21 / Compactifications of completely regular spaces

Abstract. *In this section, we discuss those spaces, S, which can be densely embedded in a compact Hausdorff space. Only completely regular spaces can possess this property. The process by which we determine such a compact space, αS, for S, is called compactifying S. The space, αS, is called the compactification of S. The family of all compactifications of a completely regular space can be partially ordered. The maximal compactification of S with respect to the chosen partial ordering is called the Stone-Čech compactification. We discuss methods for its construction. We will show that only locally compact spaces have a minimal compactification with respect to the chosen partial ordering. It is called the one-point compactification.*

21.1 Compactifying a space

In this section we will briefly talk about methods for "compactifying a space (S, τ_S)". This essentially means adding a set, F, of points to S, to obtain a larger set, $T = S \cup F$, and topologizing T so that (T, τ_T) is a compact Hausdorff space in which a homeomorphic copy of S appears as a dense subspace of T.

With certain bounded subspaces of $\mathbb{R}^n$, this can, sometimes, be quite easy to do. For example, if $S = [-1, 3) \cup (3, 7)$ is equipped with the subspace topology, then by simply adding the points $\{3, 7\}$ to S we obtain the set $T = S \cup \{3, 7\} = [-1, 7]$ which, when equipped with the subspace topology, is a compact Hausdorff space which densely contains a homeomorphic copy of S. In such a case, we will say that T is a *compactification* of S. A compactification of a bounded subset of $\mathbb{R}^n$ can always be obtained by taking its closure. This does not mean, however, that there are not others. If we are given a space such as $\mathbb{N}$ or $\mathbb{Q}$, it is not at all obvious how one would go about compactifying such spaces. We will show techniques which allow us to achieve this objective.

For what follows, recall that the evaluation map $e : S \to \prod_{i \in J}[a_i, b_i]$ induced by $C^*(S)$ (the set of all continuous bounded real-valued functions on S) is defined as

$$e(x) = < f_i(x) >_{i \in J} \in \prod_{i \in J}[a_i, b_i]$$

where $f_i \in C^*(S)$ and $f_i[S] \subseteq [a_i, b_i]$.

Definition 21.1 Let (S, τ_S) be a topological space and (T, τ_T) be a compact Hausdorff space. We will say that T *is a compactification of* S if S is densely embedded in T.[1]

[1] When we say "compactification of S" we always mean a Hausdorff compactification.

If S is a compact Hausdorff space, then S can be viewed as being a compactification of itself. Recall that a compact Hausdorff space is normal, and so is completely regular. Since subspaces of completely regular spaces are completely regular, then

... only a completely regular space can have a compactification.

In Theorem 14.7, we showed how any completely regular space can be compactified. In the proof of that theorem, we witnessed how an evaluation map, $e : S \to T$, induced by $C^*(S)$ embeds S into a cube,

$$T = \prod_{i\in J}[a_i, b_i]$$

There may be different ways of describing this type of compactification of S. Since each interval $[a_i, b_i]$ is homeomorphic to $[0, 1]$, then there is a homeomorphism, $h : T \to \prod_{i\in J}[0, 1]$, which maps T onto $P = \prod_{i\in J}[0, 1]$. By Tychonoff's theorem, P is guaranteed to be compact. So the function, $q : S \to P$, defined as, $q = h\circ e$, embeds S into $\prod_{i\in J}[0, 1]$. Hence $\text{cl}_P q[S]$ is a compact subspace of the product space, P, which densely contains the homeomorphic image, $q[S]$, of S.[2] So, even common topological spaces such at $\mathbb{R}$, $\mathbb{Q}$, and $\mathbb{N}$ have at least the compactification obtained by the method just described.

21.2 The Stone-Čech compactification.

We have described only one of the various ways to obtain a homeomorphic copy of the compactification, $\text{cl}_T e[S]$, of S. This particular compactification has a special name.

Definition 21.2 Let (S, τ_S) be a completely regular topological space. Let

$$e : S \to \prod_{i\in I}[a_i, b_i]$$

be the evaluation map induced by $C^*(S)$ which embeds S in the product space, $T = \prod_{i\in I}[a_i, b_i]$.

The subspace,

$$\text{cl}_T e[S]$$

is called the *Stone-Čech compactification of S*. The Stone-Čech compactification of S is uniquely (and universally) denoted by, βS.

[2]This is just one small example which shows why Tychonoff theorem deserves to be titled and why it is such an important theorem in topology.

So we see that a non-compact completely regular space, S, always has at least one compactification, namely, it's Stone-Čech compactification, βS. We will soon see that there can be more than one compactification, for the same space.

Equivalent compactifications

Given a completely regular space S, there is nothing stated up to now which would lead us to conclude that S has only one compactification. In fact most spaces we will consider will have many compactifications. Suppose we are given two compactifications for a space, S, say αS and γS. If there is a homeomorphism

$$h : \alpha S \to \gamma S$$

mapping αS onto γS such that $h(x) = x$ for all $x \in S$, then αS and γS will be considered to be *equivalent compactifications* of S. Two compactifications of the same space which have been determined to be "equivalent" in this way will be assumed to be the same compactification, topologically speaking. This equivalence is often expressed by the symbol, $\alpha S \equiv \gamma S$. For convenience, it is sometimes expressed by, $\alpha S = \gamma S$, even though, strictly speaking, αS and γS may not necessarily be equal sets.

The outgrowth of a topological space.

Given a topological space, S, and a compactification, αS, the set

$$\alpha S \backslash S$$

is referred to as being the *outgrowth of* S with respect to this particular compactification or as the *remainder* of αS. Equivalent compactifications will be considered to have the same outgrowth. We will sometimes be interested in determining whether an outgrowth satisfies certain topological properties.

Example 1 : *Compactifications of* $\mathbb{R}$. Construct two compactifications of the space, $\mathbb{R}$, one whose outgrowth contains a single point and another one whose outgrowth contains two points. We will refer to these as a "one-point compactification" and "two-point compactification" of $\mathbb{R}$.

Solution : *A two-point compactification of* $\mathbb{R}$. We know that $\arctan : \mathbb{R} \to (-\pi/2, \pi/2)$ maps $\mathbb{R}$ homeomorphically onto $(-\pi/2, \pi/2)$. Hence $\mathbb{R}$ is embedded in the compact space

$$\gamma\mathbb{R} = \mathrm{cl}_{\mathbb{R}} \arctan[\mathbb{R}] = \mathrm{cl}_{\mathbb{R}}(-\pi/2, \pi/2) = [-\pi/2, \pi/2]$$

with outgrowth $\{-\pi/2, \pi/2\}$

A one-point compactification of $\mathbb{R}$. We know that there is a homeomorphism $h : \mathbb{R} \to (0, 2\pi)$ which maps $\mathbb{R}$ onto $(0, 2\pi)$. The function,

$$h(x) = 2[\arctan(x) + \pi/2]$$

is an example. Define the homeomorphic function $f : (0, 2\pi) \to \mathbb{R}^2$ as

$$f(x) = (\sin(x), \cos(x))$$

Then $f \circ h : \mathbb{R} \to \mathbb{R}^2$ densely embeds $\mathbb{R}$ into the (closed and bounded) compact set

$$\alpha\mathbb{R} = K = \{(\sin(x), \cos(x)) : x \in (0, 2\pi)\} \ \cup \ \{(1, 0)\}$$

with the single point, $\{(1, 0)\}$, in its outgrowth.

Example 2. Let $\mathbb{R}^+ = \{x \in \mathbb{R} : x \geq 0\}$. Is there a two-point compactification of $\mathbb{R}^+$?

Solution : No. Suppose $\alpha\mathbb{R}^+ = \mathbb{R}^+ \cup \{a, b\}$. Then, since $\alpha\mathbb{R}^+$ is Hausdorff, there must be disjoint open neighborhoods B_a and B_b in $\alpha\mathbb{R}^+$ of a and b, respectively. Then $\alpha\mathbb{R}^+ \backslash \mathbb{R}^+ \subseteq B_a \cup B_b$. Since $B_a \cup B_b$ is open in $\alpha\mathbb{R}^+$ containing $\alpha\mathbb{R}^+ \backslash \mathbb{R}^+$, then

$$K = \alpha\mathbb{R} \backslash (B_a \cup B_b)$$

is a compact subset of $\mathbb{R}^+$. Then there exists $k \in \mathbb{R}^+$ such that $K \subseteq [0, k]$. Then $[k, \infty) = [(B_a \cup B_b) \cap \mathbb{R}^+] \backslash [0, k)$. We see that $[k, \infty)$ is a connected subspace of $\mathbb{R}^+$, while $[(B_a \cup B_b) \cap \mathbb{R}^+] \backslash [0, k)$ is not. So we have a contradiction. So $\mathbb{R}^+$ cannot have a two-point compactification.

Example 3. *There is no three-point compactification of* $\mathbb{R}$. Prove this statement.

Solution : Suppose that

$$\alpha\mathbb{R} = \mathbb{R} \cup \{a, b, c\}$$

represents a compactification of $\mathbb{R}$ with outgrowth $\alpha\mathbb{R} \backslash \mathbb{R} = \{a, b, c\}$. Let $\mathbb{R}^+ = \{x \in \mathbb{R} : x \geq 0\}$ and $\mathbb{R}^- = \{x \in \mathbb{R} : x \leq 0\}$. Then

$$\begin{aligned} \alpha\mathbb{R} &= \mathrm{cl}_{\alpha\mathbb{R}}\mathbb{R} \\ &= \mathrm{cl}_{\alpha\mathbb{R}}(\mathbb{R}^+ \cup \mathbb{R}^-) \\ &= \mathrm{cl}_{\alpha\mathbb{R}}\mathbb{R}^+ \cup \mathrm{cl}_{\alpha\mathbb{R}}\mathbb{R}^- \\ &= \mathbb{R}^+ \cup \mathbb{R}^- \cup \{a, b, c\} \end{aligned}$$

By the previous example neither $\mathbb{R}^+$ nor $\mathbb{R}^-$ can have a two-point (nor a three-point) outgrowth. So there is no three-point compactification of $\mathbb{R}$.

We can easily generalize the statement in the previous example to "If the compactification $\alpha\mathbb{R}$ has a finite outgrowth, then $\alpha\mathbb{R} \backslash \mathbb{R}$ must be either a singleton set or a doubleton set."

21.3 A partial ordering of Hausdorff compactifications of a space.

Let's gather together all compactifications of a completely regular space, S, as follows,

$$\mathscr{C} = \{\alpha_i S : i \in I\}$$

We will partially order the family, $\mathscr{C}$, by defining "$\preceq$" in the following way:

> *If $\alpha_i S$ and $\alpha_j S$ belong to $\mathscr{C}$, we will write*
>
> $$\alpha_i S \preceq \alpha_j S$$
>
> *if and only if there is a continuous function $f : \alpha_j S \to \alpha_i S$, mapping $\alpha_j S \backslash S$ onto $\alpha_i S \backslash S$ which fixes the points of S.*

Notation. Suppose two compactifications αS and γS are such that $\alpha S \preceq \gamma S$, then by definition, there is a continuous function $f : \gamma S \to \alpha S$, mapping $\gamma S \backslash S$ onto $\alpha S \backslash S$ which fixes the points of S. We will represent this continuous function f as,

$$\pi_{\gamma \to \alpha} : \gamma S \to \alpha S$$

The function $\pi_{\gamma \to \alpha}$ explicitly expresses which compactification is larger than (or equal to) the other. Note that a pair of compactifications need not necessarily be comparable in the sense that one need not necessarily by "less than" the other.

21.4 On C^*-embedded subsets of a topological space.

Given a subset T of a topological space, S, and a continuous bounded real-valued function $f : T \to \mathbb{R}$, it is not guaranteed that there is a bounded continuous function $g : S \to \mathbb{R}$ such that $g|_T = f$ on T. If there is, then we will say that

"*g is a continuous extension of f from T to S*"

This motivates the following definition.

Definition 21.3 Let (S, τ) be a topological space and U be a proper non-empty subset of S. We say that U is *C^*-embedded* in S if every real-valued bounded function, $f \in C^*(U)$, extends to a function, $g \in C^*(S)$, in the sense that $g|_U = f$.[3] In this case we will say that...

$$g : S \to \mathbb{R} \text{ is a } \textit{continuous extension} \text{ of } f : U \to \mathbb{R} \text{ from } U \text{ to } S$$

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[3]There is an analogous definition for "C-embedded" studied later: We say that U is *C-embedded* in S if every real-valued function, $f \in C(U)$, continuously extends to a function, $f^* \in C(S)$

The notion of "C^*-embedding" is closely related to completely regular spaces and their Stone-Čech compactification. For this reason, we will discuss this property in depth now (even though C^*-embeddings may be discussed in other contexts). The following theorem shows that, for any completely regular space S, S is C^*-embedded in its Stone-Čech compactification, βS. That is,

...every function, $f \in C^*(S)$, extends continuously to a function, $f^\beta \in C(\beta S)$

Theorem 21.4 Let S be a completely regular space. Then S is C^*-embedded in βS.

Proof: If $f \in C^*(S)$, let I_f be the range of f. Let

$$T = \prod_{f\in C^*(S)} \mathrm{cl}_{\mathbb{R}} I_f$$

Then the evaluation map, $e : S \to T$ embeds S in T. Recall that, by definition,

$$\beta S = \mathrm{cl}_T e[S] \subseteq \prod_{f\in C^*(S)} \mathrm{cl}_{\mathbb{R}} I_f$$

Suppose $g \in C^*(S)$.

We are required to show that $g : S \to \mathbb{R}$ extends continuously to some function, $g^\beta : \beta S \to \mathbb{R}$.

If π_g is the g^{th}-projection map, then

$$\pi_g : \prod_{f\in C^*(S)} \mathrm{cl}_{\mathbb{R}} I_f \to \mathrm{cl}_{\mathbb{R}} I_g$$

where $\beta S = \mathrm{cl}_T e[S]$ and so,

$$\pi_g|_{\beta S} : \beta S \to \mathrm{cl}_{\mathbb{R}} I_g$$

maps βS into $\mathrm{cl}_{\mathbb{R}} I_g$. Let $g^\beta = \pi_g|_{\beta S}$. Then $g^\beta[\beta S] = g^\beta[\mathrm{cl}_T e[S]] = \mathrm{cl}_{\mathbb{R}} g[S] \subseteq \mathrm{cl}_{\mathbb{R}} I_g$.

It follows that $g^\beta : \beta S \to \mathrm{cl}_{\mathbb{R}} I_g$ and, since $g[S] \subseteq I_g$, for $x \in S$, $g^\beta|_S(x) = g(x)$.

So g^β is a continuous extension of g from S to βS.

Note that, in the case where S is a compact space, $e[S]$ is a compact space densely embedded in βS and so $\beta S \backslash S = \varnothing$. Then S and βS are homeomorphic.

The above theorem guarantees that every real-valued continuous bounded function, f, on a completely regular space, S, extends to a continuous function $f^\beta : \beta S \to \mathbb{R}$. The function f^β is the *extension of* f from S to βS. Recall (from Theorem 9.8) that continuity guarantees that two continuous functions which agree on a dense subset D of a Hausdorff space, S, must agree on all of S. So there can only be one extension,

f^β, of f from S to βS.

We will soon show (in Theorem 21.9) that, if αS is a compactification of S and S is C^*-embedded in αS, then αS must be the compactification, βS. That is,

...βS is the only compactification in which S is C^*-embedded

A generalization of the extension, $f \to f^\beta$.

The above theorem can be generalized a step further. Suppose $C(S, K)$ denotes all continuous functions mapping S into a compact set K. We show that every function f in $C(S, K)$ extends to a function $f^{\beta(K)} \in C(\beta S, K)$. Note that neither f nor $f^{\beta(K)}$ need be real-valued. The space, K, represents any compact set which contains the image of S under f.

Theorem 21.5 Let S be a completely regular (non-compact) space and $g : S \to K$ be a continuous function mapping S into a compact Hausdorff space, K. Then g extends uniquely to a continuous function, $g^{\beta(K)} : \beta S \to K$.

Proof: We are given that $g : S \to K$ continuously maps the completely regular space, S, into the compact Hausdorff space K. We are required to show that g extends to $g^{\beta(K)} : \beta S \to K$.

Since K is compact Hausdorff it is completely regular; hence there exists a function (the evaluation map) which embeds the compact set K in $V = \prod_{i\in J}[0, 1]$. Since V contains a topological copy of K let us view K as a subset of V.

Since $g : S \to K$, then, for every $x \in S$, $g(x) = <g_i(x)>_{i\in J} \in K \subseteq [0, 1]^J$.

Since S is C^*-embedded in βS, then, for each $i \in J$, $g_i : S \to [0, 1]$ extends to $g_i^\beta : \beta S \to [0, 1]$.

We define the function $g^{\beta(K)} : \beta S \to V$ as

$$g^{\beta(K)}(x) = <g_i(x)>_{i\in J} \in V = \prod_{i\in J}[0, 1]$$

Since g_i^β is continuous on βS, for each i, then $g^{\beta(K)} : \beta S \to \prod_{i\in J}[0, 1]$ is continuous on βS.

See that $g^{\beta(K)}[\beta S] = g^{\beta(K)}[\mathrm{cl}_{\beta S}(S)] = \mathrm{cl}_V(g[S]) \subseteq \mathrm{cl}_V(K) = K$.

Since $g^{\beta(K)}|_S = g$, then $g : S \to K$ continuously extends to $g^{\beta(K)} : \beta S \to K$ on βS. As required.

Given a completely regular topological space S, and $T = \prod_{i\in I}[a_i, b_i]$ we now see that the evaluation function $e_{C^*(S)} : S \to \prod_{i\in I}[a_i, b_i]$ (which homeomorphically embeds a copy of S into T) then continuously extends to βS as follows:

$$\begin{aligned} e^{\beta}_{C^*(S)}[\beta S] &= e^{\beta}_{C^*(S)}[\mathrm{cl}_{\beta S} S] \\ &= \mathrm{cl}_T e_{C^*(S)}[S] \end{aligned}$$

where

$$e^{\beta}_{C^*(S)}(x) = < f^{\beta}(x) >_{f\in C^*(S)}$$

So we can represent the Stone-Čech compactification, βS, of S as being equivalent to

$$e^{\beta}_{C^*(S)}[\beta S]$$

The maximal compactification, βS.

Suppose αS is any compactification of S possibly distinct from βS. We will now show that, in the partially ordered family, $\mathscr{C}$, of all compactifications of S, $\alpha S \preceq \beta S$. Showing this requires that we produce a continuous function $\pi_{\beta\to\alpha} : \beta S \to \alpha S$ such that $\pi_{\beta\to\alpha}(x) = x$, for all $x \in S$. If we can prove this, then we will have shown that βS is the unique maximal compactification of a completely regular space, S.

Theorem 21.6 If αS is a compactification of S, then $\alpha S \preceq \beta S$.

Proof: By Theorem 21.5, the identity map $i : S \to \alpha S$, extends to a continuous function,

$$i^* : \beta S \to \alpha S$$

Then $S \subseteq i^*[\beta S] \subseteq \alpha S$, where $i^*[\beta S]$ is compact, hence closed in αS. Since S is dense in αS, , then the open set $\alpha S \backslash i^*[\beta S]$ must be empty. So the continuous function,

$$i^*[\beta S] = i^*[\mathrm{cl}_{\beta S} S] = \mathrm{cl}_{\alpha S} i^*[S] = \mathrm{cl}_{\alpha S} S = \alpha S$$

maps βS onto αS. So $\alpha S \preceq \beta S$, as required.

Then for any compactification αS, there is the continuous function

$$\pi_{\beta\to\alpha} : \beta S \to \alpha S$$

which maps βS onto αS where $\pi_{\beta\to\alpha}$ fixes the points of S.

More on C^*-embedded subsets of a space S.

We present a miscellany of results which will help us more easily recognize a C^*-embedded subset, T of a space S. We will return to our discussion of compactification immediately following this.

The simplest example of a C^*-embedded subset of $\mathbb{R}$ is a compact subset of $\mathbb{R}$.

Theorem 21.7 If K is a compact subset of $\mathbb{R}$, then K is C^*-embedded in $\mathbb{R}$.

Proof: Let K be a compact subset of $\mathbb{R}$ and $f : K \rightarrow \mathbb{R}$ be a continuous function on K. Since every continuous real-valued function is bounded on a compact subset, then $f \in C^*(K) = C(K)$. Suppose

$$u = \sup\{K\} \text{ and } v = \inf\{K\}$$

Since K is closed and bounded in $\mathbb{R}$, then u and v belong to K. Suppose $g : \mathbb{R} \rightarrow \mathbb{R}$ is a function such that

$$\begin{aligned} g &= f \text{ on } K \\ g(x) &= f(u), \text{ if } x \geq u \\ g(x) &= f(v), \text{ if } x \leq v \end{aligned}$$

It is easily verified that g is a continuous extension of f from K to $\mathbb{R}$. Then K is C^*-embedded in $\mathbb{R}$.

Example 4. The set $\mathbb{N}$ is C^*-embedded in $\mathbb{R}$. One way of visualizing this is to plot the points of $\{(n, f(n)) : n \in \mathbb{N}\}$ of a function $f \in C^*(\mathbb{N})$ in the Cartesian plane $\mathbb{R}^2$ and join every pair of successive points $(n, f(n))$ and $(n+1, f(n+1))$ by a straight line. This results in a continuous curve representing a continuous function g on $\mathbb{R}$ which extends f.

Urysohn's extension theorem.

The following theorem often referred to as *Urysohn's extension theorem* provides an important and useful tool for recognizing C^*-embedded sets.[4]

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[4]The Urysohn's extension theorem should not be confused with the *Urysohn's lemma*. Urysohn's lemma states that "The topological space (S, τ_S) is *normal* if and only if given a pair of disjoint non-empty closed sets, F and W, in S there exists a continuous function $f : S \rightarrow [0, 1]$ such that, $F \subseteq f^{\leftarrow}[\{0\}]$ and $W \subseteq f^{\leftarrow}[\{1\}]$"

Theorem 21.8 *Urysohn's extension theorem:* Let T be a subset of the completely regular space S. Then T is C^*-embedded in S if and only if pairs of sets which can be completely separated by some function in $C^*(T)$ can also be separated by some function in $C^*(S)$.

Proof: ($\Rightarrow$) Suppose T is C^*-embedded in S and U and V are completely separated subsets of T. Then there exists $f \in C^*(T)$ such that $U \subseteq f^{\leftarrow}(0)$ and $V \subseteq f^{\leftarrow}(1)$. Then by hypothesis f extends to $f^* \in C^*(S)$. Then $U \subseteq f^{\leftarrow *}(0)$ and $V \subseteq f^{\leftarrow *}(1)$. So U and V are completely separated in S.

($\Leftarrow$) Suppose that pairs of sets which can be completely separated by some function in $C^*(T)$ can also be separated by some function in $C^*(S)$.

Let f_1 be a function in $C^*(T)$. We are required to show that there exists a function $g \in C^*(S)$ such that $g|_T = f_1$.

Since f_1 is bounded on T, then there exists, $k \in \mathbb{R}$, such that $|f_1(x)| \leq k$ for all $x \in T$. Then $f_1 \leq k = 3r_1 = 3 \cdot \left[\frac{k}{2} \cdot \left(\frac{2}{3}\right)^1\right] = k$ (Where $r_1 = \frac{k}{2} \cdot \left(\frac{2}{3}\right)^1$)

We now inductively define a sequence of functions $\{f_n\} \subseteq C^*(T)$. For $n = 1, 2, 3, \ldots$, there exists $f_n \in C^*(T)$ such that $-3r_n \leq f_n(x) \leq 3r_n$ where,

$$3r_n = 3 \cdot \left[\frac{k}{2} \cdot \left(\frac{2}{3}\right)^n\right] = k \cdot \left(\frac{2}{3}\right)^{n-1} \quad \text{(Where } r_n = \tfrac{k}{2} \cdot \left(\tfrac{2}{3}\right)^n\text{)}$$

For this n, let $U_n = f^{\leftarrow}{}_n[\,[-3r_n, -r_n]\,]$ and $V_n = f^{\leftarrow}{}_n[\,[r_n, 3r_n]\,]$.

We see that U_n and V_n are completely separated in T. [5]

By hypothesis, U_n and V_n are completely separated in S. This means there exists $g_n \in C^*(S)$ such that, $g_n[S] \subseteq [-r_n, r_n]$, $g_n[U_n] = \{-r_n\}$ and $g_n[V_n] = \{r_n\}$ (where $r_n = \left[\frac{k}{2} \cdot \left(\frac{2}{3}\right)^n\right]$). So the sequence, $\{g_n : n = 1, 2, 3, \ldots\}$, thus constructed is well-defined in $C^*(S)$.

We now inductively define the sequence $\{h_n\} \subseteq C^*(T)$ initiating the process with $h_1 = f_1$ and continuing with

$$h_{n+1} = h_n - g_n|_T$$

Then for each n,

$$|h_{n+1}| \leq 2r_1 = 2 \cdot \frac{k}{2} \cdot \left(\frac{2}{3}\right)^n = 3 \cdot \frac{k}{2} \cdot \left(\frac{2}{3}\right)^{n+1} = 3r_{n+1}$$

[5] To see this: the function $h_n = (-r_n \vee f_n) \wedge r_n)$ has $U_n \subseteq Z(h_n - (-r_n))$ and $V_n \subseteq Z(h_n - r_n)$.

So $g_n|_T = h_n - h_{n+1}$. Define $g : S \to \mathbb{R}$ as the series

$$g(x) = \sum_{n \in \mathbb{N} \setminus \{0\}} g_n(x)$$

See that $g(x)$ is continuous on S: Since $|g_n(x)| \leq \frac{k}{2}\left(\frac{2}{3}\right)^n$, and $\sum_{n \in \mathbb{N}\setminus\{0\}} \frac{k}{2}\left(\frac{2}{3}\right)^n$ is a converging geometric series, then $\sum_{n \in \mathbb{N}\setminus\{0\}} g_n(x)$ converges uniformly to $g(x)$. Since each $g_n(x)$ is continuous on S, then $g \in C^*(S)$. So g is a continuous on S.

Also see that, $g|_T = f_1$:

$$\begin{aligned} g|_T(x) &= \lim_{m\to\infty} \sum_{n=1}^{m} g_n|_T(x) \\ &= \lim_{m\to\infty} (h_1(x) - h_2(x)) + (h_2(x) - h_3(x)) + \cdots + (h_m(x) - h_{m+1}(x)) \\ &= \lim_{m\to\infty} h_1(x) - h_{m+1}(x) \\ &= h_1(x) = f_1(x) \quad {\scriptstyle (\text{Since } \lim_{m\to\infty} 3r_{m+1} = 0)} \end{aligned}$$

Then, $g|_T = f_1$ so f_1 extends continuously from T to S. We are done.

Example 5. Use Urysohn's extension lemma to show that any compact subset, K, of a completely regular space, S, is C^*-embedded in S.

Solution: Let U and V be disjoint subsets of the compact set, K, which are completely separated in K. Then there is a function $f \in C(K)$ such that $U \subseteq A = f^{\leftarrow}[\{0\}]$ and $V \subseteq B = f^{\leftarrow}[\{1\}]$. Both A and B are disjoint closed subsets of compact K and so are compact sets. Then A and B are compact in the completely regular space, S. Then A and B are completely separated in S. So U and V are completely separated in S. By Urysohn's extension lemma, K is C^*-embedded in S. We conclude that ...

> ... *any compact subset, K, of a completely regular space, S, is C^*-embedded in S.*

Uniqueness of βS.

We are now able to prove that, up to equivalence, the Stone-Čech compactification of S is the only compactification in which S is C^*-embedded. By this we mean that, if S is C^*-embedded in the compactification, γS, of S, then γS is equivalent to βS. So the symbol, βS, is strictly reserved for the Stone-Čech compactification of S.

Theorem 21.9 The completely regular space S is C^*-embedded in the compactification, γS if and only if $\gamma S \equiv \beta S$.

Proof: We are given that S is completely regular.

($\Leftarrow$) Suppose $\gamma S \equiv \beta S$. Then there is a homeomorphism, $h : \gamma S \to \beta S$, mapping γS onto βS such that $h(x) = x$, for all $x \in S$. We are required to show that S is C^*-embedded in γS.

If $f \in C^*(S)$, then f extends to $f^\beta : \beta S \to \mathbb{R}$. Then $f^\beta \circ h : \gamma S \to \mathbb{R}$. Define $f^\gamma = f^\beta \circ h$. We see that $f^\gamma : \gamma S \to \mathbb{R}$ is the continuous extension of f from S to γS. Then S is C^*-embedded in γS.

($\Rightarrow$) Suppose S is C^*-embedded in γS. We are required to show that γS and βS are equivalent compactifications.

Let $i : S \to S$ be the identity map. Then, by Theorem 21.5, i extends to $i^* : \beta S \to \gamma S$. Also, just as shown in the proof of theorem 21.5, $i^{\leftarrow}$ extends to $i^{\leftarrow\wedge} : \gamma S \to \beta S$. Then $i^{\leftarrow} \circ i$ and $i \circ i^{\leftarrow}$ are both identity maps on S and, since S is dense in both βS and γS, respectively, then $i^* \circ i^{\leftarrow\wedge}$ and $i^{\leftarrow\wedge} \circ i^*$ are identity maps on γS and on βS, respectively. Then both i^* and $i^{\leftarrow\wedge}$ are homeomorphisms. Hence γS and βS are equivalent compactifications.

Example 6. Show that if F is a closed subset of a metric space S, then F is C^*-embedded in S.

Solution: Let F be a closed subset of the metric space S. We will set up the solution so that we can invoke the Urysohn extension theorem.

Let A and B be completely separated in F. Then, by definition, there is a function f in $C^*(F)$ such that $A \subseteq Z(f)$ and $B \subseteq Z(f-1)$. Then $\mathrm{cl}_F A \subseteq Z(f)$ and $\mathrm{cl}_F B \subseteq Z(f-1)$. Since F is closed in S, then so are the disjoint sets $\mathrm{cl}_F A$ and $\mathrm{cl}_F B$. It is shown on page 215 that in metric spaces closed subsets are the same as zero-sets. So $\mathrm{cl}_F A$ and $\mathrm{cl}_F B$ are disjoint zero-sets in S, say $\mathrm{cl}_F A = Z(k)$ and $\mathrm{cl}_F B = Z(g)$ in S. If

$$h = \frac{|k|}{|k| + |g|}$$

on S, $\mathrm{cl}_F A = Z(h)$ and $\mathrm{cl}_F B = Z(h-1)$ in S. So A and B are completely separated in S. By Urysohn's extension lemma every closed subset of a metric space is C^*-embedded.

Because of this, it is useful to remember that,

... any closed subset of $\mathbb{R}$ is C^*-embedded in $\mathbb{R}$

21.5 Associating compactifications to subalgebras of $C^*(S)$.

To each compactification αS we can associate a subset, $C_\alpha(S)$, of $C^*(S)$ defined as

$$C_\alpha(S) = \{f|_S \in C^*(S) : f \in C(\alpha S)\}$$

That is, $f \in C_\alpha(S)$ if and only if f extends to $f^\alpha \in C(\alpha S)$. If αS and γS are two compactifications of S such that $\alpha S \preceq \gamma S$ it is normal to wonder how $C_\alpha(S)$ compares to $C_\gamma(S)$ in $C^*(S)$.

Theorem 21.10 Let αS and γS be two compactifications of S such that $\alpha S \preceq \gamma S$. Let $C_\alpha(S)$ denote the set of all real-valued continuous bounded functions on S that extend to αS. Let $C_\gamma(S)$ denote the set of all real-valued continuous bounded functions on S that extend to γS. Then $C_\alpha(S) \subseteq C_\gamma(S)$.

Proof : We are given that $\alpha S \preceq \gamma S$. Then there is a continuous function $\pi_{\gamma\to\alpha} : \gamma S \to \alpha S$ such that $\pi_{\gamma\to\alpha}(x) = x$ on S.

Suppose $t \in C_\alpha(S)$. It suffices to show that $t \in C_\gamma(S)$. Then there a function $t^\alpha : \alpha S \to \mathbb{R}$ is such that $t^\alpha|_S = t$. Define the function $g : \gamma S \to \mathbb{R}$ as

$$g = t^\alpha \circ \pi_{\gamma\to\alpha}$$

Since $\pi_{\gamma\to\alpha} : \gamma S \to \alpha S$ and $t^\alpha : \alpha S \to \mathbb{R}$ are both continuous, then g is continuous on γS and

$$g|_S(x) = (t \circ \pi_{\gamma\to\alpha})(x) = t(x)$$

So $t = g|_S \in C_\gamma(S)$. Hence

$$\alpha S \preceq \gamma S \quad \Rightarrow \quad C_\alpha(S) \subseteq C_\gamma(S)$$

as required.

Equivalent functions in $C^*(S)$.

In what follows we will refer to pairs of functions in $C^*(S)$ which are "equivalent".

We define a particular subset, $C_\omega(S)$, of $C^*(S)$ as follows:

$$C_\omega(S) = \{f \in C^*(S) : f^\beta \text{ is constant on } \beta S \backslash S\}$$

The subset $C_\omega(S)$ is easily seen to be closed under addition, multiplication, scalar multiplication and contains all constant functions. We say that it is a *subalgebra* of $C^*(S)$.

We will say that two functions f and g in $C^*(S)$ are *equivalent functions in* $C^*(S)$ if

$$f - g \in C_\omega(S)$$

Equivalent functions f and g are sometimes expressed by the notation

$$f \cong g$$

The next theorem shows that there are as many compactifications of S as there are subalgebras of $C^*(S)$ of a certain type.

Suppose (S, τ) is a completely regular space and $\mathscr{F}$ is a *subalgebra* of $C^*(S)$ which contains $C_\omega(S)$ and separates points and closed sets. We can invoke Theorem 7.17 and Theorem 10.16 to obtain a compactification,

$$\alpha S = e^{\beta}_{\mathscr{F}}[\beta S] = \text{cl}_T e_{\mathscr{F}}[S]$$

of S, generated by $\mathscr{F}$. We can associate to the compactification αS, the set $C_\alpha(S) \subseteq C^*(S)$ of real-valued continuous functions. In the following theorem we show conditions that $\mathscr{F}$ must satisfy to guarantee that $\mathscr{F} = C_\alpha(S)$.

Theorem 21.11 Let (S, τ) be a completely regular space and $\mathscr{F}$ be a *subalgebra* of $C^*(S)$ which separates points and closed sets in S. Suppose $\mathscr{F}$ generates the compactification

$$\alpha S = e^{\beta}_{\mathscr{F}}[\beta S] = \text{cl}_T e_{\mathscr{F}}[S]$$

If $\mathscr{F}$ satisfies the properties:

1) $\mathscr{F}$ contains $C_\omega(S)$.
2) every $f \in C_\alpha(S)$ is equivalent to some function $g \in \mathscr{F}$.

then $\mathscr{F} = C_\alpha(S)$.

Proof: We are given that $\mathscr{F}$ separates points and closed sets of S and $C_\omega(S) \subseteq \mathscr{F}$. Then

$$e^{\beta}_{\mathscr{F}}[\beta S] = \text{cl}_T e_{\mathscr{F}}[S] = \alpha S$$

is a compactification of S (generated by $\mathscr{F}$) where $C_\alpha(S) = \{f|_S : f \in C(\alpha S)\}$. We are also given that every function in $C_\alpha(S)$ is equivalent to some function in $\mathscr{F}$.

We are required to show that $\mathscr{F} = C_\alpha(S)$.

Claim #1: $\mathscr{F} \subseteq C_\alpha(S)$. Note that $e_{\mathscr{F}}^{\beta\leftarrow} : \alpha S \to \beta S$. Let $f \in \mathscr{F}$. We define the real-valued function $f^\alpha : \alpha S \to \mathbb{R}$ as

$$f^\alpha = f^\beta \circ e_{\mathscr{F}}^{\beta\leftarrow}$$

We see that $f^\alpha \in C(\alpha S)$ and so is a continuous extension of $f \in \mathscr{F}$. We then conclude that $\mathscr{F} \subseteq C_\alpha(S)$, as claimed.

Claim #2: $C_\alpha(S) \subseteq \mathscr{F}$. Suppose $g \in C_\alpha(S)$. Then, by hypothesis, there is a function $f \in \mathscr{F}$ such that $g - f = h \in C_\omega(S)$. Then $g = f + h$. Since both f and h belong to the subalgebra $\mathscr{F}$, then $g \in \mathscr{F}$. We can then conclude that $C_\alpha(S) \subseteq \mathscr{F}$, as claimed.

Then $C_\alpha(S) = \mathscr{F}$, as required.

The converse is easily seen to be true. That is, if $\mathscr{F} = C_\alpha(S)$, then $\mathscr{F}$ satisfies the two given properties.

The reader can expect to encounter, a bit later, similar theorem statements, one of which is called *The Stone-Weierstrass theorem* in 30.3, the other is Theorem 30.4.[6]

Suppose γS is a compactification of S and $C_\gamma(S) = \{f|_S \in C(S) : f \in C(\gamma S)\}$. That is, $C_\gamma(S)$ is the set of all function, f, in $C^*(S)$ which extend to $f^\gamma : \gamma S \to \mathbb{R}$. We have shown that $\gamma S \preceq \beta S$ and $C_\gamma(S)$ is a subalgebra of $C^*(S)$. We have also seen that there is a continuous map $\pi_{\beta\to\gamma} : \beta S \to \gamma S$ which fixes the points of S.

In the following lemma we show that we can express the function $\pi_{\beta\to\gamma} : \beta S \to \gamma S$ in a form which better describes the mechanism behind the function itself.

Lemma 21.12 Let γS be a compactification of the space S. Let $\mathscr{G} = C_\gamma(S)$. Then

$$\pi_{\beta\to\gamma} = e_{\mathscr{G}}{}^{\gamma\leftarrow} \circ e_{\mathscr{G}}{}^{\beta}$$

where $e_{\mathscr{G}}$ is the evaluation map generated by $\mathscr{G}$.

[6] The Stone-Weierstrass theorem states: "Let S be a compact topological space. Let $\mathscr{F}$ be a complete subring of $C^*(S)$ which contains the constant functions. If $\mathscr{F}$ separates the points of S, then $\mathscr{F} = C^*(S)$".

A consequence of the Stone-Weierstrass statement is the Theorem 30.4 which roughly states that:
"If the set, $C^*(S)$ contains a subring, $\mathscr{F}$, which is complete and contains $C_\omega(S)$, then $\mathscr{F} = C_\alpha(S)$ for some compactification, αS, of S."

Proof: If $f \in C_\gamma(S)$, for $x \in \beta S$, $f^\beta(x) = (f^\gamma \circ \pi_{\beta\to\gamma})(x)$. Then, for $x \in \beta S$,

$$\begin{aligned} e_{\mathscr{G}}{}^\beta(x) &= \; < f^\beta(x) >_{f\in C_\gamma(S)} \\ &= \; < (f^\gamma \circ \pi_{\beta\to\gamma})(x) >_{f\in C_\gamma(S)} \\ &= \; < f^\gamma(\pi_{\beta\to\gamma}(x)) >_{f\in C_\gamma(S)} \\ &= e_{\mathscr{G}}{}^\gamma(\pi_{\beta\to\gamma}(x)) \\ &= (e_{\mathscr{G}}{}^\gamma \circ \pi_{\beta\to\gamma})(x) \end{aligned}$$

Then $e_{\mathscr{G}}{}^\beta = (e_{\mathscr{G}}{}^\gamma \circ \pi_{\beta\to\gamma})$ on βS. Then $\pi_{\beta\to\gamma} = e_{\mathscr{G}}{}^{\gamma\leftarrow} \circ e_{\mathscr{G}}{}^\beta$.

21.6 Limits of z-ultrafilters in βS.

For what follows, recall the definitions of terms related to z-filters in 14.10.

Suppose $\mathscr{Z} = \{Z(f) : f \in M \subseteq C^*(S)\}$ is a free z-ultrafilter in the locally compact Hausdorff space S, where M is the corresponding free maximal ideal in $C^*(S)$. Let

$$\mathscr{Z}^* = \{\mathrm{cl}_{\beta S} Z(f) : f \in M\}$$

denote a family of closures of the elements in $\mathscr{Z}$. Since βS is compact Hausdorff and $\mathscr{Z}$ is a filter, $\mathscr{Z}^*$ satisfies the finite intersection property. Then $\mathscr{Z}^*$ must have non-empty intersection in βS. Then it is fixed and so must have a unique limit point,

$$\{p\} = \cap\{\mathrm{cl}_{\beta S} Z(f) : f \in M\}$$

in $\beta S \backslash S$. We clearly have $\mathrm{cl}_{\beta S} Z(f) \subseteq Z(f^\beta)$ for each $f \in M \subseteq C^*(S)$. Since $f^\beta|_{Z(f)}$ agrees with f^β on $\mathrm{cl}_{\beta S} Z(f)$, then its extension to $Z(f^\beta)$ agrees with f^β on $Z(f^\beta)$. So

$$\mathrm{cl}_{\beta S} Z(f) = Z(f^\beta) \qquad (*)$$

So,

> "*...for any free z-ultrafilter, $\mathscr{Z} = \{Z(f) : f \in M \subseteq C^*(S)\}$ in $Z[S]$, we can write,*
>
> $$\{p\} = \cap\{cl_{\beta S} Z(f) : f \in M\} = \cap\{Z(f^\beta) : p \in Z(f^\beta)\}$$
>
> *where $p \in \beta S \backslash S$. So the points in $\beta S \backslash S$ are precisely the unique limits of a unique free z-ultrafilter in $Z[S]$.*"

Of course, if $\mathscr{Z}$ is a fixed z-ultrafilter in $Z[S]$, then

$$\{p\} = \cap\{Z(f) : f \in M\}$$

for some p in S.

Theorem 21.13 Suppose T is a dense subset of the completely regular space S. Then the following are equivalent:

a) The subset, T, is C^*-embedded in the space S

b) Disjoint zero-sets in T have disjoint closures in S.

c) Every point of S is the limit of a unique z-ultrafilter on T.

Proof: We are given that S is completely regular.

(a $\Rightarrow$ b) Suppose T is a C^*-embedded dense subset of S. Let $h, g \in C^*(T)$ be such that $Z(h)$ and $Z(g)$ are disjoint zero-sets of T. Then $Z(h)$ and $Z(g)$ are completely separated in T.[7]

By Urysohn's extension theorem, $Z(h)$ and $Z(g)$ are completely separated in S. Then there exists $t \in C^*(S)$ such that $Z(h) \subseteq Z(t)$ and $Z(g) \subseteq Z(t-1)$. Then $\text{cl}_S Z(h) \subseteq Z(t)$ and $\text{cl}_S Z(g) \subseteq Z(t-1)$. We can then conclude that $\text{cl}_S Z(h) \cap \text{cl}_S Z(g) = \varnothing$, as required.

(b $\Rightarrow$ c) Suppose that disjoint zero-sets in T have disjoint closures in S. Let Z_1 and Z_2 be disjoint zero-sets in T.

Claim #1: We claim that $\text{cl}_S(Z_1 \cap Z_2) = \text{cl}_S Z_1 \cap \text{cl}_S Z_2$.

Proof of claim: Of course, $LHS \subseteq RHS$ is always true. Suppose, on the other hand, that $x \in \text{cl}_S Z_1 \cap \text{cl}_S Z_2$.

We are required to show that $x \in \text{cl}_S(Z_1 \cap Z_2)$. For any zero-set neighborhood, Z, of x, $Z \cap Z_1$ is dense in $Z \cap \text{cl}_S Z_1$ so

$$x \in \text{cl}_S(Z \cap Z_1) \cap \text{cl}_S(Z \cap Z_2) \neq \varnothing \quad (*)$$

Now both $Z \cap Z_1$ and $Z \cap Z_2$ are zero-sets so, by hypothesis, $(Z \cap Z_1) \cap (Z \cap Z_2)$ cannot be empty. So $Z \cap (Z_1 \cap Z_2) \neq \varnothing$. Since Z is any zero-set neighborhood of x, this means that $x \in \text{cl}_S(Z_1 \cap Z_2)$.

We conclude $\text{cl}_S(Z_1 \cap Z_2) = \text{cl}_S Z_1 \cap \text{cl}_S Z_2$, as claimed.

Claim #2: We claim that each point in $S \backslash T$ is the limit of a unique z-ultrafilter on T.

Proof of claim: Let $y \in T$. Then y belongs to the closure of a zero-set, Z, in T. Hence y is the limit point of a z-ultrafilter, $\mathscr{Z}$, in T. Now, if y is also the limit of another z-ultrafilter, $\mathscr{Z}_1$, in T, then $\mathscr{Z}_1$, will contain a zero-set Z_1 which will not intersect

[7] To see this, note that $Z(|h| + |g|) = \varnothing$ in T and so for the function, $k(x) = |h(x)|/[|h(x)| + g(x)|]$, $Z(h) \subseteq Z(k)$ and $Z(g) \subseteq Z(k-1)$.

some zero-set, Z_2, of $\mathscr{X}$. Then $y \in \mathrm{cl}_S Z_1 \cap \mathrm{cl}_S Z_2 = \mathrm{cl}_S(Z_1 \cap Z_2) = \varnothing$. We can only conclude that every point in $S \backslash T$ is the limit of a unique z-ultrafilter in T. As claimed.

(c $\Rightarrow$ a) We are given that $S \backslash T$ is a set of limits of free z-ultrafilters on T. Since $\beta T \backslash T$ is the set of all limit points of free z-ultrafilters on T, we can say that

$$S \backslash T \subseteq \beta T \backslash T$$

Now, since S is dense in βS and T is dense in S, then T is dense in βS so we can view βS as a compactification, say γT, of T with outgrowth

$$\gamma T \backslash T = (\beta S \backslash S) \cup (S \backslash T)$$

Then for the function $\pi_{\beta \rightarrow \gamma} : \beta T \rightarrow \gamma T$, we have, $\pi_{\beta \rightarrow \gamma}[\beta T \backslash T] = \gamma T \backslash T$ where,

$$\pi_{\beta \rightarrow \gamma}|_S(x) = x$$

Then, for $f \in C^*(T)$ define $f^* : S \rightarrow \mathbb{R}$ as

$$f^*(x) = f^{\beta} \circ \pi_{\beta \rightarrow \gamma}|_S^{\leftarrow}(x)$$

where f^* is seen to be continuous on S and $f^*|_S = f$ on T. So f^* is a continuous extension of f from T to S. We conclude that T is C^*-embedded in S.

21.7 Pseudocompact spaces revisited.

Recall (from definition 17.11) that a topological space is said to be *pseudocompact* if every continuous real-valued function on S is bounded. That is, if $C(S) = C^*(S)$.

Although the pseudocompact property has a simple and easily understood definition, it turns out that, when not compact, such spaces are not easily recognizable. It will be helpful to obtain a few characterizations. In the following theorem, we show that, for locally compact Hausdorff spaces, pseudocompact spaces are precisely those spaces, S, where $\beta S \backslash S$ does not contain a zero-set.

Theorem 21.14 A locally compact Hausdorff space S is pseudocompact if and only if no zero set Z in $Z[\beta S]$ is entirely contained in $\beta S \backslash S$.

Proof: Let S be a locally compact Hausdorff space.

($\Rightarrow$) Suppose S is a pseudocompact space and $Z(f^\beta) \in Z[\beta S]$. We are required to show that $Z(f^\beta) \cap S \neq \varnothing$.

Suppose not. That is, suppose, $Z(h) \subseteq \beta S \backslash S$ (where $h \in C(\beta S)$. Then, since $h|_S$ is not zero in S, we can define the function $g = 1/h|_S$ and so $g \in C(S)$. Let $z \in Z(h)$. Since z belongs to $\mathrm{cl}_{\beta S}S$, then there is a sequence, $\{x_i\}$ in S, which converges to z. By continuity, the corresponding sequence, $\{h(x_i)\}$ in $\mathbb{R}$, must converge to $h(z) = 0$. So g is unbounded on S, which contradicts the hypothesis which states that all real-valued continuous functions on S are bounded. So $Z(h)$ must intersect with S, as required.

($\Leftarrow$) Suppose now that, for any $Z \in Z[\beta S]$, $Z \cap S \neq \varnothing$. We are required to show that S is pseudocompact.

Suppose S is not pseudocompact. Then $C(S)$ contains an unbounded function g. Let

$$f = |g| \wedge k$$

where $k > 0$. Then f is continuous real-valued unbounded on S. Then for each $n \in \mathbb{N}$, there exists $x_n \in S$ such that $f(x_n) \in (n, \infty)$. Then $h = 1/f$ is a continuous well-defined real-valued bounded function on S. Since βS is compact $\{x_n : n \in \mathbb{N}\}$ has a converging subsequence $\{x_{n_i} : i \in \mathbb{N}\}$ with limit, say $x \in \beta S \backslash S$. Since h is continuous and bounded on S, h extends to $h^\beta : \beta S \to \mathbb{R}$ with $h^\beta(x) = 0$. Then $Z(h^\beta) \subseteq \beta S \backslash S$, a contradiction of our hypothesis. Then S is pseudocompact.

It is interesting to note that, in the above theorem, a property of the outgrowth, $\beta S \backslash S$ characterizes a property of the space S.

We point out one easy consequence of the above theorem.

Suppose S is locally compact and completely regular. If S is pseudocompact and $k = f^\beta(x) \in f^\beta[\beta S \backslash S]$, then there exists $y \in Z(f^\beta - k) \cap S$, so $f^\beta(y) = k \in f[S]$. So $f^\beta[\beta S \backslash S] \subseteq f[S]$.

Conversely, suppose $f^\beta[\beta S \setminus S] \subseteq f[S]$. Suppose $Z(f^\beta - t) \cap \beta S \setminus S \neq \varnothing$. Then there exists $y \in \beta S \backslash S$ such that $f^\beta(y) = t$ and $x \in S$ such that $f(x) = t$. Then $x \in Z(f^\beta - t) \cap S \neq \varnothing$. So S is pseudocompact.

> "*The locally compact completely regular space, S, is pseudocompact if and only if, for every $f \in C^*(S)$, $f^\beta[\beta S \backslash S] \subseteq f[S]$.*"

21.8 The one-point compactification.

It was shown in Theorem 18.8, that every locally compact Hausdorff space is completely regular. Hence a locally compact Hausdorff space, S, has at least one compactification, namely, βS. This compactification is maximal when compared to all others in the family, $\mathscr{C}$, of all compactifications. Does the family $\mathscr{C}$ contain a minimal element? That is, does $\mathscr{C}$ have a compactification, γS, such that $\gamma S \preceq \alpha S$, for all compactifications, αS in $\mathscr{C}$? The answer will depend on the space, S. In a previous chapter of the book, we, in fact, provided an answer to this question, as we shall soon see.

In Theorem 18.7, we showed that given a locally compact Hausdorff space (S, τ) and a point $\omega \notin S$, we can construct a larger set,

$$\omega S = S \cup \{\omega\}$$

By first defining, $\mathscr{B}_\omega = \{U \cup \{\omega\} : U \in \tau \text{ and } S \backslash U \text{ is compact }\}$, we then define a topology, τ_ω, on ωS as follows:

$$\tau_\omega = \tau \ \cup \ \mathscr{B}_\omega$$

We then showed that $(\omega S, \tau_\omega)$ is a compact space which densely contains S. Then $(\omega S, \tau_\omega)$ satisfies the definition of a compactification of S. Furthermore, and quite importantly, we show in Theorem 18.7 that ωS is Hausdorff if and only if S is locally compact.

So a non-compact, locally compact Hausdorff space, S, has a compactification, ωS, which may be different from βS. We formally define it.

Definition 21.15 Let (S, τ) be a locally compact Hausdorff topological space, ω be a point not in S and $\omega S = S \cup \{\omega\}$. If

$$\mathscr{B}_\omega = \{U \cup \{\omega\} : U \in \tau \text{ and } S \backslash U \text{ is compact }\}$$

and $\tau_\omega = \tau \ \cup \ \mathscr{B}_\omega$ then $(\omega S, \tau_\omega)$ is called...

...the one-point compactification of S [8]

We will, more succinctly, denote the one-compactification of S by,

$$\omega S = S \cup \{\omega\}$$

---

[8] The one-point compactification of S is also referred to as the *Alexandrov compactification* of S, named after the soviet mathematician, Pavel Alexandrov, (1896-1982).

Since we have chosen the symbol ωS as notation, then, to be consistent with the notation used up to now, we should let

$$C_\omega(S) = \{f|_S : f \in C(\omega S)\}$$

But the symbol, $C_\omega(S)$, has already been used in another sense on page 420 where $C_\omega(S)$ is used to represent the set of all $f \in C^*(S)$ such that f^β is constant on $\beta S \backslash S$. We verify that these are the same set. For $f \in C_\omega(S)$ if and only if,

$$f^\beta[\beta S \backslash S] = f^\beta[\, \pi^{\leftarrow}_{\beta \to \omega}[\{\omega\}]\,] = \{f^\omega(\omega)\} \subset \mathbb{R}$$

Then $C_\omega(S)$ separates points and closed sets of S.[9]

It is also worth emphasizing the fact that

> "...for a space S to have a one-point compactification which is Hausdorff, S must be locally compact".

We can even say more. Amongst all the completely regular spaces, S, the only ones that are open in any compactification are the ones where S is locally compact. We will prove this now.

Theorem 21.16 Let S be a completely regular topological space and αS be any compactification of S. Then S is open in αS if and only if S is locally compact.

Proof : We are given that S is completely regular and αS is a compactification of S.

($\Rightarrow$) Suppose S is open in αS. Since the space S is the intersection of the open set S and the closed set αS, by Theorem 18.3, S is locally compact.

($\Leftarrow$) Suppose S is locally compact in αS. By Theorem 18.3 part d), S is the intersection of an open subset, U, and a closed subset, F. Since $F = \alpha S$ and S is dense in αS, then S is open in αS, as required.

If S is locally compact, then the map, $\pi_{\beta \to \omega} : \beta S \to \omega S$, collapses the set $\beta S \backslash S$ down to the singleton set $\{\omega\}$. More generally, if αS is any compactification of S, then $\pi_{\alpha \to \omega}$ continuously collapses the outgrowth $\alpha S \backslash S$ down to $\{\omega\}$ and fixes the points of S. So for any compactification αS,

$$\omega S \preceq \alpha S \ \text{ and } \ C_\omega(S) \subseteq C_\alpha(S)$$

[9] Had we known this fact before we would has seen that, in the statement of Theorem 21.11, the condition 1) is redundant.

Theorem 21.17 *Uniqueness of ωS.* Let S be a locally compact completely regular topological space. Suppose αS and γS are both compactifications of S which contain only one point in their compact extension. Then they are equivalent compactifications.

Proof: We are given that S is locally compact completely regular. Suppose $\alpha S \backslash S = \{\omega_\alpha\}$ and $\gamma S \backslash S = \{\omega_\gamma\}$, both singleton sets.

Consider the map, $h : \alpha S \to \gamma S$, where $h(x) = x$ on S and $h(\omega_\alpha) = \omega_\gamma$. Then h is one-to-one and onto. It will suffice to show that h maps open neighborhoods to open neighborhoods.

Let U_α be any open neighborhood of a point in αS. If $U_\alpha \subseteq S$, then U_α is open in S. Then $h[U_\alpha] = U_\alpha$. Since S is locally compact it is open in γS and $U_\alpha = U_\alpha \cap S$ is open in γS.

Suppose $\omega_\alpha \in U_\alpha$. Since $\alpha S \backslash U_\alpha$ is compact and $h|_S$ is continuous, then $h[\alpha S \backslash U_\alpha]$ is a compact. Since h is one-to-one, $h[\alpha S \backslash U_\alpha] = h[\alpha S] \backslash h[U_\alpha] = \gamma S \backslash h[U_\alpha]$, a compact set which doesn't contain ω_γ. So $h[U_\alpha]$ is open. Hence $h : \alpha S \to \gamma S$ is a homeomorphism. We can conclude that $\alpha S \equiv \gamma S$.

From this theorem we can conclude that the one-point compactification is unique, up to equivalence. We now consider a few examples involving compactifications of a space.

Example 7. Suppose that S is locally compact and its one-point compactification, ωS, of S is metrizable. Show that S must be second countable.

Solution : Suppose ωS is metrizable. In the proof of theorem 15.8 it is shown that countably compact metric spaces are separable. By Theorem 5.11, a separable metric space is second countable. Since ωS is compact and so is countably compact, then, by combining these two results we obtain that ωS is second countable. By theorem, 5.13, subspaces of second countable spaces are second countable. So S is second countable. As required.[10]

It may happen that βS and ωS are the same compactification. We provide an example where they are not equivalent.

Example 8. Consider the set $S = (0, 1]$ equipped with the usual subspace topology. Determine ωS. Show that the one-point compactification, $\omega S = [0, 1]$, of S is not

[10] The converse of the statement in this example is true. That is, "Locally compact second countable spaces have a metrizable one-point compactification" has been proven. But its proof is fairly involved. So we will not show it here.

equivalent to βS.

Solution : Since [0, 1] is a compact set which densely contains S, and the one-point compactification is unique, then $\omega S = [0,1]$. Note that, since the function $f(x) = \sin\frac{1}{x}$ is a bounded continuous function on S, then it extends to βS. (Plot $f(x) = \sin\frac{1}{x}$.) Verify that f does not extend continuously to $[0,1]$.

Then $[0,1]$ cannot be the Stone-Čech compactification of S.

The following theorem provides an example of a space, S, such that $\beta S = \omega S$.

Theorem 21.18 If S is the ordinal space $[0,\omega_1)$ (where ω_1 is the first uncountable ordinal), then $\beta S \backslash S = \{\omega_1\}$ and so $\beta S = \omega S = [0,\omega_1]$, the one-point compactification of S.

Proof : Given: The space, S, is the ordinal space $[0,\omega_1)$. Then $\omega S = [0,\omega_1]$ is its one-point compactification. So S is completely regular and locally compact.

We are required to show that $\beta S = \omega S = [0,\omega_1]$.

In the example on page 311, it is shown that $S = [0,\omega_1)$ is countably compact but non-compact. In Theorem 15.9 it is shown that, if S is countably compact every function in $C(S)$ is bounded and so has a compact image in $\mathbb{R}$.

Let $f \in C^*(S) = C(S)$. Then

$$f^{\beta}[\beta S] = f^{\beta}[\mathrm{cl}_{\beta S} S] = \mathrm{cl}_{\mathbb{R}} f[S] = f[S] \subseteq \mathbb{R}$$

To show that $\beta S = \omega S$, it suffices to show that $f^{\beta}[\beta S \backslash S]$ is a singleton set in $f[S]$. Suppose q and q^* are two points in $f^{\beta}[\beta S \backslash S] \subseteq f[S] \subseteq \mathbb{R}$.

We claim: That $q = q^*$.

Proof of claim: Express $f[S]$ as a net

$$N = \{f(\alpha) : \alpha \in S = [0,\omega_1)\}$$

Both q and q^* are accumulation points of the net $N = f[S]$. So, for each $n \in \mathbb{N}$, both $B_{1/2n}(q)$ and $B_{1/2n+1}(q^*)$ each contain a cofinal subset of the tail end of the net N. We can then choose, $\{\alpha_n : n \in \mathbb{N}\}$ strictly increasing in S such that $f(\alpha_{2n}) \in B_{1/2n}(q)$ and $f(\alpha_{2n+1}) \in B_{1/2n+1}(q^*)$. If $\sup\{\alpha_n\} = \kappa$, then $\sup\{\alpha_{2n}\} = \kappa = \sup\{\alpha_{2n+1}\}$. Hence

$$\lim_{n\to\infty} f(\alpha_{2n}) = q = f(\kappa) = q^* = \lim_{n\to\infty} f(\alpha_{2n+1})$$

So $q = q^*$ as claimed.

From this we can conclude that, for all $f \in C^*(S)$, f^{β} is constant on $\beta S \backslash S$. Then $\beta S = [0,\omega_1] = \omega S$, the one-point compactification of S.

In the following statement an *n-sphere* , S^n, refers to a subset of $\mathbb{R}^{n+1}$ defined as $\{\vec{x} \in \mathbb{R}^{n+1} : \|\vec{x}\| = 1\}$. For example, S_2 is a subset of $\mathbb{R}^3$ where $(x, y, z) \in S^2$ if and only if $\|(x, y, z)\| = 1$. We will show that, the 2-sphere, S^2 is homeomorphic to $\omega\mathbb{R}^2$.

Theorem 21.19 Let S^n denote the n-sphere in $\mathbb{R}^{n+1}$. Then the "punctured n-sphere", S^n minus a single point, $(S^n \backslash \{p\})$, is homeomorphic to $\mathbb{R}^n$. Then, S^n is homeomorphic to the one-point compactification, $\omega\mathbb{R}^n$, of $\mathbb{R}^n$.

Proof: We are given that S^n is the n-sphere which is a subset of $\mathbb{R}^{n+1}$ and that $p = (0, 0, 0, \ldots, 0, 1)$ is a point in $\mathbb{R}^{n+1}$ which belongs to S^n. We will first show that $S^n \backslash \{p\}$ is homeomorphic to $\mathbb{R}^n$. We achieve this by showing that the function $g : S^n \backslash \{p\} \to \mathbb{R}^n$ defined as

$$g(x_1, x_2, \ldots, x_n, x_{n+1}) = \frac{1}{1 - x_{n+1}}(x_1, x_2, \ldots, x_{n-1}, x_n)$$ [11]

maps $S^n \backslash \{p\}$ homeomorphically onto $\mathbb{R}^n$.

Since $p = (0, 0, \ldots, 1)$, then $1 - x_{n+1} \neq 0$, so g is well-defined.

Let $a = (a_1, a_2, \ldots, a_{n+1})$ and $b = (b_1, b_2, \ldots, b_{n+1})$ be distinct points in $S^n \backslash \{p\}$. Then $a_k \neq b_k$ for at least one $k \in \{1, 2, \ldots, n+1\}$. Then

$$\frac{1}{1 - a_{n+1}}(a_1, a_2, \ldots, a_{n-1}, a_n) \neq \frac{1}{1 - b_{n+1}}(b_1, b_2, \ldots, b_{n-1}, b_n)$$

So g is one-to-one. The function g is also verified to be onto $\mathbb{R}^n$.

To prove continuity of $g : S^n \backslash \{p\} \to \mathbb{R}^n$, it suffices to show that $\pi_i \circ g$ is continuous for each $i \in \{1, \ldots, n\}$, and invoke 7.11. See that

$$\begin{aligned}
(\pi_i \circ g)(x_1, x_2, \ldots, x_n, x_{n+1}) &= \pi_i \left[\frac{1}{1 - x_{n+1}}(x_1, x_2, \ldots, x_{n-1}, x_n) \right] \\
&= \frac{x_i}{1 - x_{n+1}} \\
&= \frac{\pi_i(x_1, x_2, \ldots, x_n)}{1 - x_{n+1}} \\
&= \left(\frac{\pi_i}{1 - x_{n+1}} \right)(x_1, x_2, \ldots, x_n)
\end{aligned}$$

[11] The function, g, is known as a *stereographic projection*

so $\pi_{i}\circ g$ is continuous for each i.

Also, since π is an open map, the function g is an open map.

Then g maps $S^n\backslash\{p\}$ homeomorphically onto $\mathbb{R}^n$, as we claimed earlier.

We now define a function $g^* : S_n \to \omega\mathbb{R}^n$ and show it is a homeomorphism.

Let $g^* : S_n \to \omega\mathbb{R}^n$ be a function mapping the one-point compactification, S^n, of $S^n\backslash\{p\}$ onto the one-point compactification, $\omega\mathbb{R}^n = \mathbb{R}\cup\{\omega\}$, of $\mathbb{R}^n$ which is defined as

$$\begin{aligned} g^*|_{S^n\backslash\{p\}}(x) &= g(x) \\ g^*(p) &= \omega \end{aligned}$$

If U is an open subset of $\omega\mathbb{R}^n$ which doesn't contain ω, then $g^{*\leftarrow}[U]$ is open in $S^n\backslash\{p\}$, and so is open in S^n.

If U is an open subset of $\omega\mathbb{R}^n$ which contains ω then $g^{*\leftarrow}[U] = \{p\}\cup g^{\leftarrow}[U\cap\mathbb{R}^n]$. Now, $\omega\mathbb{R}^n\backslash U$ is compact in $\omega\mathbb{R}^n$ and so is compact in $\mathbb{R}^n$. Then $g^{\leftarrow}[\omega\mathbb{R}^n\backslash U]$ is compact in S^n. So, $g^{*\leftarrow}[U]$ is an open neighborhood of p in S^n. So g^* is continuous on S^n.

Similarly, g^* maps open subsets of S^n to open subsets of $\omega\mathbb{R}^n$.

So g^* is a homeomorphism between the compact sets S^n and $\omega\mathbb{R}^n$.

We also showed along the way that $g^*|_{S_n\backslash\{p\}} = g$ maps the punctured n-sphere, $S^n\backslash\{p\}$, homeomorphically onto $\mathbb{R}^n$.

21.9 Topic: Cardinality of some common Stone-Čech compactifications.

We know the cardinality of the most common sets we encounter (such as $\mathbb{N}$, $\mathbb{Q}$ and $\mathbb{R}^n$). We can sometimes determine the cardinality of associated sets such as their Stone-Čech compactification. We know the cardinality, $|\mathbb{N}|$, of the set $\mathbb{N}$ is $\aleph_0$, while $|\mathbb{R}| = c = 2^{\aleph_0}$. In the following theorem we compute the cardinalities of $\beta\mathbb{N}$, $\beta\mathbb{Q}$ and $\beta\mathbb{R}$. This is good practice in working with these particular compactifications.

Theorem 21.20 The cardinality, $|\beta\mathbb{N}|$, of the set $\beta\mathbb{N}$ is 2^c.

Proof: We are given the compactification, $\beta\mathbb{N}$, of $\mathbb{N}$.

Claim #1. We first claim that $|\beta\mathbb{N}| \geq 2^c$.

Proof of claim #1: In Theorem 7.10, it is shown that, since $[0,1]$ is separable, then the product space, $K = \prod_{i\in\mathbb{R}}[0,1]$, with $|\mathbb{R}| = c$ factors, is also separable. This means that there is a countably infinite set, D, contained in K. There then exist a function

$$g : \mathbb{N} \to D \subset K$$

which indexes the elements of D. Note that g is continuous on $\mathbb{N}$ and densely embeds D in K. By Tychonoff's theorem, K is compact. See that g extends continuously from $\mathbb{N}$ to

$$g^{\beta(K)} : \beta\mathbb{N} \to K$$

Then

$$\begin{aligned} g^{\beta(K)}[\beta\mathbb{N}] &= g^{\beta(K)}[\mathrm{cl}_{\beta\mathbb{N}}\mathbb{N}] \\ &= \mathrm{cl}_K g^{\beta(K)}[\mathbb{N}] \\ &= \mathrm{cl}_K D \\ &= K \end{aligned}$$

Now $|K| = |\prod_{i\in\mathbb{R}}[0,1]| = c^c = 2^c$. (See footnote)[12]. Since $\beta\mathbb{N}$ is the domain of the function $g^{\beta(K)}$ (which could possibly not be one-to-one), then

$$|\beta\mathbb{N}| \geq |K| = 2^c$$

as claimed.

Claim #2. That $|\beta\mathbb{N}| \leq |K| = 2^c$.

Proof of claim #2: We know that each function in $C^*(\mathbb{N})$ can be seen as a sequence $\{x_i : i \in \mathbb{N}\}$ in $\mathbb{R}^{\mathbb{N}}$. Then

$$|C^*(\mathbb{N})| = |\mathbb{R}^{\mathbb{N}}| = c^{\aleph_0} = c \quad {}_{\text{(See footnote)}}{}^2$$

That is, if $I = |C^*(\mathbb{N})|$, then

$$C^*(\mathbb{N}) = \{f_i : \mathbb{N} \to \mathbb{R} : i \in I\}$$

contains $I = c$ distinct functions.

Let $T = \prod_{i\in I}[a_i, b_i] \subseteq \prod_{i\in I}\mathbb{R}$ where, by Tychonoff theorem, T is compact.

Recall that $e_{C^*(\mathbb{N})} : \mathbb{N} \to \prod_{i\in I}\mathbb{R}$ is the evaluation map generated by $C^*(\mathbb{N})$, explicitly defined as

$$e_{C^*(\mathbb{N})}(n) = \langle f_i(n)\rangle_{f_i\in C^*(\mathbb{N})} \in e_{C^*(\mathbb{N})}[\mathbb{N}] \subseteq T \subseteq \prod_{i\in I}\mathbb{R}$$

[12] The proof of $|\prod_{i\in\mathbb{R}}[0,1]| = c^c = 2^c$ is shown in an example of *Section* 24.2 of *Set theory: An introduction to Axiomatic Reasoning*, R. André, in which we compute the cardinality of $\mathbb{R}^{\mathbb{R}}$

[2] The proof of $|\mathbb{R}^{\mathbb{N}}| = c$ is shown in theorem 25.2 of *Set theory: An introduction to Axiomatic Reasoning*, R. André

Then $e_{C^*(\mathbb{N})}[\mathbb{N}] \subseteq T = \prod_{i\in I}[a_i, b_i]$, and $|T| \leq |\mathbb{R}|^I = c^c = 2^c$. So,

$$\begin{array}{rcl} e^{\beta}_{C^*(\mathbb{N})}[\beta\mathbb{N}] & = & e^{\beta}_{C^*(\mathbb{N})}[\mathrm{cl}_{\beta S}\mathbb{N}] \\ & = & \mathrm{cl}_T e_{C^*(\mathbb{N})}[\mathbb{N}] \\ & \subseteq & T \end{array}$$

So $|\beta\mathbb{N}| \leq |T| \leq 2^c$. This establishes our second claim.

Since $|\beta\mathbb{N}| \leq 2^c$ and $|\beta\mathbb{N}| \geq 2^c$, then $|\beta\mathbb{N}| = 2^c$. We are done.

Now, $\mathbb{N}$ contains countably many points and so $|\beta\mathbb{N}\backslash\mathbb{N}| = 2^c$. Since we can associate to each point in $\beta\mathbb{N}\backslash\mathbb{N}$ a unique free z-ultrafilter in $Z[\mathbb{N}]$, the above theorem confirms that there are 2^c free z-ultrafilters in $Z[\mathbb{N}]$.

The cardinality of $\beta\mathbb{N}$ will help us determine the cardinalities of $\beta\mathbb{R}$ and $\beta\mathbb{Q}$.

Theorem 21.21 The sets $\beta\mathbb{N}$, $\beta\mathbb{R}$ and $\beta\mathbb{Q}$ each have a cardinality equal to 2^c.

Proof : We have already shown that $|\beta\mathbb{N}| = 2^c$.

Claim #1 : $|\beta\mathbb{Q}| \leq 2^c$.

Proof of claim #1. We know $\mathbb{N}$ and $\mathbb{Q}$ are countable so both have cardinality $\aleph_0$. Then there exists a one-to-one function, $f : \mathbb{N} \to \beta\mathbb{Q}$, mapping $\mathbb{N}$ onto $\mathbb{Q} \subseteq \beta\mathbb{Q}$. Since $\mathbb{N}$ is discrete f is continuous on $\mathbb{N}$. By Theorem 21.5, $f : \mathbb{N} \to \beta\mathbb{Q}$ extends to

$$f^{\beta(\beta\mathbb{Q})} : \beta\mathbb{N} \to \mathrm{cl}_{\beta\mathbb{Q}} f[\mathbb{N}] = \mathrm{cl}_{\beta\mathbb{Q}}\mathbb{Q} = \beta\mathbb{Q}$$

Then

$$f^{\beta(\beta\mathbb{Q})}[\beta\mathbb{N}] = f^{\beta(\beta\mathbb{Q})}[\mathrm{cl}_{\beta\mathbb{N}}\mathbb{N}] = \mathrm{cl}_{\beta\mathbb{Q}} f[\mathbb{N}] = \mathrm{cl}_{\beta\mathbb{Q}}\mathbb{Q} = \beta\mathbb{Q}$$

So $|\beta\mathbb{Q}| \leq |\beta\mathbb{N}| = 2^c$. This establishes claim #1.

Claim #2 : $|\beta\mathbb{R}| \leq |\beta\mathbb{Q}|$.

Proof of claim #2. Since $\mathbb{Q}$ is dense in $\mathbb{R}$ and $\mathbb{R}$ is dense in $\beta\mathbb{R}$, then it is dense in $\beta\mathbb{R}$; then $\mathrm{cl}_{\beta\mathbb{R}}\mathbb{Q} = \beta\mathbb{R}$. Consider the continuous inclusion function

$$i : \mathbb{Q} \to \mathrm{cl}_{\beta\mathbb{R}}\mathbb{Q} = \beta\mathbb{R}$$

By Theorem 21.5, $i : \mathbb{Q} \to \mathrm{cl}_{\beta\mathbb{R}}\mathbb{Q} = \beta\mathbb{R}$ extends to

$$i^{\beta(\beta\mathbb{R})} : \beta\mathbb{Q} \to \beta\mathbb{R}$$

Then

$$i^{\beta(\beta\mathbb{R})}[\beta\mathbb{Q}] = i^{\beta(\beta\mathbb{R})}[\mathrm{cl}_{\beta\mathbb{Q}}\mathbb{Q}] = \mathrm{cl}_{\beta\mathbb{R}} i[\mathbb{Q}] = \mathrm{cl}_{\beta\mathbb{R}}\mathbb{Q} = \beta\mathbb{R}$$

So $|\beta\mathbb{R}| \leq |\beta\mathbb{Q}|$. This establishes claim #2.

Up to now we have shown that $|\beta\mathbb{R}| \leq |\beta\mathbb{Q}| \leq |\beta\mathbb{N}| = 2^c$.

Claim #3 : $|\beta\mathbb{R}| \geq |\beta\mathbb{N}|$.

Proof of claim #3. We know that $\mathbb{N}$ is C^*-embedded in $\mathbb{R}$. (See example on page 415 or Theorem 21.8.) Then, if $i : \mathbb{N} \to \beta\mathbb{N}$ is the continuous inclusion map, since $\mathbb{N}$ is C^*-embedded in $\mathbb{R}$, $i : \mathbb{N} \to \beta\mathbb{N}$ extends continuously to

$$i^* : \mathbb{R} \to \beta\mathbb{N}$$

Also, $i^* : \mathbb{R} \to \beta\mathbb{N}$ extends continuously to $i^{*\beta} : \beta\mathbb{R} \to \beta\mathbb{N}$. Then

$$\begin{aligned} \beta\mathbb{N} &= \mathrm{cl}_{\beta\mathbb{N}}\mathbb{N} \\ &= \mathrm{cl}_{\beta\mathbb{N}} i[\mathbb{N}] \\ &\subseteq \mathrm{cl}_{\beta\mathbb{N}} i^*[\mathbb{R}] \quad \text{(Since } i \text{ embeds } \mathbb{N} \text{ in } i^*[\mathbb{R}]) \\ &= i^{*\beta}[\mathrm{cl}_{\beta\mathbb{R}}\mathbb{R}] \\ &= i^{*\beta}[\beta\mathbb{R}] \end{aligned}$$

Since $i^{*\beta}[\beta\mathbb{R}]$ contains $\beta\mathbb{N}$, , then $|\beta\mathbb{N}| \leq |\beta\mathbb{R}|$. This establishes claim #3.

Combining the results in the three claims above we conclude that $|\beta\mathbb{R}| = |\beta\mathbb{Q}| = |\beta\mathbb{N}| = 2^c$ as required.

21.10 Compactifying a subset T of $S \subseteq \beta S$.

If T is a non-compact subset of S, it is interesting to reflect on how $\mathrm{cl}_{\beta S}T$ compares with βT. Does it make sense to say that $\beta T \subseteq \beta S$? We examine this question in the following example.

Example 9. Let T be a non-empty subspace of a completely regular space, S. Show that

$$\mathrm{cl}_{\beta S}T \text{ is equivalent to } \beta T$$

Solution: We are given that $T \subseteq S$. Since subspaces of completely regular spaces are completely regular, then T is completely regular. Let $i : T \to \beta S$ be the identity function which embeds T into βS. By Theorem 21.5, $i : T \to \beta S$ extends continuously to $i^{\beta(\beta S)} : \beta T \to \beta S$. Then

$$\begin{aligned}\beta T &= i^{\beta(\beta S)}[\beta T] \\ &= i^{\beta(\beta S)}[\mathrm{cl}_{\beta T} T] \\ &= \mathrm{cl}_{\beta S} i[T] \\ &= \mathrm{cl}_{\beta S} T\end{aligned}$$

So $\mathrm{cl}_{\beta S} T$ is equivalent to βT.

Suppose now that F is a *closed* non-empty subset of the completely regular space S. The statement in the above example will allow us to say something more about F.

To see this, suppose $f \in C^*(S)$. As stated in the example,

$$\mathrm{cl}_{\beta S} F = \beta F$$

So F is C^*-embedded in $\mathrm{cl}_{\beta S} F$. Since F is closed in S, then

$$(\mathrm{cl}_{\beta S} F) \setminus F \subseteq \beta S \setminus S$$

So $f : F \to \mathbb{R}$ extends to $f^\beta : \beta F \to \mathbb{R}$. Since βF is compact in βS, then f^β extends to $f^{\beta^*} : \beta S \to \mathbb{R}$. So

$$f^{\beta^*}|_S : S \to \mathbb{R}$$

is a continuous extension of f on F to $f^{\beta^*}|_S$ on S (via f^{β^*} on βS).

Then F is C^*-embedded in S. We conclude that,

> "*If F is a closed non-empty subset of a completely regular space S, then F is C^*-embedded in S.*"

21.11 The zero-sets of $\beta\mathbb{N}$ are clopen.

Zero-sets in $\beta\mathbb{N}$ play a role in the solution of the following example. If Z is a zero-set in $\mathbb{N}$, then it is clopen in $\mathbb{N}$. It is easily seen that any zero-set in $\mathbb{N}$ is a zero-set of a characteristic function, $g : \mathbb{N} \to \{0, 1\}$, on $\mathbb{N}$. Since g extends to $g^\beta : \beta\mathbb{N} \to \{0, 1\}$, $g^{\beta\leftarrow}(0) = Z(g^\beta) = \mathrm{cl}_{\beta\mathbb{N}} Z$ is a clopen zero-set in $\beta\mathbb{N}$. As well, $g^{\beta\leftarrow}(1) = \beta\mathbb{N} \setminus Z(g^\beta)$ is a clopen zero-set in $\beta\mathbb{N}$. So, for every zero-set $Z(g)$ in $\mathbb{N}$, $Z(g^\beta)$ is clopen in $\beta\mathbb{N}$. Hence zero-sets of $\beta\mathbb{N}$ are clopen in $\beta\mathbb{N}$.

Example 10. The compactification, $\beta\mathbb{N}$, is easily seen to be separable ($\mathbb{N}$ is a dense subset of $\beta\mathbb{N}$.). Show that $\beta\mathbb{N}\setminus\mathbb{N}$ is *not* separable.

Solution: Suppose that $\beta\mathbb{N}\backslash\mathbb{N}$ is separable. Then, $\beta\mathbb{N}\backslash\mathbb{N}$ contains a dense countable subset
$$D = \{x_i : i \in \mathbb{N}\}$$
Since the cardinality of $\beta\mathbb{N}$ is 2^c (shown above), we can fix two points
$$n^* \in (\beta\mathbb{N}\backslash\mathbb{N})\backslash D \text{ and } n \in \mathbb{N}$$
Now, for each $i \in \mathbb{N}$, $\{x_i\}$ and $\{n^*, n\}$ form disjoint closed subsets of $\beta\mathbb{N}$.
Since $\beta\mathbb{N}$ is normal, for each $x_i \in D$, there is a clopen zero-set, $Z_i = Z(g^\beta)$ such that
$$x_i \in Z_i \quad \text{and} \quad \{n^*, n\} \subseteq \beta\mathbb{N}\backslash Z_i$$
Then $\{\beta\mathbb{N}\backslash Z_i : i \in \mathbb{N}\}$ is a family of zero-set clopen neighborhoods of $\{n^*, n\}$.
If $W = \cap\{\beta\mathbb{N}\backslash Z_i : i \in \mathbb{N}\}$, then W is a countable intersection of zero-sets and so is, itself, a zero-set which contains $\{n^*, n\}$ (see page 209) which does not intersect D. Since $n^* \in \beta\mathbb{N}\backslash\mathbb{N} \cap W$, then $W \cap \beta\mathbb{N}\backslash\mathbb{N}$ is a non-empty clopen subset of $\beta\mathbb{N}\backslash\mathbb{N}$ which does not intersect D. Since D is dense in $\beta\mathbb{N}\backslash\mathbb{N}$ we have a contradiction. So $\beta\mathbb{N}\backslash\mathbb{N}$ is not separable.

Since subspaces of separable metrizable spaces were shown to separable (see page 90), then $\beta\mathbb{N}$ cannot be a metrizable space.

Concepts review:

1. Suppose S is a topological space and T is a compact Hausdorff space. What does it mean to say that T is a compactification of S?

2. If S has a compactification, αS, what separation axiom is guaranteed to be satisfied by S?

3. Given a completely regular space S let $e : S \to \pi_{i\in I}[a_i, b_i]$ be the evaluation map on S induced by $C^*(S)$. Give a definition of the Stone-Čech compactification of S which involves this evaluation map.

4. What does it mean to say that the two compactifications of S, αS and γS, are equivalent compactifications?

5. If $\mathscr{C} = \{\alpha_i S : i \in I\}$ denotes the family of all compactifications of S. Define a partial ordering of $\mathscr{C}$.

6. If U is a subset of the topological space S, what does it mean to say that U is C^*-embedded in S?

7. If S is C^*-embedded in the compactification, αS, of S what can we say about αS?

8. Suppose S is completely regular and $g : S \to K$ is a continuous function mapping S into a compact Hausdorff space K. For which compactifications, αS, does the following statement hold true: "the function g extends to a continuous function $g^* : \alpha S \to K$"?

9. Suppose S is locally compact and Hausdorff. Define the one-point compactification, ωS, of S.

10. What can we say about those subspaces of a compactification, αS, which are locally compact? What can we say about those subspaces of a compactification, αS, which are open in αS?

11. What is the Stone-Čech compactification of the ordinal space $[0, \omega_1)$?

12. State a characterization of the pseudocompact property stated in this chapter.