Point-set topology with topics

Basic general topology for graduate studies (Edition 2024)
by Robert André

For December 2024 revised version of the book see

`https://www.math.uwaterloo.ca/~randre/sets/newbook230825.pdf`

Errata page

In section:

5.8 Topic : Hereditary topological properties.

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Definition 5.12 A topological property, say P, of a space (S, τ_S) is said to be a *hereditary topological property* provided every subspace, (T, τ_T), of S also has P.

Example 15. Show that metrizability is a hereditary property.

Solution: Suppose (S, τ) is metrizable. Then there exists a metric ρ such that (S, τ) and (S, ρ) have the same open sets. Suppose $T \subseteq S$ has the subspace topology and $\rho_t : T \times T \to \mathbb{R}$ is the subspace metric on T. Then (T, τ_t) and (T, ρ_t) have the same open sets and so T is metrizable. So subspaces of metrizable spaces are metrizable.

"First countable" is another example of a hereditary topological property.

However, the reader may want to verify that, if S is separable and V is an *open* subspace of S, then V is separable. But, in general, separability is not a hereditary property. (An example supporting this fact is found later in the book on page **??**.[9])

We now show that the "second countable" property is hereditary.

Theorem 5.13 Suppose (S, τ_S) is a second countable topological space. Then any non-empty subspace of S is also second countable. So "second countable" is a hereditary property.

Proof: Suppose (S, τ_S) has a countable base $\mathscr{B} = \{B_i : i \in \mathbb{N}\}$. Suppose (T, τ) is a non-empty subspace of S. Let U be an open subset of T. Then there exists an open subset U^* of S such that $U = U^* \cap T$. Then there exists $N \subseteq \mathbb{N}$ such that $U^* = \cup\{B_i : i \in N\}$. Then $U = \cup\{B_i \cap T : i \in N\}$. So $\mathscr{B}_T = \{B_i \cap T : i \in \mathbb{N}\}$ is a countable basis of T. Hence T inherits the second countable property from its superset S.

"Separable metrizable" is hereditary.

It is immediately worth noting that the above results allow us to conclude that *subspaces of separable metrizable spaces are separable*. That is, the "separable metrizable" property is hereditary. To see this, simply note that if T is a subspace of the a separable metrizable space S then T is metrizable (by the above example). We claim that T is separable: From theorem 5.11 , the metrizable space, S, must be second countable. By theorem 5.13, the second countable property is hereditary. So T must be both metrizable and second countable. Since second countable spaces are separable (by theorem 5.10, then T is a both metrizable and separable.

So, for example, should one want to argue that the irrationals, $\mathbb{J}$ (equipped with the usual topology), forms a separable space, it suffices to justify that $\mathbb{R}$ is both metrizable and second countable.

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[9]Where it is shown that $\beta\mathbb{N}\backslash\mathbb{N}$ is not separable even though $\beta\mathbb{N}$ is known to be separable.

In section:

6.8 Topic: The Pasting lemma and a generalization.

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Note that, in the above theorem, if the members of the collection $\mathscr{F}$ are all *open subsets* (rather than all closed as hypothesized in the theorem) the family $\mathscr{F}$ need not be locally finite for the statement to hold true. That is...

> *Let S and T be topological spaces and $\{O_i : i \in I\}$ be a collection of open subsets of S which covers all of S. Let $f : S \to T$ be a function. Then, $f : S \to T$ is a continuous function on S if and only if the restriction, $f|_{O_i}$, of f to O_i is continuous for each $i \in I$.*

It is left as an easy exercise for the reader to verify that this holds true.

Corollary 6.19 . . .

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In section:

13.6 Other properties of filters.

Remove Theorem 13.11

In section:

14.3 Properties of compact subsets.

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Example 6. Suppose S and T are topological spaces. We know that projection maps on product spaces are open maps. (See theorem on page 124)

Show that, in the case where the space T is compact then the projection map, $\pi_1 : S \times T \to S$, is a closed map.

Solution : Let K be a closed subset of $S \times T$. We are required to show that $\pi_1[K]$ is closed. It then suffices to show that $S \setminus \pi_1[K]$ is open in S.

Let $u \in S\setminus\pi_1[K]$. Then $(\{u\} \times T) \cap K = \varnothing$. Since T is compact then we easily see that $\{u\} \times T$ is compact. Since K is closed in $S \times T$, for each $(u, x) \in \{u\} \times T$ there is an open neighborhood, $V_u^x \times U_x$, which does not meet K. In this way we obtain an open cover

$$\{V_u^x \times U_x : x \in T\}$$

of $\{u\} \times T$ which then has a finite subcover

$$\{V_u^{x_i} \times U_{x_i} : x_i \in F \subseteq T\}$$

Then $\cap\{V_u^{x_i} \times U_{x_i} : x_i \in F \subseteq T\}$ forms an open neighborhood of u which does not intersect $\pi_1[K]$. So $S\setminus\pi_1[K]$ is open in S, as claimed. We conclude that $\pi_1 : S \times T \to S$ is a closed projection map.

In section:

17.2 Example of a compact space which is not sequentially compact.

Example 1. Let $S = [0, 1]^{[0,1]}$ be equipped with the *product topology*. That is, we view S as $\prod_{i\in[0,1]}[0, 1]_i$.

(a) Show that S is compact, hence countably compact.

(b) Show that, in spite of its compactness, S is not sequentially compact.

Solution : We are given that the space $S = [0, 1]^{[0,1]}$ viewed as $\prod_{i\in[0,1]}[0, 1]_i$ is equipped with the product topology.

(a) Since $[0,1]$ is compact and given the fact that any product space of compact sets is compact (by Tychonoff theorem), then $S = \prod_{i\in[0,1]}[0,1]_i$ is compact. Since any compact set is countably compact, then S is also countably compact.

(b) We are given that $S = [0,1]^{[0,1]}$ is equipped with the product topology. That is, we view S as $\prod_{j\in[0,1]}[0,1]_j$, a compact set. We will construct a sequence in S which has no converging subsequence.

Suppose each element, x, of $[0,1]$ is expressed in its binary expansion form, $[0,1]_2$. For $n \in \mathbb{N}\backslash\{0\}$ we will define the function $f_n : [0,1]_2 \to \{0,1\}$ as,

$$f_n(x) = \text{ “the } n^{th} \text{ digit in the binary expansion of } x\text{”}$$

to form a sequence $\{f_1, f_2, f_3, f_4, \ldots\}$ each mapping $[0,1]$ into $\{0,1\}$.
For example, given a particular value of $x = 0.1011101010\ldots \in [0,1]_2$

$$f_1(x) = 1,\; f_2(x) = 0,\; f_3(x) = 1,\; f_4(x) = 1\ldots$$

with which we form the ordered sequence, $\{f_1(x), f_2(x), f_3(x), f_4(x), \ldots\} = \{1,0,1,1,\ldots\}$.

Then

$$T \;=\; \{f_n : n = 1,2,3,\ldots\}$$

is a sequence of functions each mapping $[0,1]$ into $\{0,1\}$. So $T \subset \prod_{i\in[0,1]_2}\{0,1\}$.

We will show that T cannot have a convergent subsequence.

For suppose $\{f_{n_k} : k = 1,2,3,\ldots\}$ is a subsequence of T which converges to the function $f \in \prod_{i\in[0,1]_2}\{0,1\}$. Then, for every $x \in [0,1]_2$, $\{f_{n_k}(x) : k = 1,2,3,\ldots\}$ must converge to $f(x)$.

We will choose $q \in [0,1]_2$ so that $f_{n_{2k}}(q) = 0$ and $f_{n_{2k-1}}(q) = 1$.

But the subsequence,

$$\{f_{n_1}(q), f_{n_2}(q), f_{n_3}(q), \ldots,\} = \{1,0,1,0,1,\ldots\}$$

clearly, does not converge (when it should converge to $f(q)$).

So the sequence of functions T in the compact space $\prod_{i\in[0,1]_2}\{0,1\}$ does not have a convergent subsequence.

So S cannot be sequentially compact.

In section:

22.5 More on equivalent singular compactifications.

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Add Examples 9 and 10.

In the following example we see how we can use a singular function to construct, from a rectangle, a cylindrical shell.

Example 9. Consider the non-compact subspace, $S = [0, 2\pi] \times (0, 2\pi)$ of $\mathbb{R}^2$ equipped with the usual topology. Consider the function $f : S \to [0, 2\pi]$ defined as $f[\{x\} \times (0, 2\pi)] = \{x\}$. Verify that the function f is continuous and that the singular set, $S(f)$, of f is the closed interval, $[0, 2\pi]$. Then verify that $f[S] \subseteq [0, 2\pi]$ so that $S \cup_f S(f)$ is a singular compactification of S. Also verify that the singular compactification of S, induced by f, is (topologically speaking) a closed and bounded cylindrical shell with radius 1.

Solution : For $x \in [0, 2\pi]$, $f^{\leftarrow}[B_\varepsilon(x)] = B_\varepsilon(x) \times [0, 2\pi]$ is open in S and so f is continuous on S. See that, for all $x \in [0, 2\pi]$, $\mathrm{cl}_S f^{\leftarrow}[B_\varepsilon(x)]$ is not compact in S so $S(f) = [0, 2\pi]$. Since $f[S]$ is a (proper) subset of $S(f) = [0, 2\pi]$ then f is a singular map and so $S \cup_f [0, 2\pi]$ is a singular compactification of S.

Then what geometric representation can we provide for $S \cup_f [0, 2\pi]$? Well, let's consider the point x in $S(f)$ viewed as an element of the compactification $S \cup_f S(f)$. An open neighborhood base of x in $S \cup_f S(f)$ would look something like this

$$\{\ B_\varepsilon(x)\ \cup\ [\ f^{\leftarrow}[B_\varepsilon(x)] \setminus [0, 2\pi] \times [\delta, 2\pi - \delta]\ ] : \varepsilon, \delta > 0\ \}$$

where $[0, 2\pi] \times [\delta, 2\pi - \delta]$ is seen to be a compact subset of S. So $S(f)$ appears to be the edge which provides the material necessary to seal together the bottom and the top edges of the rectangle $[0, 2\pi] \times (0, 2\pi)$ to form a cylindrical shell.

Example 10. Consider the non-compact subspace, $S = [0, 2\pi] \times (0, 2\pi)$ of $\mathbb{R}^2$ equipped with the usual topology. Consider the function $g : S \to [0, 2\pi]$ defined as

$$g[\{x\} \times (0, 2\pi)] = \{2\pi - x\}$$

Verify that that $S \cup_g S(g)$ (where $S(g) = [0, 2\pi]$) is a singular compactification.
Also, if $f : S \to [0, 2\pi]$ is the function as defined in the previous example, show that $S \cup_f S(f)$ and $S \cup_g S(g)$ are not equivalent compactifications (in spite of the fact that

$S(f) = S(g) = [0, 2\pi]$).

Solution : The proof that $S \cup_g S(g)$ is a singular compactification of S mimics the proof which appears in the previous example for $S \cup_f S(f)$.

Suppose $\gamma S = S \cup_f S(f)$ and $\alpha S = S \cup_g S(g)$. We verify that γS and αS are not equivalent compactifications.

Suppose $\gamma S \equiv \alpha S$. Then there exists a homeomorphism $\pi_{\gamma\to\alpha} : \gamma S \to \alpha S$ which fixes the points of S. Then $\pi_{\gamma\to\alpha}|_{\gamma S\setminus S}$ maps $[0, 2\pi]$ homeomorphically onto $[0, 2\pi]$ and so is monotone. Suppose without loss of generality that it is increasing and so maps 0 to 0 and 2π to 2π.

Consider the open ball $B = B_\varepsilon(2\pi)$ in $[0, 2\pi]$. Let

$$D = \pi_{\gamma\to\alpha}[B]$$

an open subset of $S(g)$. Then

$$\begin{aligned} f^{\leftarrow}[B] &= (2\pi - \varepsilon, 2\pi] \times (0, 2\pi) \\ g^{\leftarrow}[D] &= [0, \delta) \times (0, 2\pi) \end{aligned}$$

We can of course choose ε so that $f^{\leftarrow}[B] \cap g^{\leftarrow}[D] = \varnothing$.

Recall that $g : S \to S(g)$ extends to $g^\alpha : \alpha S \to S(g)$ such that g^α fixes the points of $S(g)$, so we have

$$\begin{aligned} g^{\alpha\leftarrow}[D] &= D \cup g^{\leftarrow}[D] \quad \text{(An open subset of } \alpha S) \\ \pi_{\gamma\to\alpha}[B \cup f^{\leftarrow}[B]] &= D \cup f^{\leftarrow}[B] \quad \text{(An open subset of } \alpha S) \end{aligned}$$

Then

$$\begin{aligned} (D \cup g^{\leftarrow}[D]) \cap (D \cup f^{\leftarrow}[B]) &= D \cup (g^{\leftarrow}[D]) \cap f^{\leftarrow}[B]) \\ &= D \cup \varnothing \\ &= D \end{aligned}$$

so D is an open subset of αS which is contained in $\alpha S \setminus S$. A contradiction! So $\gamma S \not\equiv \alpha S$. [5]

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[5]See that the compactification $S \cup_g S(g)$ constructed in this way is a Möbius strip.

In section:

24.1 Realcompact space: Definitions and characterizations.

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Definition 24.1: Let S be a completely regular space.

a) If $f \in C(S)$ . . .

All through subsection 24.1 replace the expression "locally compact and Hausdorff" with "completely regular".

In section:

24.3 The *Hewitt-Nachbin realcompactification* of a space S.

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Add:

Theorem 24.7 Suppose S is completely regular. The space S is realcompact if and only if S is homeomorphic to a closed subspace of a power of $\mathbb{R}$.

Proof: We are given that S is completely regular. Then S is densely contained into its realcompactification υS. Then every (real-valued) function f in $C(S)$ extends continuously to (a real-valued function) $f^{\upsilon} \in C(\upsilon S)$. Let $\mathscr{F} = C(S)$. Then the evaluation map $e_{\mathscr{F}} : S \to \prod_{f\in\mathscr{F}} \mathbb{R}_f$ extends continuously to the evaluation map on υS defined as

$$e^{\upsilon}_{\mathscr{F}}(x) =< f^{\upsilon}(x) >_{f\in\mathscr{F}} \in \prod_{f\in\mathscr{F}} \mathbb{R}_f$$

Since υS is completely regular (given that $\upsilon S \subseteq \beta S$) then $C(\upsilon S)$ separates points and closed sets of υS. Then $e^{\upsilon}_{\mathscr{F}}$ maps υS homeomorphically onto a subset of $\prod_{f\in\mathscr{F}} \mathbb{R}_f$

(as argued in the embedding theorem I in 7.17).

Claim: We claim that $e^{\upsilon}_{\mathscr{F}}[\upsilon S]$ is a closed subset of $\prod_{f\in\mathscr{F}}\mathbb{R}_f$.

Proof of claim: Recall that the evaluation map,

$$e^{f(\omega)}_{C(\beta S,\omega\mathbb{R})}:\beta S\rightarrow\prod_{f\in C(\beta S,\omega\mathbb{R})}\omega\mathbb{R}_f$$

maps βS continuously onto a *compact* subset, say K, of $\prod_{f\in C(\beta S,\omega\mathbb{R})}\omega\mathbb{R}_f$ with the function defined as

$$e^{\beta(\omega)}_{C(\beta S,\omega\mathbb{R})}(x) \;=<f^{\beta(\omega)}(x)>_{f\in C(\beta S,\omega\mathbb{R})}\in K$$

Let $\mathscr{G}=\{\pi_f:f\in\mathscr{F}\}$ where $\pi_f:K\rightarrow\mathbb{R}_f$ is defined as

$$\pi_g[<f^{\upsilon}(x)>_{f\in\mathscr{F}}]=g^{\upsilon}(x)\in\mathbb{R}_g$$

See that $\mathscr{G}$ separates points and closed sets of K (check!) and that the function $g:K\rightarrow\prod_{f\in\mathbb{R}_f}$ defined as,

$$g(x)=e_{\mathscr{G}}(x)\;=<\pi_f(x)>_{\pi_f\in\mathscr{G}}=<f^{\upsilon}(x)>_{f\in\mathscr{F}}=\;e^{\upsilon}_{\mathscr{F}}(x)\in\prod_{f\in\mathscr{F}}\mathbb{R}_f$$

so $g\circ e^{\beta(\omega)}_{C(\beta S,\omega\mathbb{R})}[\beta S]=g[K]=e^{\upsilon}_{\mathscr{F}}[\upsilon S]$ is closed in $\prod_{f\in\mathscr{F}}\mathbb{R}_f$, as claimed.

Let $A=\prod_{f\in\mathscr{F}}\mathbb{R}$. We then have

$$\begin{aligned}e_{\mathscr{F}}[S] &\subseteq \mathrm{cl}_A e_{\mathscr{F}}[S]\\ &\subseteq e^{\upsilon}_{\mathscr{F}}[\upsilon S] \quad \text{(Since } e^{\upsilon}_{\mathscr{F}}[\upsilon S] \text{ is closed in } A)\\ &= e^{\upsilon}_{\mathscr{F}}[\mathrm{cl}_{\upsilon S}S]\\ &\subseteq \mathrm{cl}_A e_{\mathscr{F}}[S] \quad \text{(By continuity of } e^{\upsilon}_{\mathscr{F}}.)\end{aligned}$$

So

$$e^{\upsilon}_{\mathscr{F}}[\upsilon S]=\mathrm{cl}_A e_{\mathscr{F}}[S]$$

We conclude,

$$\begin{aligned}S=\upsilon S &\Leftrightarrow e^{\upsilon}_{\mathscr{F}}[\upsilon S]=\mathrm{cl}_A e_{\mathscr{F}}[S]=e_{\mathscr{F}}[S]\\ &\Leftrightarrow e_{\mathscr{F}}[S]=\mathrm{cl}_A e_{\mathscr{F}}[S]\subseteq A\end{aligned}$$

We conclude that S is realcompact if and only if the homeomorphism, $e_{\mathscr{F}}$, maps S onto a closed subset of a power of $\mathbb{R}$.

In section:

29.2 A base for a uniformity.

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Example 3. Consider the set of real numbers $\mathbb{R}$. For $\kappa > 0$, let

$$B_\kappa = \{(a, b) \in \mathbb{R} \times \mathbb{R} : |a - b| < \kappa\}$$

Verify that the collection,

$$\mathscr{B} = \{B_\kappa : \kappa > 0\}$$

forms a base for some uniformity on $\mathbb{R}$.

Solution : We verify that $\mathscr{B}$ satisfies the four base properties for a uniformity.

U1. Since $(x, x) \in B_\kappa$ for all κ, then $\mathscr{B}$ satisfies U1.

U2*. For κ_1 and κ_2 larger than 0, let $\kappa_3 = \min\{\kappa_1, \kappa_2\}$. If $(x, y) \in B_{\kappa_3}$, then (x, y) are strictly within a distance of κ_3 from each other. So $(x, y) \in B_{\kappa_1} \cap B_{\kappa_2}$. Then $B_{\kappa_3} \subseteq B_{\kappa_1} \cap B_{\kappa_2}$. It follows that $\mathscr{B}$ satisfies U2*.

U3. Let $B_\kappa \in \mathscr{B}$. We claim there exists λ such that $B_\lambda \circ B_\lambda \subseteq B_\kappa$.

Let $\lambda = \kappa/4$. Recall that

$$U \circ V = \{(u, v) : (u, y) \in V \text{ and } (y, v) \in U \quad \text{for some } y \in \text{im } V.\}$$

Let $(u, v) \in B_{\kappa/4} \circ B_{\kappa/4}$. We claim that $|u - v| < \kappa$.

See that $(u, v) \in B_{\kappa/4} \circ B_{\kappa/4}$ implies that there exists $z \in \text{ im } B_{\kappa/4}$ such that $(u, z) \in B_{\kappa/4}$ and $(z, v) \in B_{\kappa/4}$.

Then $|u - z| < \kappa/4$ and $|z - v| \in \kappa/4$. We have,

$$\begin{aligned} |u - v| &\leq |u - z| + |z - v| \\ &< \kappa/4 + \kappa/4 \\ &= \kappa/2 < \kappa \end{aligned}$$

Then $(u, v) \in B_\kappa$. So $B_{\kappa/4} \circ B_{\kappa/4} \subseteq B_\kappa$.

We conclude that $\mathscr{B}$ satisfies U3.

U4. Let $B_\kappa \in \mathscr{B}$. Since $B_\kappa = \{(x,y) : |x-y| < \kappa\} = \{(x,y) : |y-x| < \kappa\} = B_\kappa^{-1}$, then $\mathscr{B}$ satisfies U4.

Then

$$\mathscr{B} = \{B_\kappa : \kappa > 0\}$$

is a base which generates a uniformity on $\mathbb{R}$. Then the uniformity on $\mathbb{R}$ is

$$\mathscr{U} = \{V \in \mathscr{P}(\mathbb{R} \times \mathbb{R}) : B_\kappa \subseteq V \text{ for some } \kappa > 0\}$$

In section:

29.4 The uniform topology, $\tau_{\mathscr{U}}$, generated by a uniformity, $\mathscr{U}$, on a set.

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Theorem 29.8 Let S be a non-empty set, $\mathscr{U}$ be a uniformity on S and $\mathscr{B}$ be a base for $\mathscr{U}$. For $x \in S$, and B in $\mathscr{U}$, let $B(x)$ denote the image of $\{x\}$ (in S) under the relation, B. Let

$$\tau_{\mathscr{U}} = \{U \in \mathscr{P}(S) : \text{ for each } x \in U \text{ there exists } B \in \mathscr{U} \text{ such that } B(x) \subseteq U\}$$

Then $\tau_{\mathscr{U}}$ is a topology on S.

Add Theorem 29.10:

Theorem 29.10: Let S be a non-empty set and $\mathscr{U}$ be a uniformity on S. Let $\tau_{\mathscr{U}}$ be the topology on S generated by the uniformity, $\mathscr{U}$. If $B \in \mathscr{U}$ and $x \in S$, let $B(x)$ denote the image of $\{x\}$ under B. Let T be a subset of S and $\text{int}_S T$ be the non-empty interior of T with respect to the uniform topology, $\tau_{\mathscr{U}}$, on S. Then $x \in \text{int}_S T$ if and only if there is some $B \in \mathscr{U}$ such that $B(x) \subseteq T$. Hence

$$\mathscr{B}(x) = \{B(x) : B \in \mathscr{U}\}$$

forms a base for the neighborhood system of x.

Proof: Let

$$M = \{x \in S : B(x) \subseteq T\}$$

We claim that M is open with respect to the uniform topology. By theorem 29.8, to show this, it suffices to verify that, for every x in M, there is a $B \in \mathscr{U}$ such that $B(x) \subseteq M$.

Proof of claim: Suppose $x \in M$. This means there is some $B \in \mathscr{U}$ such that $x \in B(x) \subseteq T$. By U3, there exists $V \in \mathscr{U}$ such that $V \circ V \subseteq B$. Suppose $z \in V(x)$. Then

$$V(z) \subseteq V \circ V(x) \subseteq B(x) \subseteq T$$

Since $V(z) \subseteq T$ then $z \in M$. So every element of $V(x)$ belongs to M. We conclude that $V(x) \subseteq M$. We have shown that for any $x \in M$ there is a $V \in \mathscr{U}$ such that $V(x) \subseteq T$. By theorem 29.8, M is open with respect to the uniform topology. This establishes the claim.

Since M is the largest open subset of S which is contained in T then $M = \text{int}_S T$.

We can then conclude that for any neighborhood T of x in S there is a $B \in \mathscr{U}$ such that $x \in B(x) \subseteq T$. So for each $x \in S$, $\mathscr{B}(x) = \{B(x) : B \in \mathscr{U}\}$ forms a base for a neighbourhood system of x.

. . .

At the end of Example 6:

See that,

$$\begin{aligned} x &\in (x - \varepsilon/4, x + \varepsilon/4) \\ &= \pi_2 \left[(\{x\} \times S) \cap B_{\varepsilon/2} \right] \\ &= (B_{\varepsilon/2})(x) \\ &\subseteq (x - \varepsilon, x + \varepsilon) \\ &\subseteq A \end{aligned}$$

Then $x \in (B_{\varepsilon/2})(x) \subseteq (x - \varepsilon, x + \varepsilon) \subseteq A$.

We conclude that $\tau \subseteq \tau_{\mathscr{U}}$, as claimed.

So $\tau_{\mathscr{U}} = \tau$.
