} \in \mathcal{C}_{l_0,l_1,l_2})/\mathbf{E}(X_1(\mathcal{P}) \mid \mathcal{P} \in \mathcal{C}_{l_0,l_1,l_2})$ by counting the ordered pairs of pairings $(\mathcal{P}, \mathcal{P}')$ such that $\mathcal{P}, \mathcal{P}' \in \mathcal{C}_{l_0,l_1,l_2}$ and $\mathcal{P}'$ is obtained by applying an S_3-operation to $\mathcal{P}$ without any creation or destruction of loops or double pairs. Then the number of such S_3-switching operations that can be applied to $\mathcal{P}$ is $X_1 M_1(L) + O(X_1 d_{\max}^2 + X_1 l_1)$. The number of such inverse S_3-operations that can be applied to $\mathcal{P}$ is $2tX_3 + O(X_3 d_{\max}^2 + X_3(l_0 + l_2))$. So the ratio $\mathbf{E}(X_3(\mathcal{P}) \mid \mathcal{P} \in \mathcal{C}_{l_0,l_1,l_2})/\mathbf{E}(X_1(\mathcal{P}) \mid \mathcal{P} \in \mathcal{C}_{l_0,l_1,l_2})$ equals the right hand side of (4.3). Similar analysis of the S_4-switching and its inverse shows that the ratio $\mathbf{E}(X_4(\mathcal{P}) \mid \mathcal{P} \in \mathcal{C}_{l_0,l_1,l_2})/\mathbf{E}(X_3(\mathcal{P}) \mid \mathcal{P} \in \mathcal{C}_{l_0,l_1,l_2})$ equals the right hand side of (4.4).

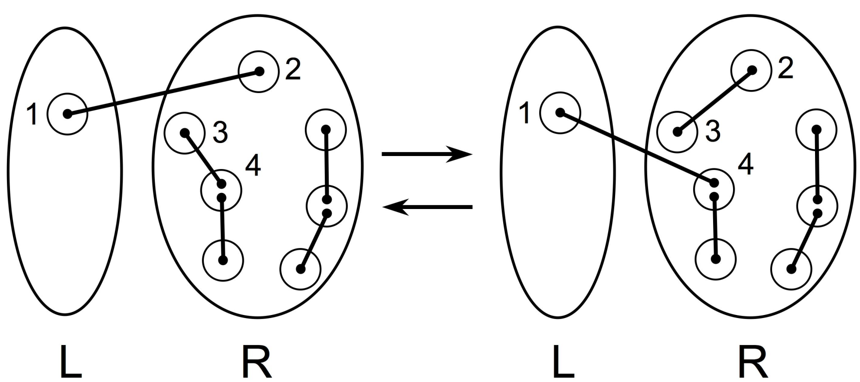

Figure 10: S_3-switching and its inverse

On the other hand, by (4.6), for any $\mathcal{P} \in \mathcal{C}_{l_0,l_1,l_2}$, $A_1(\mathcal{P}) + 2A_2(\mathcal{P}) + A_3(\mathcal{P}) = M_2(R)(1 + O((\ell d_{\max} + l_0 d_{\max}^2)/M_2(R)))$. Thus,

$$\begin{aligned}
&\mathbf{E}(A_1^2 \mid \mathcal{P} \in \mathcal{C}_{l_0,l_1,l_2}) + 2\mathbf{E}(A_1A_2 \mid \mathcal{P} \in \mathcal{C}_{l_0,l_1,l_2}) + \mathbf{E}(A_1A_3 \mid \mathcal{P} \in \mathcal{C}_{l_0,l_1,l_2}) \\
&\quad = \mathbf{E}(A_1(A_1 + 2A_2 + A_3) \mid \mathcal{P} \in \mathcal{C}_{l_0,l_1,l_2}) \\
&\quad = \mathbf{E}(A_1 \mid \mathcal{P} \in \mathcal{C}_{l_0,l_1,l_2}) M_2(R)(1 + O((\ell d_{\max} + l_0 d_{\max}^2)/M_2(R))) \\
&\quad = a_1(l_0, l_1, l_2) M_2(R)(1 + O((\ell d_{\max} + l_0 d_{\max}^2)/M_2(R))). \qquad (4.7)
\end{aligned}$$

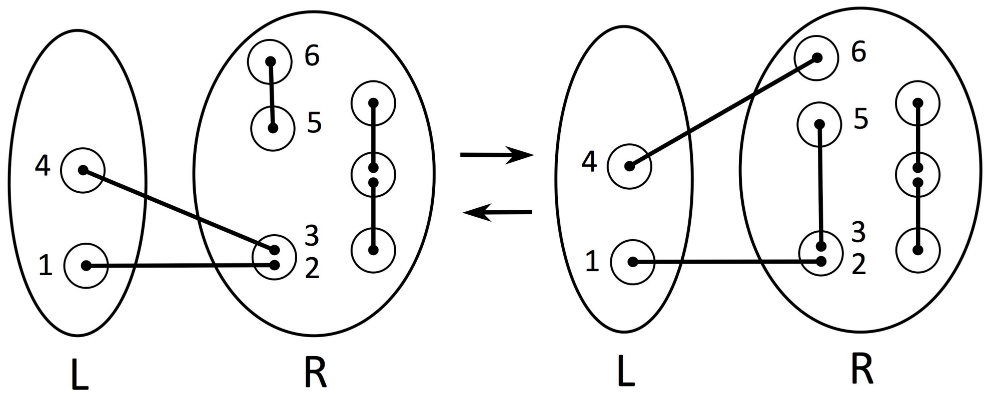

Figure 11: S_4-switching and its inverse

We also have

$$X_1 = A_1^2 + O(A_1 d_{\max}^2), \quad X_3 = A_1 A_2 + O(A_1 d_{\max}^2), \quad X_4 = A_1 A_3 + O(A_1 d_{\max}^2), \tag{4.8}$$

where the error terms in (4.8) account for the number of ordered pairs of simple 2-directed paths that are not vertex disjoint. Let $a_1 = a_1(l_0, l_1, l_2)$. Taking the conditional expectation on both sides of each equation in (4.8), we obtain

$$\begin{aligned} \mathbf{E}(X_1 \mid \mathcal{P} \in \mathcal{C}_{l_0,l_1,l_2}) &= \mathbf{E}(A_1^2 \mid \mathcal{P} \in \mathcal{C}_{l_0,l_1,l_2}) + O(a_1 d_{\max}^2), \\ \mathbf{E}(X_3 \mid \mathcal{P} \in \mathcal{C}_{l_0,l_1,l_2}) &= \mathbf{E}(A_1 A_2 \mid \mathcal{P} \in \mathcal{C}_{l_0,l_1,l_2}) + O(a_1 d_{\max}^2), \\ \mathbf{E}(X_4 \mid \mathcal{P} \in \mathcal{C}_{l_0,l_1,l_2}) &= \mathbf{E}(A_1 A_3 \mid \mathcal{P} \in \mathcal{C}_{l_0,l_1,l_2}) + O(a_1 d_{\max}^2). \end{aligned}$$

Combining this with (4.7) we have

$$\begin{aligned} &\mathbf{E}(X_1 \mid \mathcal{P} \in \mathcal{C}_{l_0,l_1,l_2}) + 2\mathbf{E}(X_3 \mid \mathcal{P} \in \mathcal{C}_{l_0,l_1,l_2}) + \mathbf{E}(X_4 \mid \mathcal{P} \in \mathcal{C}_{l_0,l_1,l_2}) \\ &= a_1 M_2(R)(1 + O((\ell d_{\max} + l_0 d_{\max}^2 + d_{\max}^2)/M_2(R))). \end{aligned}$$

So part (i) follows from an argument similar to that used for Lemma 4.1 and (4.8). Similarly, by analysing two switching operations similar to those of S_3-switching and S_4-switching, except that the extra 2-path is of type 3, we can show that the ratio $\mathbf{E}(X_5 \mid \mathcal{P} \in \mathcal{C}_{l_0,l_1,l_2})/\mathbf{E}(X_4 \mid \mathcal{P} \in \mathcal{C}_{l_0,l_1,l_2})$ equals the right hand side of (4.3), and $\mathbf{E}(X_2 \mid \mathcal{P} \in \mathcal{C}_{l_0,l_1,l_2})/\mathbf{E}(X_5 \mid \mathcal{P} \in \mathcal{C}_{l_0,l_1,l_2})$ equals the right hand side of (4.4). By the fact that

$$X_5 = A_2 A_3 + O(A_3 d_{\max}^2), \quad X_4 = A_1 A_3 + O(A_3 d_{\max}^2), \quad X_2 = A_3^2 + O(A_3 d_{\max}^2),$$

and

$$\begin{aligned} &\mathbf{E}(X_2 \mid \mathcal{P} \in \mathcal{C}_{l_0,l_1,l_2}) + 2\mathbf{E}(X_5 \mid \mathcal{P} \in \mathcal{C}_{l_0,l_1,l_2}) + \mathbf{E}(X_4 \mid \mathcal{P} \in \mathcal{C}_{l_0,l_1,l_2}) \\ &= a_3(l_0, l_1, l_2) M_2(R)(1 + O((\ell d_{\max} + l_0 d_{\max}^2 + d_{\max}^2)/M_2(R))), \end{aligned}$$

together with Lemma 4.1(ii), part (ii) follows from an argument similar to that in part (i) and the proof of Lemma 4.1(ii). ∎

5 Synthesis

We are now ready to substitute the values of the variables a_i and b_i determined in Section 4 in the ratios determined in Section 3, and from there to prove the main theorem. The reader should not be surprised at how the separate cases combine to give the same resulting formulae with the desired error terms; the definitions of the cases and the choices of switchings for each case were carefully designed to achieve this. Before we proceed to the next lemma and the proof of Theorem 2.1, we specify the values of ζ_i, $i = 0, 1, 2$, first introduced above Corollary 3.7.

Choose ζ_i sufficiently large such that the following conditions hold:

$$(d_{\max} + \ln M)\left(\frac{d_{\max}^2 + \ln M}{M} + \frac{d_{\max}^2 + \ln M}{\zeta_1(d_{\max}^4 + d_{\max}^2 \ln^2 M)} + \frac{d_{\max}^3 + d_{\max}^2 \ln M}{\zeta_0(d_{\max}^5 + d_{\max}^3 \ln^2 M)}\right) < 1, \tag{5.1}$$

$$(d_{\max}^2 + \ln M)\left(\frac{d_{\max}^2 + \ln M}{M} + \frac{d_{\max}^3 + d_{\max} \ln M}{\zeta_2(d_{\max}^5 + d_{\max}^3 \ln^2 M)} + \frac{d_{\max}^3 + d_{\max} \ln M}{\zeta_0(d_{\max}^5 + d_{\max}^3 \ln^2 M)}\right) < 1, \tag{5.2}$$

$$(d_{\max}^2 + \ln M)\left(\frac{d_{\max}^2}{M} + \frac{d_{\max}^2 + \ln M}{\zeta_1(d_{\max}^4 + d_{\max}^2 \ln^2 M)} + \frac{d_{\max}^3 + d_{\max} \ln M}{\zeta_0(d_{\max}^5 + d_{\max}^3 \ln^2 M)}\right) < 1. \tag{5.3}$$

This completes the definition of k_i above Corollary 3.7. Note that by the definition of k_i, we always have $k_0 = O(d_{\max} + \ln M)$ and $k_i = O(d_{\max}^2 + \ln M)$ for $i = 1, 2$.

Lemma 5.1 *Assume $d_{\max}^4 = o(M)$ and $\ell = l_0 + l_1 + l_2 = o(M)$. Assume $M_2(R)/d_{\max}^3$ is sufficiently large. Then*

$$(i)\ \frac{|\mathcal{C}_{l_0,l_1,l_2}|}{|\mathcal{C}_{l_0-1,l_1,l_2}|} = \frac{\mu_0}{l_0}\left(1 + O\left(\frac{l_1}{M} + \frac{d_{\max}^2 + l_0 + l_2}{t} + \frac{(l_1 + l_2)d_{\max} + l_0 d_{\max}^2}{M_2(R)}\right)\right),\ l_0 \geq 1;$$

$$(ii)\ \frac{|\mathcal{C}_{0,l_1,l_2}|}{|\mathcal{C}_{0,l_1-1,l_2}|} = \frac{\mu_1}{l_1}\left(1 + O\left(\frac{d_{\max}^2 + l_1 + l_2}{M} + \frac{d_{\max}^3 + l_1 d_{\max}}{M_2(L)} + \frac{(l_1 + l_2)d_{\max}}{M_2(R)}\right)\right),\ l_1 \geq 1;$$

$$(iii)\ \frac{|\mathcal{C}_{0,0,l_2}|}{|\mathcal{C}_{0,0,l_2-1}|} = \frac{\mu_2}{l_2}\left(1 + O\left(\frac{d_{\max}^2}{M} + \frac{d_{\max}^2 + l_2}{t} + \frac{l_2 d_{\max} + d_{\max}^3}{M_2(R)}\right)\right),\ l_2 \geq 1.$$

Proof. Clearly $0 \leq M_1(L)/M \leq 1/2$ since $M_1(R) \geq M_1(L)$. We discuss two cases according to the ratio $M_1(L)/M$.

Case 1: $M_1(L)/M \leq 1/4$.

Here t, which was defined as $(M_1(R) - M_1(L))/2$, is $\Theta(M)$. By part (i) of Lemmas 3.9–3.11 and 4.1–4.2, and recalling (2.1)–(2.3), we obtain the following, with some of the bounds on error terms explained below.

$$\begin{aligned}
\frac{|\mathcal{C}_{l_0,l_1,l_2}|}{|\mathcal{C}_{l_0-1,l_1,l_2}|} &= \frac{\mu_0}{l_0}\left(1 + O\left(\frac{d_{\max}^2 + l_0 + l_2}{M} + \frac{(l_1 + l_2)d_{\max} + l_0 d_{\max}^2}{M_2(R)}\right)\right), \\
\frac{|\mathcal{C}_{0,l_1,l_2}|}{|\mathcal{C}_{0,l_1-1,l_2}|} &= \frac{\mu_1}{l_1}\left(1 + O\left(\frac{d_{\max}^2 + l_2}{M} + \frac{d_{\max}^3 + l_1 d_{\max}}{M_2(L)}\frac{(l_1 + l_2)d_{\max}}{M_2(R)}\right)\right), \\
\frac{|\mathcal{C}_{0,0,l_2}|}{|\mathcal{C}_{0,0,l_2-1}|} &= \frac{\mu_2}{l_2}\left(1 + O\left(\frac{d_{\max}^2 + l_2}{M} + \frac{d_{\max}^3 a_1}{b_1} + \frac{l_2 d_{\max} + d_{\max}^2}{M_2(R)}\right)\right).
\end{aligned}$$

For the second equation, note that error terms involving l_0 do not appear since $l_0 = 0$, and similarly $l_0 = l_1 = 0$ for the third equation.

Case 2: $1/4 < M_1(L)/M \le 1/2$.

Here $M_1(L) = \Theta(M)$. By part (ii) of Lemmas 3.9–3.11 and 4.1–4.2, we obtain the following, with some error terms explained below.

$$\begin{aligned}
\frac{|\mathcal{C}_{l_0,l_1,l_2}|}{|\mathcal{C}_{l_0-1,l_1,l_2}|} &= \frac{\mu_0}{l_0}\left(1+O\left(\frac{d_{\max}^2+l_1}{M}+\frac{d_{\max}^2+l_0+l_2}{t}+\frac{(l_1+l_2)d_{\max}+l_0d_{\max}^2}{M_2(R)}\right)\right),\\
\frac{|\mathcal{C}_{0,l_1,l_2}|}{|\mathcal{C}_{0,l_1-1,l_2}|} &= \frac{\mu_1}{l_1}\left(1+O\left(\frac{d_{\max}^2+l_1}{M}+\frac{d_{\max}^3+l_1d_{\max}}{M_2(L)}+\frac{(l_1+l_2)d_{\max}}{M_2(R)}\right)\right),\\
\frac{|\mathcal{C}_{0,0,l_2}|}{|\mathcal{C}_{0,0,l_2-1}|} &= \frac{\mu_2}{l_2}\left(1+O\left(\frac{d_{\max}^2}{M}+\frac{d_{\max}^2+l_2}{t}+\frac{d_{\max}^3a_3}{b_3}+\frac{l_2d_{\max}+d_{\max}^2}{M_2(R)}\right)\right).
\end{aligned}$$

Parts (i) and (ii) follow by combining the two cases. To complete the proof of part (iii), we show that $a_1/b_1 = O(M_2(R)^{-1})$ when $M_1(L) \le M/4$ and $a_3/b_3 = O(M_2(R)^{-1})$ when $M_1(L) > M/4$.

First consider $M_1(L) \le M/4$. Considering a_1/b_1, we have the following two cases.

Case A: $M_2(R) \le \zeta_0(d_{\max}^5 + d_{\max}^3 \ln^2 M)$. Then $k_2 = d_{\max}^2$ according to its redefinition after Lemma 3.6. Since $M_2(R)/d_{\max}^3$ can be made arbitrarily large by the present lemma's assumption, the error terms $l_2d_{\max}/M_2(R)$ and $d_{\max}^3/M_2(R)$ in Lemmas 4.1(i) and 4.2(i) are sufficiently small, e.g. smaller than 1/2. It follows that $a_1 = \Omega(M_2(R))$ and $b_1 = \Theta(a_1^2)$, and so $a_1/b_1 = O(M_2(R)^{-1})$.

Case B: $M_2(R) > \zeta_0(d_{\max}^5 + d_{\max}^3 \ln^2 M)$. Then for any $l_2 \le k_2 = O(d_{\max}^2 + \ln M)$, as defined in (3.2), the error terms in Lemmas 4.1(i) and 4.2(i) are arbitrarily small. Thus $a_1 = \Omega(M_2(R))$, $b_1 = \Theta(a_1^2)$, and $a_1/b_1 = O(M_2(R)^{-1})$.

On the other hand, assuming $M_1(L) > M/4$, a similar argument shows that $a_3/b_3 = O(M_2(R)^{-1})$. ∎

Proof of Theorem 2.1. Recall that $\mathbf{P}(\mathbf{d})$ denotes the probability that a random pairing $\mathcal{P} \in \mathcal{M}(L,R,\mathbf{d})$ corresponds to a simple B-graph, and $U(m)$ denotes the number $m!/\big((m/2)!2^{m/2}\big)$ of pairings of m points. The total number of pairings in $\mathcal{M}(L,R,\mathbf{d})$ is thus $[M_1(R)]_{M_1(L)}U(M_1(R)-M_1(L))$. Since each simple B-graph corresponds to $\prod_{i=1}^n d_i!$ pairings in $\mathcal{M}(L,R,\mathbf{d})$, we have

$$g(L,R,\mathbf{d}) = \frac{M_1(R)!\mathbf{P}(\mathbf{d})}{2^{(M_1(R)-M_1(L))/2}((M_1(R)-M_1(L))/2)!\prod_{i=1}^n d_i!},$$

and it only remains to show that $\mathbf{P}(\mathbf{d}) = e^{-\mu_0-\mu_1-\mu_2}(1+O(d_{\max}^4/M))$.

If $M_2(R) = O(d_{\max}^3)$, we have $\mu_i = O(d_{\max}^4/M)$ for $i = 0,1,2$. Then by Corollary 3.4, $\mathbf{P}(\mathbf{d}) = 1 - O(d_{\max}^4/M)$ and we are done. So we may assume

$$M_2(R)/d_{\max}^3 > C \tag{5.4}$$

for any fixed (large) C. (Note we assume throughout that $d_{\max} > 0$ since otherwise there is nothing to prove.) By Corollary 3.8, it is enough to show

$$\sum_{l_2=0}^{k_2}\sum_{l_1=0}^{k_1}\sum_{l_0=0}^{k_0}|\mathcal{C}_{l_0,l_1,l_2}| = |\mathcal{C}_{0,0,0}|e^{\mu_0+\mu_1+\mu_2}(1+O(d_{\max}^4/M)). \tag{5.5}$$

Iterating the ratio in Lemma 5.1(i), for any fixed $l_0 \le k_0$, $l_1 \le k_1$ and $l_2 \le k_2$, we get

$$\frac{|\mathcal{C}_{l_0,l_1,l_2}|}{|\mathcal{C}_{0,l_1,l_2}|} = \frac{\mu_0^{l_0}}{l_0!}\left(1 + O((d_{\max}^2 + l_0 + l_2)/t + l_1/M + ((l_1 + l_2)d_{\max} + l_0 d_{\max}^2)/M_2(R))\right)^{l_0}. \quad (5.6)$$

First we sum over l_0. Here we assume $t \ge 1$, since otherwise $B_0 = 0$, which will trivially give the desired conclusion (5.10) stated below. Recalling the definition (3.2) of k_i and its redefinition after Corollary 3.6, we have $k_0 = O(d_{\max} + \ln M)$ and for $i = 1, 2$, $k_i = O(d_{\max}^2 + \ln M)$. Consider the following two cases.

Case 1: $M_2(R) \le \zeta_0(d_{\max}^5 + d_{\max}^3 \ln^2 M)$ or $2t \le \zeta_1(d_{\max}^4 + d_{\max}^2 \ln^2 M)$.

Here, by the redefinition of k_i, we have $k_0 = d_{\max}$ and $k_2 = d_{\max}^2$, so $(l_0+l_2)/t = O(d_{\max}^2/t)$ and $((l_1+l_2)d_{\max}+l_0 d_{\max}^2)/M_2(R) = O((d_{\max}^3+l_1 d_{\max})/M_2(R))$. Recalling also the definition (2.1) of μ_0 as $tM_2(R)/M_1(R)^2$, and noting $M_1(R) = \Omega(M)$ and $M_2(R) = O(d_{\max}M)$, we have from Lemma 5.1(i) that for $1 \le l_0 \le k_0$ and all relevant l_1 and l_2,

$$\frac{|\mathcal{C}_{l_0,l_1,l_2}|}{|\mathcal{C}_{l_0-1,l_1,l_2}|} = \frac{1}{l_0}\big(\mu_0 + O(d_{\max}^3/M + d_{\max} l_1/M)\big).$$

Hence (bounding $d_{\max}^3$ by $d_{\max}^4$ for consistency with the later argument),

$$\sum_{l_0=0}^{k_0} \frac{|\mathcal{C}_{l_0,l_1,l_2}|}{|\mathcal{C}_{0,l_1,l_2}|} = \sum_{l_0=0}^{k_0} \frac{\big(\mu_0 + O(d_{\max}^4/M + d_{\max} l_1/M)\big)^{l_0}}{l_0!}. \quad (5.7)$$

Note that for any $x = o(1)$,

$$\sum_{l_0=k_0+1}^{\infty} \frac{(\mu_0 + O(x))^{l_0}}{l_0!} = \sum_{l_0=k_0+1}^{\infty} \frac{\big(O(\mu_0)\big)^{l_0} + \big(O(x)\big)^{l_0}}{l_0!} = \big(O(\mu_0)\big)^{k_0+1}/(k_0+1)! + O(x), \quad (5.8)$$

using $\mu_0 = o(k_0)$, which is implied by $\mu_0 = O((d_{\max}^5 + d_{\max}^3 \ln^2 M)/M) = o(d_{\max})$. As in this case $k_0 = d_{\max} \ge 1$, by Stirling's formula, we obtain

$$\frac{(O(\mu_0))^{k_0+1}}{(k_0+1)!} = \left(\frac{O(\mu_0)}{d_{\max}+1}\right)^{d_{\max}+1} = \left(\frac{O(\mu_0)}{d_{\max}}\right)^2 = O(d_{\max}^4/M). \quad (5.9)$$

Combining (5.7)–(5.9) and setting $x = d_{\max}^4/M + d_{\max} l_1/M$, we obtain

$$\begin{aligned}\sum_{l_0=0}^{k_0} \frac{|\mathcal{C}_{l_0,l_1,l_2}|}{|\mathcal{C}_{0,l_1,l_2}|} &= \exp\big(\mu_0 + O(d_{\max}^4/M + d_{\max} l_1/M)\big) + O\big((d_{\max}^4 + d_{\max} l_1)/M\big) \\ &= \exp(\mu_0)\big(1 + O(d_{\max}^4/M + d_{\max} l_1/M)\big).\end{aligned}$$

Case 2: $M_2(R) > \zeta_0(d_{\max}^5 + d_{\max}^3 \ln^2 M)$ and $2t > \zeta_1(d_{\max}^4 + d_{\max}^2 \ln^2 M)$.

Here $k_0 = O(\ln M + d_{\max})$, $k_i = O(\ln M + d_{\max}^2)$ for $i = 1, 2$. By the choice of ζ_0 and ζ_1 in (5.1) and (5.2), we find that $l_0((d_{\max}^2 + l_0 + l_2)/t + l_1/M + ((l_1 + l_2)d_{\max} + l_0 d_{\max}^2)/M_2(R)) = O(1)$ provided $l_i \le k_i$ for $i = 0, 1, 2$. So, from (5.6),

$$\begin{aligned}\sum_{l_0=0}^{k_0} \frac{|\mathcal{C}_{l_0,l_1,l_2}|}{|\mathcal{C}_{0,l_1,l_2}|} &= \sum_{l_0=0}^{k_0} \frac{\mu_0^{l_0} \exp(O(l_0(d_{\max}^2/t + (l_0 + l_2)/t + l_1/M + \alpha_0)))}{l_0!} \\ &= \sum_{l_0=0}^{k_0} \frac{\mu_0^{l_0}}{l_0!} + O\left(\sum_{l_0=0}^{k_0} \frac{\mu_0^{l_0}}{l_0!} l_0 \left(\frac{d_{\max}^2 + l_2}{t} + \frac{l_1}{M} + \frac{(l_1 + l_2)d_{\max}}{M_2(R)}\right)\right) \\ &\quad + O\left(\sum_{l_0=0}^{k_0} \frac{\mu_0^{l_0}}{l_0!} l_0^2 \left(\frac{1}{t} + \frac{d_{\max}^2}{M_2(R)}\right)\right).\end{aligned}$$

Note also that $k_0 \ge 8\eta(R) \ge 16\mu_0$, and $k_0 \ge \ln M$. So

$$\sum_{l_0=k_0+1}^{\infty} \frac{\mu_0^{l_0}}{l_0!} = O\left(\frac{(k_0/16)^{k_0}}{k_0!}\right) = O\big((e/16)^{k_0}\big) = o(M^{-1}).$$

Also, of course, $\sum_{l_0=0}^{k_0} (\mu_0^{l_0}/l_0!) l_0 \le \mu_0 e^{\mu_0}$ and $\sum_{l_0=0}^{k_0} (\mu_0^{l_0}/l_0!) l_0^2 \le (\mu_0^2 + \mu_0) e^{\mu_0}$. So we have

$$\begin{aligned}\sum_{l_0=0}^{k_0} \frac{|\mathcal{C}_{l_0,l_1,l_2}|}{|\mathcal{C}_{0,l_1,l_2}|} &= e^{\mu_0} - O(M^{-1}) + O\left(e^{\mu_0} \mu_0 \left(\frac{d_{\max}^2 + l_2}{t} + \frac{l_1}{M} + \frac{(l_1 + l_2)d_{\max} + d_{\max}^3}{M_2(R)}\right)\right) \\ &\quad + O\left(e^{\mu_0}(\mu_0^2 + \mu_0)\left(\frac{1}{t} + \frac{d_{\max}^2}{M_2(R)}\right)\right).\end{aligned}$$

Now using

$$\begin{aligned}\mu_0/t &= M_2(R)/M_1(R)^2 = O(d_{\max}/M), \\ \mu_0^2/t &= O(M_2(R)^2 t/M_1(R)^4) = O(d_{\max}^2/M_1), \\ \mu_0 &= O(M_2(R)/M) = O(d_{\max}), \\ \mu_0 d_{\max}^3/M_2(R) &= O(t d_{\max}^3/M^2) = O(d_{\max}^3/M), \\ \mu_0^2 d_{\max}^2/M_2(R) &= O(t^2 M_2(R) d_{\max}^2/M^4) = O(d_{\max}^3/M),\end{aligned}$$

we obtain

$$\sum_{l_0=0}^{k_0} \frac{|\mathcal{C}_{l_0,l_1,l_2}|}{|\mathcal{C}_{0,l_1,l_2}|} = e^{\mu_0}\left(1 + O\left(\frac{(l_1 + l_2)d_{\max}}{M} + \frac{d_{\max}^3}{M}\right)\right).$$

Combining the two cases, we have (for l_1 and l_2 in the appropriate range)

$$\sum_{l_0=0}^{k_0} |\mathcal{C}_{l_0,l_1,l_2}| = |\mathcal{C}_{0,l_1,l_2}| \exp(\mu_0)\left(1 + O\left(\frac{(l_1 + l_2)d_{\max}}{M} + \frac{d_{\max}^4}{M}\right)\right). \qquad (5.10)$$

We will next sum this expression over l_1. By Lemma 5.1(ii), for any fixed $l_1 \le k_1$ and $l_2 \le k_2$,

$$\frac{|\mathcal{C}_{0,l_1,l_2}|}{|\mathcal{C}_{0,0,l_2}|} = \frac{\mu_1^{l_1}}{l_1!}\left(1 + O\left(\frac{d_{\max}^2 + l_1 + l_2}{M} + \frac{d_{\max}^3 + l_1 d_{\max}}{M_2(L)} + \frac{(l_1 + l_2)d_{\max}}{M_2(R)}\right)\right)^{l_1}. \tag{5.11}$$

Case 1: $M_2(R) \le \zeta_0(d_{\max}^5 + d_{\max}^3 \ln^2 M)$ or $M_2(L) \le \zeta_2(d_{\max}^5 + d_{\max}^3 \ln^2 M)$. Then $k_1 = d_{\max}^2$, and summing over $0 \le l_1 \le k_1$ we obtain

$$\begin{aligned}
\sum_{l_1=0}^{k_1}\sum_{l_0=0}^{k_0} |\mathcal{C}_{l_0,l_1,l_2}| &= \sum_{l_1=0}^{k_1} |\mathcal{C}_{0,l_1,l_2}| e^{\mu_0}\left(1 + O\left(\frac{(l_1+l_2)d_{\max}}{M} + \frac{d_{\max}^4}{M}\right)\right) \\
&= \sum_{l_1=0}^{k_1} |\mathcal{C}_{0,0,l_2}| e^{\mu_0}\left(1 + O\left(\frac{l_2 d_{\max}}{M} + \frac{d_{\max}^4}{M}\right)\right) \\
&\quad \times \left(\mu_1\left(1 + O\left(\frac{d_{\max}^2 + l_2}{M} + \frac{d_{\max}^3}{M_2(L)} + \frac{l_2 d_{\max} + d_{\max}^3}{M_2(R)}\right)\right)\right)^{l_1} \frac{1}{l_1!} \\
&= |\mathcal{C}_{0,0,l_2}| e^{\mu_0}\left(1 + O\left(\frac{l_2 d_{\max} + d_{\max}^4}{M}\right)\right) \sum_{l_1=0}^{k_1} \frac{\left(\mu_1 + O\left((d_{\max}^4 + l_2 d_{\max}^2)/M\right)\right)^{l_1}}{l_1!},
\end{aligned}$$

using (5.10), (5.11) and

$$\mu_1 = O(M_2(L)M_2(R)/M^2) = O(d_{\max}^2), \quad l_1 \le k_1 = d_{\max}^2.$$

Similar to the summation over l_0 in case 1, we obtain

$$\sum_{l_1=0}^{k_1}\sum_{l_0=0}^{k_0} |\mathcal{C}_{l_0,l_1,l_2}| = \exp(\mu_0 + \mu_1)|\mathcal{C}_{0,0,l_2}|\left(1 + O\left(\frac{l_2 d_{\max}^2 + d_{\max}^4}{M}\right)\right). \tag{5.12}$$

Case 2: $M_2(R) > \zeta_0(d_{\max}^5 + d_{\max}^3 \ln^2 M)$ and $M_2(L) > \zeta_2(d_{\max}^5 + d_{\max}^3 \ln^2 M)$. Then by the choice of ζ_0 and ζ_2 in (5.1) and (5.3), for any $l_1 \le k_1$, $l_2 \le k_2$,

$$l_1\left(\frac{d_{\max}^2 + l_1 + l_2}{M} + \frac{d_{\max}^3 + l_1 d_{\max}}{M_2(L)} + \frac{(l_1 + l_2)d_{\max}}{M_2(R)}\right)$$

is bounded. Estimating error terms similar to Case 2 of the earlier summation over l_0, we obtain the same result (5.12).

For summing over l_2, the argument is similar, and the final result is (5.5) as required. ∎

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