Lec 9, 2010 (part 1)

Note Title 02/06/2010

Last time, we stated ① ② ③ are equivalent:

① $\forall N_1, N_2 \quad \chi^{(1)}(N_1 \otimes N_2) = \chi^{(1)}(N_1) + \chi^{(1)}(N_2)$

② $\forall \rho_1, \rho_2 \quad E_F(\rho_1 \otimes \rho_2) = E_F(\rho_1) + E_F(\rho_2)$

③ $\forall N_1, N_2 \quad S_{min}(N_1 \otimes N_2) = S_{min}(N_1) + S_{min}(N_2)$

Define $\underset{\text{given}}{\chi_\rho}(N) = \max_{\substack{\{p_i, \rho_i\} \\ \text{s.t. } \sum p_i \rho_i = \rho}} S(N(\rho)) - \sum_i p_i S(N(\rho_i))$

Let ⓪ $\forall \rho_1 \rho_2 N_1 N_2, \quad \chi_{\rho_1 \otimes \rho_2}(N_1 \otimes N_2) = \chi_{\rho_1}(N_1) + \chi_{\rho_2}(N_2)$

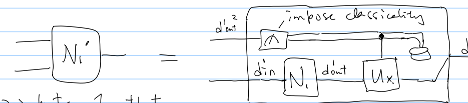
Plan: $\underbrace{① \Rightarrow ③}_{\text{simple}} \underbrace{\Rightarrow ⓪}_{\text{real pain}} \underbrace{\Rightarrow ①}_{\text{missing}}$, $\quad \underbrace{② \Leftrightarrow ⓪}_{\text{simple}}$

① ⇒ ③ Add of $\chi^{(1)}$ ⇒ Add of S_{min}

Given any N_1 & N_2, want $S_{min}(N_1 \otimes N_2) = S_{min}(N_1) + S_{min}(N_2)$

Idea: construct N_1' out of N_1, N_2' out of N_2

and use $\chi^{(1)}(N_1' \otimes N_2') = \chi^{(1)}(N_1') + \chi^{(1)}(N_2')$



2 inputs, 1 output

where $\frac{1}{d_{out}'^2} \sum_{x=1}^{d_{out}'^2} U_x \eta U_x^+ = \frac{I}{d_{out}'}$

$\therefore N_1'(|x\rangle\langle x| \otimes \rho) = U_x N_1(\rho) U_x^+$

Recall optimal ensemble has only pure states.

Because of special structure of N_1'

the ith signal state should look like $|x_i\rangle|\psi_i\rangle$

Also, to max $\chi^{(1)}(N_1')$:

① max $\underbrace{S(\text{average output})}$ ② min ave $\underbrace{S\left(\begin{matrix}\text{individual}\\ \text{outputs}\end{matrix}\right)}$

This can be maximized by choosing x_i uniformly while taking $|\psi_i\rangle$ the same $\forall i$

ith output: $U_{x_i} N_1(|\psi_i\rangle\langle\psi_i|) U_{x_i}^+$

entropy $= S(N_1(|\psi_i\rangle\langle\psi_i|))$

$\therefore$ take $|\psi_i\rangle = |\psi_1\rangle \ \forall i$

where $\rho = |\psi_1\rangle\langle\psi_1|$ min $S(N_1(\rho))$

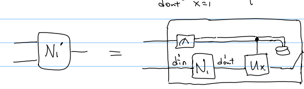
simultaneously optimized

$\therefore \chi^{(1)}(N_1) = \log(d_{out}') - \min_{|\psi\rangle} S(N_1(|\psi_1\rangle\langle\psi_1|))$

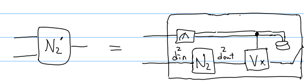
Similarly construct N_2' out of N_2 in the same way.

(Note: N_1, N_2 can have diff input / output dims.)

$\frac{1}{d_{out}'^2} \sum_{x=1}^{d_{out}'^2} U_x \eta U_x^+ = \frac{I}{d_{out}'}$



$N_1'(|x\rangle\langle x| \otimes \rho_1) = U_x N_1(\rho) U_x^+$



$N_2'(|x\rangle\langle x| \otimes \rho_2) = V_x N_2(\rho) V_x^+$

$\frac{1}{d_{out}^2} \sum_{x=1}^{d_{out}^{2\,2}} V_x \eta V_x^+ = \frac{I}{d_{out}^2}$

So, consider $N_1' \otimes N_2'$ (which is also $(N_1 \otimes N_2)'$):

out The optimal ensemble has $d_{out}^{1\,2} \times d_{out}^{2\,2}$ states

of the form $|x_1\rangle|x_2\rangle|\psi\rangle$, drawn uniformly.

Where $x_1 = 1, \cdots, d_{out}'^2$, $x_2 = 1, \cdots, d_{out}^{2\,2}$

$\underbrace{\rho = |\psi\rangle\langle\psi|}_{\text{lives in the joint input spaces of } N_1 \,\&\, N_2}$ min $S(N_1 \otimes N_2(\rho))$.

$\chi^{(1)}(N_1 \otimes N_2) = \log d_{out}^1 + \log d_{out}^2 - \min_{|\psi\rangle} S(N_1 \otimes N_2(|\psi\rangle\langle\psi|))$

But $\chi^{(1)}(N_1) = \log d_{out}^1 - \min_{|\psi_1\rangle} S(N_1(|\psi_1\rangle\langle\psi_1|))$

$\chi^{(1)}(N_2) = \log d_{out}^2 - \min_{|\psi_2\rangle} S(N_2(|\psi_2\rangle\langle\psi_2|))$

$\therefore$ If ① is true ie $\chi^{(1)}$ is additive for ANY 2 channels

Then it holds for N_1' & N_2'

$\therefore \chi^{(1)}(N_1' \otimes N_2') = \chi^{(1)}(N_1') + \chi^{(1)}(N_2')$

Cancelling the $\log(d_{out})$ terms & negating gives:

$\min_{|\psi\rangle} S(N_1 \otimes N_2(|\psi\rangle\langle\psi|)) = \min_{|\psi_1\rangle} S(N_1(|\psi_1\rangle\langle\psi_1|)) + \min_{|\psi_2\rangle} S(N_2(|\psi_2\rangle\langle\psi_2|))$

and the above holds for ANY N_1 & N_2 $\therefore$ ③ is true.

Trying $3 \Rightarrow 0$ or $3 \Rightarrow 1$:

No luck ....

Showing $2 \Leftrightarrow 0$:

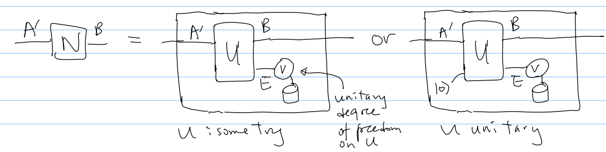
Ingredients:

(i) Stinespring dilation / isometric extension of of N:

Given N (taking sys A' to B)

$\exists\ U$ (taking sys A' to BE)

s.t. $N(\rho) = \mathrm{tr}_E\, U \rho U^\dagger$



(ii) Matsumoto, Shimono, Winter:

Given ρ and N.

$$\chi_\rho(N) = \max_{\{p_i,\ \rho_i\}} S(N(\overset{\text{fixed}}{\rho})) - \sum_i p_i S(N(\underset{\text{can be chosen pure}}{\rho_i}))$$

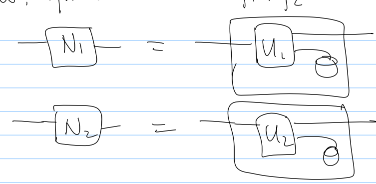
$$= S(N(\rho)) - \min_{\{p_i,\ |\psi_i\rangle\}} \sum_i p_i S(N(|\psi_i\rangle\langle\psi_i|))$$

$$= S(N(\rho)) - \underbrace{\min_{\{p_i,\ |\psi_i\rangle\}} \sum_i p_i E(U|\psi_i\rangle)}_{E_F \text{ of } \sum_i p_i U|\psi_i\rangle\langle\psi_i|U^\dagger \,=\, U\rho U^\dagger \,=\, \rho\text{-}\boxed{U}\text{-}B,E}$$

entanglement across BE

Now, given N_1, N_2, ρ_1, ρ_2

let



arbitrary on the joint input spaces

$$\chi_{\rho_1\otimes\rho_2}(N_1\otimes N_2) = \max_{\substack{\{p_i,\ \eta_i\} \\ \sum_i p_i\eta_i = \rho_1\otimes\rho_2}} \left[S(N_1\otimes N_2(\rho_1\otimes\rho_2)) - \sum_i p_i S(N_1\otimes N_2(\eta_i)) \right]$$

// MSW

$$S(N_1\otimes N_2(\rho_1\otimes\rho_2)) - E_F(U_1\otimes U_2(\rho_1\otimes\rho_2))$$

$$\parallel$$

$$S(N_1(\rho_1)) + S(N_2(\rho_2)) - E_F(U_1\rho_1U_1^\dagger \otimes U_2\rho_2U_2^\dagger)$$

$$\parallel$$

$$\chi_{\rho_1}(N_1) + \chi_{\rho_2}(N_2) + \left[\begin{array}{l} E_F(U_1\rho_1U_1^\dagger) + E_F(U_2\rho_2^\dagger U_2^\dagger) \\ - E_F(U_1\rho_1U_1^\dagger \otimes U_2\rho_2U_2^\dagger) \end{array} \right]$$

$\therefore$ If 2 holds (E_F additivity) then 0 holds $\forall\ N_1\ N_2\ \rho_1\ \rho_2$.

Converse: given σ_1 & σ_2, we can always find ρ_1, ρ_2, U_1, U_2 s.t. $\sigma_1 = U_1\rho_1U_1^\dagger$, $\sigma_2 = U_2\rho_2U_2^\dagger$

$\therefore\ 2 \Leftrightarrow 0$ But unfortunately, does not give $2 \Leftrightarrow 1$ since in $\chi^{(1)}$ we cannot control what ρ is.

Claim of $1 \Rightarrow 2$ were made based on finding a channel for which one can control the optimal average output but no luck in my reading (with tight time constraints).