Supplement to notes for lec 1:

① References:

- Relating: independence & low weight errors, def of QECC, discretization: private comm with Daniel Gottesman

- QECC criterion:
  - Ekert & Macchiavello 1996
  - Knill & Laflamme 1996
  - Bennett, DiVincenzo, Smolin, Wootters 1996
  - Nielsen, Barnum, Caves, Schumacher 1997

  Our version is due to Nielson et al, streamlined with concepts & defs by Gottesman.

- 9-bit Shor code:
  - Shor
  - lecture notes by Cleve for QIC 710

② Definition of weight and identity map if the dimension is increased by a channel:

Let $N_i : B(\mathbb{C}^d) \longrightarrow B(\mathbb{C}^{d'})$, $d' \geq d$.

Let $\tilde{I} = \begin{array}{c} \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \\ & & \\ & & \end{bmatrix} \\ \leftarrow d \rightarrow \end{array} \Big\updownarrow d'$, $\tilde{\mathcal{I}}(\rho) = \tilde{I} \rho \tilde{I}^{\dagger}$

Then the weight of $A = A_1 \otimes A_2 \otimes \cdots \otimes A_m$ where each $A_i : \mathbb{C}^d \to \mathbb{C}^{d'}$ can be defined as the number of A_i's not equal to $\tilde{I}$.

Likewise, $\| N_i - \tilde{\mathcal{I}} \|_\diamond$ is a measure of how close N_i is to the identity map.

The theorems on approx indep errors by low weight error channel go through without major changes.

(3) A slight simplification of P(ii) of part 2, lec 1, starting from the 3rd bullet point:

- $\because$ $|\psi\rangle, |\phi\rangle$ lin indep,

  $|\psi\rangle = a_0 |\phi\rangle + a_1 |\phi^\perp\rangle$ for some $a_0, a_1 \in \mathbb{C}$, $a_1 \neq 0$, $\langle \phi^\perp | \phi \rangle = 0$.

  Applying $\langle \phi^\perp|$ to the last eqn:

  $$\sqrt{c_{E,|\psi\rangle}}\ a_1\ |S_{E,|\psi\rangle}\rangle = \sqrt{c_{E,\eta(|\phi\rangle+|\psi\rangle)}}\ a_1\ |S_{E,\eta(|\phi\rangle+|\psi\rangle)}\rangle$$

  $\because a_1 \neq 0$, $\sqrt{c_{E,|\psi\rangle}} = \sqrt{c_{E,\eta(|\phi\rangle+|\psi\rangle)}} = \sqrt{c_{E,|\phi\rangle}}$

  $|S_{E,|\psi\rangle}\rangle = |S_{E,\eta(|\phi\rangle+|\psi\rangle)}\rangle = |S_{E,|\phi\rangle}\rangle$

  by reversing the roles of $|\psi\rangle, |\phi\rangle$

- $\therefore$, $\forall |\psi\rangle$, $VEU|\psi\rangle = \sqrt{c_E}\ |\psi\rangle |S_E\rangle$

  indep of $|\psi\rangle$

- To show $(\neq)$:

  $\forall |\psi\rangle, |\phi\rangle \in \mathbb{C}^k$, $\forall E_i, E_j \in \mathcal{E}$,

  $$\left(\langle\phi| U^\dagger E_i^\dagger V^\dagger\right)\left(V E_j U |\psi\rangle\right) = \underbrace{\sqrt{c_{E_i}}\sqrt{c_{E_j}}\ \langle S_{E_i} | S_{E_j}\rangle}_{c_{ij}}\ \langle \phi|\psi\rangle$$

  $$\parallel$$

  $$\langle \phi | U^\dagger E_i^\dagger E_j U |\psi\rangle$$

  Recall $P = UU^\dagger$, $\therefore U^\dagger P = U^\dagger$, $PU = U$, $U^\dagger P U = I$ on $\mathbb{C}^k$,

  $\boxed{U}$ $\boxed{U^\dagger}$

  $$\langle \phi | U^\dagger P E_i^\dagger E_j P U |\psi\rangle = c_{ij}\ \langle \phi | U^\dagger P U |\psi\rangle$$

  $$\therefore\quad P E_i^\dagger E_j P = c_{ij}\ P.$$