

Preface

This text was prepared to serve as an introduction to the study of general topology. Most students in mathematics are required, at some point in their study, to have knowledge of the fundamentals of general topology and master topological techniques that may be useful in their area of specialization. “Trust us, not only will you be glad to be skilled at using these tools, but a lot of concepts you study in your field of interest will eventually be seen as specific cases investigated in a more general context in general topology” they are told. It sometimes occurs that some students develop a particular fascination for general topology in spite of it occasionally being described as being “no longer in fashion anymore”. Recall that mathematicians such as George Cantor and Felix Hausdorff, were also told that some of the mathematics they spent time investigating was “not in fashion”. Reasons for the continued study of this topic often go beyond the simple perception that it is a practical or useful tool. The intricate beauty of the mathematical structures that are derived in this field become its main attraction. This is, of course, how this writer perceives the subject and served as the main motivation to prepare a textbook that will help the reader enjoy its study. Of course, one would naturally hope that an author would write a book only about something he or she feels passionate about. I often heard some students describe general topology as being “hard”. Well, some of it is. But I have often thought that maybe this perception was developed because they did not approach it quite in the right way. In this text, we try to improve on the ways that students are introduced to it.

But first, I should at least write a few words about the mathematical content of this textbook. The choice of content as well as the order and pace of the presentation of the concepts found in the text were developed with senior math undergraduate or math graduate students in mind. The targeted reader will have been exposed to some mathematical rigor to a level normally found in an introduction to mathematical analysis texts or as presented in an introduction to linear algebra or abstract algebra texts. The first two sections of Part I consist mostly of a review in the form of a summarized presentation of very basic ideas on normed vector spaces and metric spaces. These are meant to ease the reader into the main subject matter of general topology (in chapter 3 to 20 of Parts II to VI). Parts II to VI normally form the core material contained in most, one or two semester, Basic General Topology course. Once we have worked through the most fundamental concepts of topology in chapters one to twenty, the reader will be exposed to brief introductions to more specialized or advanced topics. These are presented in Part VII in the form of a sequence of chapters many of which can be read or studied, independently, or in short sequences of two or three chapters, provided the student has mastered chapters 3 to 20.

Each chapter is followed by a list of *Concepts review* type questions. These questions highlight for students the main ideas presented in that section and will help test their understanding of these concepts. The answers to all *Concept review* questions are in the

main body of the text. Attempting to answer these questions will help the student discover essential notions which are often overlooked when first exposed to these ideas. Reading a section provides a certain level of understanding, but answering questions, even simple ones, related to its content requires a much deeper understanding. The efforts required in answering correctly such questions leads the student to the ability to solve more complex problems in the Exercise sections. If the student desires a more in-depth study of a topic in Part VII, there are many excellent topology books that can satisfy this need.

Textbook examples will serve as solution models to most of the exercise questions at the end of each section.

In certain sections, we make use of elementary set theory. A student who feels a bit rusty when facing the occasional references to set theory notions may want to review some of these. For convenience, a summary of the main set theory concepts appear at the end of the text in the form of an appendix to the book. A more extensive coverage of naive set theory is offered in the book “Set theory: An introduction to Axiomatic Reasoning” by this writer. It is highly recommended and will serve as an excellent companion to this book.

As we all know, any textbook, when initially published, will contain some errors, some typographical, others in spelling or in formatting and, what is even more worrisome, some mathematical. Critical or alert readers of the text can help weed out the most mistakes by communicating suggestions and comments directly to the author. I hold a particular debt of gratitude to Dr. Eliza Wajch whose keen eye detected a few mathematical inaccuracies, thus sparing this writer from embarrassment and, more importantly, protecting future readers from being subjected to them in following editions.

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