QIC 890 / CO781 / CS867, W24 Lecture 4, part 2.

Tracking evolution under Clifford unitaries & Pauli meas

① Evolution under Clifford unitary U:

ⓐ Suppose initial state $|\psi\rangle$ is a <u>stabilizer state</u>

(ie $|\psi\rangle \in T(S)$ where S maximal, ie with n generators in n qubits)

eg. $|0\rangle^{\otimes n}$ (S generated by $Z_1, Z_2, \ldots, Z_n$)

eg. $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ (S generated by XX, ZZ)

$\frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$ (" " " $XX, -ZZ$) this is allowed.

$\frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$ (" " " $-XX, ZZ$)

$\frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$ (" " " $-XX, -ZZ$)

From page ① lec 4, if U acts on $|\psi\rangle \in T(S)$,
then new stabilizer $S' = \{UMU^\dagger : M \in S\} =: USU^\dagger$.

More succinctly,

$$\begin{array}{ccc} Q_1 & & UQ_1U^\dagger \\ Q_2 & \longrightarrow & UQ_2U^\dagger \\ \vdots & & \vdots \\ Q_n & & UQ_nU^\dagger \\ \text{generators of } S & & \text{generators of } S' \end{array}$$

eg. $U = $ CNOT. $|\psi\rangle = |00\rangle$,

$$\begin{array}{ccccc} Z_1 & \longrightarrow & Z_1 & \sim & Z_1 \\ Z_2 & & Z_1Z_2 & & Z_2 \end{array}$$

can multiply generators to get new generator.

Of course, CNOT$|00\rangle = |00\rangle$!

eg. $U = $ CNOT, $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$

$$\begin{array}{ccc} -XX & \longrightarrow & -XI \\ ZZ & & IZ \end{array}$$

. Indeed, CNOT$|\psi\rangle = |-\rangle|0\rangle$.

(b) For unrestricted states,

eg. $|0\rangle \to U|0\rangle$ So $|0\rangle\langle 0| = \frac{1}{2}(I+Z) \to U|0\rangle\langle 0|U^\dagger = U\frac{1}{2}(I+Z)U^\dagger = \frac{1}{2}(I+UZU^\dagger)$

$|1\rangle \to U|1\rangle$ $\quad |1\rangle\langle 1| = \quad - \quad U|1\rangle\langle 1|U^\dagger \quad - \quad -$

$|+\rangle \to U|+\rangle$ $\quad |\pm\rangle\langle\pm| = \frac{1}{2}(I \pm X) \to U|\pm\rangle\langle\pm|U^\dagger = \frac{1}{2}(I \pm UXU^\dagger)$

$|-\rangle \to U|-\rangle$

So $X_i \to U X_i U^\dagger$

$Z_i \to U Z_i U^\dagger$ for $i=1,2,\dots n$ in general.

(c) Most general case: Stabilizer code encoding k qubits in n

Suppose $U \in C_n$ is applied,

- the stabilizer generators evolve as: $Q_i \to U Q_i U^\dagger$, $i=1,2,\dots,n-k$
- the encoded Pauli's evolve as: $\underbrace{\begin{array}{ll} \bar{X}_i \to U\bar{X}_i U^\dagger, & i=1,\dots,k \\ \bar{Z}_i \to U\bar{Z}_i U^\dagger, & i=1,\dots k \end{array}}_{\text{inherit com/anticom rel'n}}$

- Case (a) $k=0$, case (b) $k=n$.

Examples:

code 1 $\longrightarrow$ code 2

- 5 qubit code, $U = H^{\otimes 5}$,
  XZZX1 → ZXXZ1
  1XZZX → 1ZXXZ
  X1XZZ → Z1ZXX
  ZX1XZ → XZ1ZX

  $\bar{X} = XXXXX \to ZZZZZ$
  $\bar{Z} = ZZZZZ \to XXXXX$

- 7 qubit code, $U = H^{\otimes 7}$, or $R_{\frac{\pi}{4}}^{\otimes 7}$, or $CNOT^{\otimes 7}$:

Permute stabilizer generators but preserves stabilizer.
" logical Pauli's as logical Clifford gates.

② Evolution under measurement of $P \in \hat{P}_n$.

3 Cases: ⓐ $\pm P \in S$ (syndrome measurements)
ⓑ $P \in N(S) - S$ (measuring encoded Pauli's)
ⓒ $P \notin N(S)$ (taking encoded state out of codespace)

Case ⓐ: State is an eigenstate of P, so unchanged by meas.
$\therefore$ Stabilizer & encoded Paulis unchanged.

Measurement outcome: $\begin{cases} +1 & \text{if } P \in S \\ -1 & \text{if } -P \in S \end{cases}$

eg meas XX on $\frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$, get -1, Bell state unchanged.

Case ⓑ: $P \in N(S) - S$

Pcom with all $M \in S$

Panticom with half of $N(S)/S$.

If encoded state is $|\psi\rangle$, $\text{prob}(\underset{\text{outcome}}{b} = \pm) = \text{tr}\left[\left(\frac{I \pm P}{2}\right) |\psi\rangle\langle\psi|\right]$.

Distribution of b depends on $|\psi\rangle$.

State change: $|\psi\rangle \longrightarrow \frac{1}{2}(I + b \cdot P)|\psi\rangle$

Suppose $S \to S'$, $N(S) \to N(S')$. How to relate $S, S', N(S), N(S')$?

(i) $S \subseteq S'$. Pf: $\forall M \in S$,

$$M\left(\tfrac{1}{2}(I + b \cdot P)|\psi\rangle\right) \underset{\because M, P \text{ com}}{=} \tfrac{1}{2}(I + b \cdot P) M|\psi\rangle$$
$$\underset{\because M \in S}{=} \tfrac{1}{2}(I + b \cdot P)|\psi\rangle$$

$\therefore M$ stabilizes $\frac{1}{2}(I + b \cdot P)|\psi\rangle$.

Intuitively: P compatible with all $M \in S$.
Also meas encoded Pauli is a logical op which preserves the code space.

(ii) $b \cdot P \in S'$. (Meas reduces the encoded space.)

One interpretation is that there is now a <u>logical</u> <u>stabilizer</u> :

$$S = \boxed{\begin{array}{c} Q_1 \\ Q_2 \\ \vdots \\ Q_{n-k} \end{array}}$$

$$\bar{S} : \boxed{b \cdot P}$$ ← a logical Pauli that stabilizes the post-meas logical state

$N(S)/S \cong \hat{\bar{P}}_k$

There are situations we continue to think within the above framework (sub-logical space).

But the above is not quite our usual stabilizer framework.

(iii) How does $N(S)$ transform to $N(S')$?

Before meas: After:

$$\text{gen } S: \boxed{\begin{array}{c} Q_1 \\ Q_2 \\ \vdots \\ Q_{n-k} \end{array}} \qquad S': \boxed{\begin{array}{c} Q_1 \\ Q_2 \\ \vdots \\ Q_{n-k} \\ b\cdot P \end{array}}$$

$$\text{gen } N(S)/S: \begin{array}{|c|c|} \hline \bar{X}_1 & \bar{Z}_1 \\ \hline \bar{X}_2 & \bar{Z}_2 \\ \hline \vdots & \vdots \\ \hline \bar{X}_k & \bar{Z}_k \\ \hline \end{array} \xrightarrow[\text{logical qubit}]{\text{1 fewer}} \boxed{\ ?\ }$$

By lin alg lemma on $\bar{\tilde{P}}_k$, we know half of $N(S)/S$ anticom with P.

All elements in $N(S')$ commute with all elements in S', so elements in $N(S)$ that anticommute with P have to leave.

Procedure: ① At least one of $\bar{X}_1, \bar{X}_2, \dots, \bar{X}_k, \bar{Z}_1, \dots, \bar{Z}_k$ anti com with P.

For concreteness, say $c(\bar{X}_1, P) = 1$.

② For each of $\bar{X}_2, \dots, \bar{X}_k, \bar{Z}_2, \dots, \bar{Z}_k$

If it anticommutes with P, multiply $\bar{X}_1$ to it. Else do nothing.

Call the resulting ops $\bar{X}_2', \dots, \bar{X}_k', \bar{Z}_2', \dots, \bar{Z}_k'$.

we drop $\bar{X}_1, \bar{Z}_1 \rightarrow$ ③ $N(S')/S'$:

$$\begin{array}{|c|c|} \hline \bar{X}_2' & \bar{Z}_2' \\ \hline \bar{X}_3' & \bar{Z}_3' \\ \hline \vdots & \vdots \\ \hline \bar{X}_k' & \bar{Z}_k' \\ \hline \end{array}$$

- Let $F \in \{\bar{X}_2', \bar{X}_3', \dots, \bar{Z}_2', \dots, \bar{Z}_k'\}$.
- F commutes with $Q_1, Q_2, \dots, Q_{n-k}$.
- if $C(E,P) = 1$, then $C(E\bar{X}_1, P) = C(E,P) + C(\bar{X}_1, P) = 0$.

  $\therefore$ Procedure ② ensures F com with P.
- Since $\bar{X}_1$ com with all of $\bar{X}_2, \dots, \bar{X}_k, \bar{Z}_2, \dots, \bar{Z}_k$, the com/anticom rel'ns in $\bar{X}_2, \dots, \bar{X}_k, \bar{Z}_2, \dots, \bar{Z}_k$ stay in $\bar{X}_2', \bar{X}_3', \dots, \bar{X}_k', \bar{Z}_2', \bar{Z}_3', \dots, \bar{Z}_k'$.

Overall effect: $b \cdot P$ promoted to S'
Only $K-1$ encoded qubits remain.

remaining encoded Pauli's can be permuted

$\therefore |\psi\rangle \xrightarrow[\text{Clifford}]{\text{encoded}} \frac{1}{2}(I+bP)|\psi\rangle$

(if P anticom with N instead of $\bar{X}_1$, similar procedure using N instead of $\bar{X}_1$.)

eg. $S: \begin{cases} Q_1 = XXXX \\ Q_2 = ZZZZ \end{cases}$

$N(S)/S: \begin{cases} \bar{X}_1 = XX11, & \bar{Z}_1 = 1Z1Z \\ \bar{X}_2 = X1X1, & \bar{Z}_2 = 11ZZ \end{cases}$

$P = YY11 \quad , \quad b = +$

- We find $\bar{X}_2$ anticom with P, use $\bar{X}_2$ as special op in step ②
- Going over all $\bar{X}_i, \bar{Z}_i$ for $i \neq 2$,
  here, checking $\bar{X}_1, \bar{Z}_1$:
  $\bar{X}_1$ com with P $\therefore$ $\bar{X}_1' = \bar{X}_1 = XX11$
  $\bar{Z}_1$ anti com with P $\therefore$ $\bar{Z}_1' = \bar{Z}_1 \cdot \bar{X}_2 = (1Z1Z)\cdot(X1X1) = XZXZ$.
  NB: $\bar{X}_1', \bar{Z}_1'$ com with Q_1, Q_2, P, anti-com with one another.
- $S': \begin{cases} Q_1 = XXXX \\ Q_2 = ZZZZ \\ P = YY11 \end{cases}$
- $N(S')/S': \begin{cases} \bar{X}_1' = XX11 \\ \bar{Z}_1' = XZXZ \end{cases}$

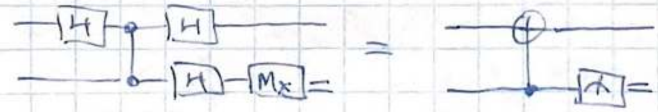
Of course, $\bar{X}_2, \bar{Z}_2$ now removed from $N(S')/S'$.

- What encoded operation has been achieved?
  $\bar{X}_1 \to \bar{X}_1$
  $\bar{Z}_1 \to \bar{Z}_1 \bar{X}_2$
  $\bar{X}_2, \bar{Z}_2$ gone
  $YY11 = \bar{X}_1 \bar{Z}_2 Q_2$ measured

Compare:

$$\begin{array}{ccccccc}
\bar{X}_1 & \longrightarrow & \bar{Z}_1 & \longrightarrow & \bar{Z}_1 & \longrightarrow & \bar{X}_1 \\
\bar{Z}_1 & \longrightarrow & \bar{X}_1 & \longrightarrow & \bar{X}_1\bar{Z}_2 & \longrightarrow & \bar{Z}_1\bar{X}_2 \\
\bar{X}_2 & \longrightarrow & \bar{X}_2 & \longrightarrow & \bar{Z}_1\bar{X}_2 & \longrightarrow & \bar{X}_1\bar{Z}_2 \\
\bar{Z}_2 & \longrightarrow & \bar{Z}_2 & \longrightarrow & \bar{Z}_2 & \longrightarrow & \bar{X}_2 \\
 & \bar{H}I & & \text{Controlled} - Z & & \bar{H}\bar{H} &
\end{array}$$

Encoded circuit:



Case (C) $P \notin N(S)$

- Then $c(P, Q_i) = 1$ for some i
- $\forall |\psi\rangle \in T(S)$, Prob (meas outcome $= +$)

$$= \mathrm{tr}\left[|\psi\rangle\langle\psi| \tfrac{1}{2}(I+P) \right]$$

$$= \tfrac{1}{2} + \tfrac{1}{2} \langle\psi|P|\psi\rangle, \quad \text{but } \langle\psi|P|\psi\rangle = \langle\psi|P\,Q_i|\psi\rangle$$

$$= -\langle\psi|Q_i P|\psi\rangle$$

$$= -\langle\psi|P|\psi\rangle = 0.$$

$$= \tfrac{1}{2}$$

$\therefore$ outcome is random and indep of $|\psi\rangle$.

Let outcome $= b \in \{+,-\}$. $\therefore$ $bP \in S'$.

Reorganize S and $N(S)/S$ as follows.

$\forall j \neq i$, if $c(Q_j, P) = \begin{cases} 0 \\ 1 \end{cases}$ then $\begin{matrix} Q_j' = Q_j \\ Q_j' = Q_j Q_i \end{matrix}$

$\forall j = 1, \ldots, k$ if $c(\bar{X}_j, P) = \begin{cases} 0 \\ 1 \end{cases}$ then $\bar{X}_j' = \begin{matrix} \bar{X}_j \\ \bar{X}_j Q_i \end{matrix}$

if $c(\bar{Z}_j, P) = \begin{cases} 0 \\ 1 \end{cases}$ then $\bar{Z}_j' = \begin{matrix} \bar{Z}_j \\ \bar{Z}_j Q_i \end{matrix}$

$$S: \begin{matrix} Q_1' \\ \vdots \\ Q_{i-1}' \\ Q_i \\ Q_{i+1}' \\ \vdots \\ Q_{n-k} \end{matrix} \xrightarrow{\text{meas } P} S': \begin{matrix} Q_1' \\ \vdots \\ Q_{i-1} \\ b\cdot P \\ Q_{i+1}' \\ \vdots \\ Q_{n-k} \end{matrix}$$

$$N(S)/S: \begin{matrix} \bar{X}_1' \\ \vdots \\ \bar{X}_k' \\ \bar{Z}_1' \\ \vdots \\ \bar{Z}_k' \end{matrix} \qquad N(S')/S': \begin{matrix} \bar{X}_1' \\ \vdots \\ \bar{X}_k' \\ \bar{Z}_1' \\ \vdots \\ \bar{Z}_k'. \end{matrix}$$

all but Q_i com with P!

Recipe:
- Starting from S, find Q_i anticom with P.
- For any Q_j, $j \neq i$ or encoded Pauli, if anti com w/ P multiply Q_i to it.
- Replace Q_i by $b\cdot P$

eg. 5qubit code, meas $P = YZ111$

$Q_1 = X Z Z X 1$

$Q_2 : 1 X Z Z X$

$Q_3 : X 1 X Z Z$

$Q_4 : Z X 1 X Z$

$\bar{X} : X X X X X$

$\bar{Z} : Z Z Z Z Z$

Q_i anticom with P. Use Q_1 as "Q_i in the recipe".

$C(Q_1, P) = 1$

$C(Q_2, P) = 1$

$C(Q_3, P) = 1$

$C(Q_4, P) = 0$

$C(\bar{X}, P) = 0$

$C(\bar{Z}, P) = 1$

$$\therefore S' = \begin{cases} bP = b \cdot YZ111 \\ Q_2' = Q_2 Q_1 = XY1YX \\ Q_3' = Q_3 Q_1 = 1ZYYZ \\ Q_4' = Q_4 = ZX1XZ \end{cases}$$

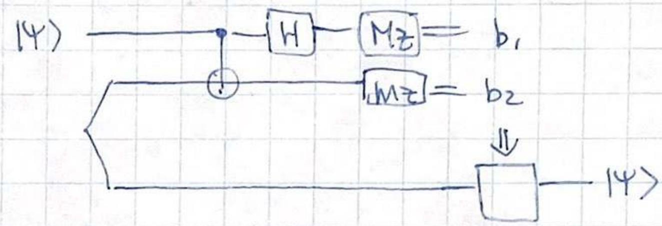
$$N(S')/S' = \begin{cases} \bar{X}' = \bar{X} = XXXXX \\ \bar{Z}' = \bar{Z} Q_1 = Y11YZ \end{cases}$$

In case ©, meas $P \notin N(S)$ changes the "code".

<u>Gottesman–Knill theorem</u>: Given initial stabilizer state, circuit of m Clifford operations, and Pauli measurements, outcome statistics can be simulated in $O(n^3 \cdot m)$ time.

Can improve to $O(n^2 m)$ time.

Example: teleportation. $n=3$



S: $Q_1 = 1XX$ $\quad$ $Q_2 = 1ZZ$ $\xrightarrow{CNOT_{12}}$ $1XX$, ZZZ $\xrightarrow{H_1}$ $1XX$, XZZ

$N(S)/S$: $\bar{X} = X11$ $\quad$ $\bar{Z} = Z11$ $\qquad$ $XX1$, $Z11$ $\qquad$ $ZX1$, $X11$

XZZ, $ZX1$, $X11$ } anti com with $Z11$

$\downarrow$ meas $Z11$, take $Q_i = XZZ$

$1XX$, $b_1 \cdot Z11$, $ZX1$, $1ZZ$

$\longleftarrow$ meas $1Z1$, take $Q_i = 1XX$

$b_2 \; 1Z1$
$b_1 \; Z11$

multiplication by stabilizer

$b_1 \; 11X \sim Z1X = (ZX1)(1XX)$

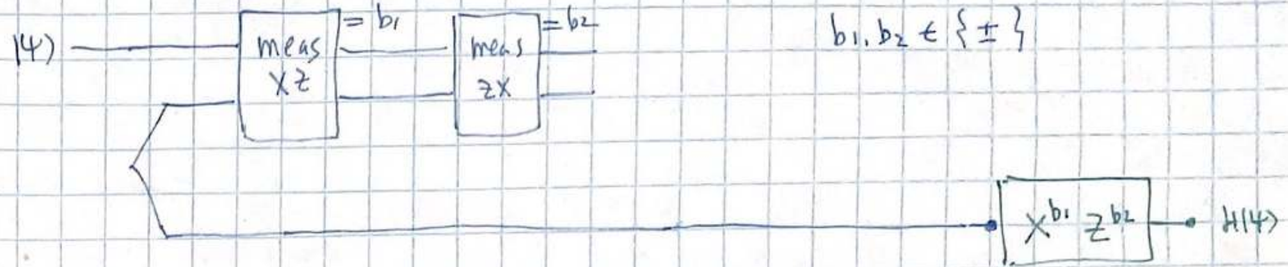
$b_2 \; 11Z \sim 1ZZ$

i.e. "Encoded" qubit $\otimes \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \longrightarrow |b_1\rangle|b_2\rangle \otimes$ "Encoded" qubit.

$Z^{b_1} X^{b_2}$

$11X$
$11Z$

eg. Gate teleportation of H. (apply H, then Bell meas, combined as new meas):



$b_1, b_2 \in \{\pm\}$

Stab gens:	1XX 1ZZ	$\longrightarrow$ b_1 XZ1 1ZZ	$\longrightarrow$ b_1 XZ1 b_2 ZX1	b_1 XZ1 b_2 ZX1	b_1 XZ1 b_2 ZX1
				$\sim$	
Logical Pauli's:	X11 Z11	X11 ZXX	XZZ ZXX	b_1 11Z b_2 11X	11Z 11X

$\therefore$ H is applied.

eg. $\bar{H}$ on 5-qubit code:

① Prepare logical $\frac{1}{\sqrt{2}}(|\bar{0}\rangle|\bar{0}\rangle + |\bar{1}\rangle|\bar{1}\rangle)$, meas $G_i \otimes I$, $I \otimes G_i$ for $i = 1,2,3,4$

meas $\bar{X} \otimes \bar{X}$, $\bar{Z} \otimes \bar{Z}$

$\quad\quad = X^{\otimes 10}$, $= Z^{\otimes 10}$

note: correcting physical Pauli's equiv to getting +1 outcomes.

② meas $\bar{X} \otimes \bar{Z}$, $\bar{Z} \otimes \bar{X}$ on first 2 code blocks.

③ perform $\bar{X}^{b_1} \bar{Z}^{b_2} = (X^{\otimes 5})^{b_1} (Z^{\otimes 5})^{b_2}$ on last code block.

Generally: application & meas of Pauli's give encoded Clifford for the general stabilizer code.

- Finally, meas also gives code switching, gauge fixing etc for stabilizer / subsys codes, Floquet codes, MBQC..., etc... Stab framework makes it easy to characterize reversible meas.