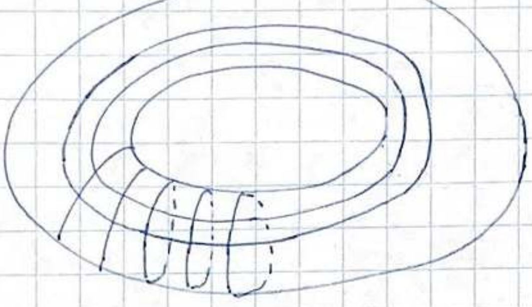
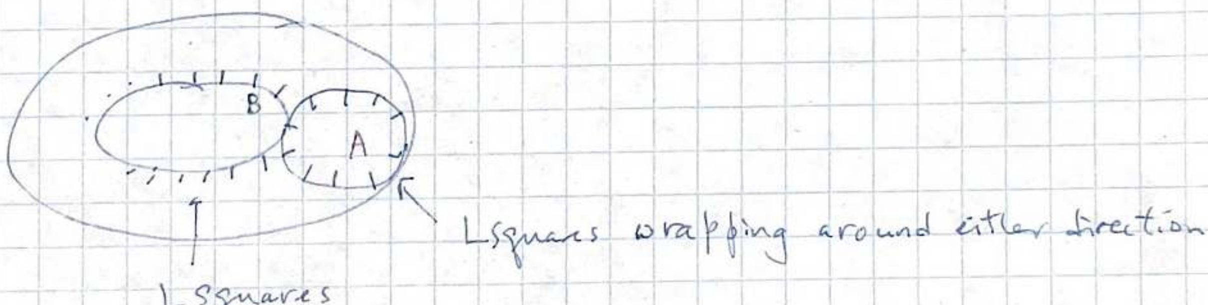
QIC 890, CO 781, CS 867   W 2024   Lecture 6.

Toric code (Kitaev 1997, FTQC by Anyons):



torus, add a square lattice
note: no boundary



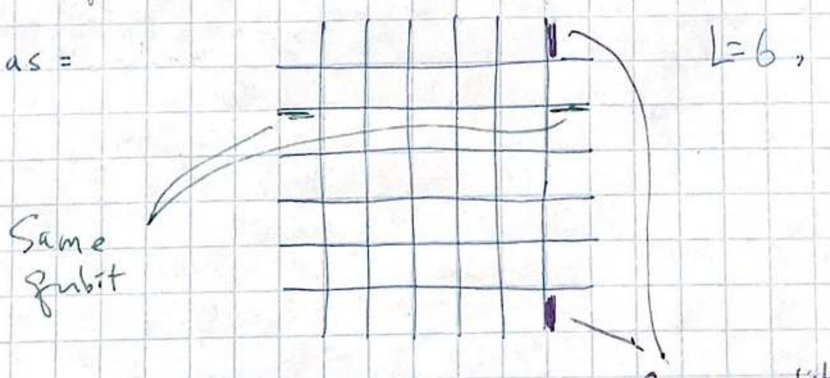
Now associate each edge with a physical qubit.

For the A type circles: L qubits / circle
L circles

Similarly for the B type circles.

Total $2L^2$ qubits.

Can draw as:



$L=6$, L^2 vertices
L^2 faces.

Define a CSS code:

X stabilizers: for each vertex v:  X on qubits on edges incident to v, I elsewhere

"star operator".

Z stabilizers: for each face f:  Z on qubits on edges incident to f, I elsewhere

"plaquette operator"

NB: labelled as A_s, B_p in Kitaev.

use X_v, Z_p here (subscripts are $\underline{\underline{\text{not}}}$ integers, different than the usage X_i for X on the i-th qubit.)

① $\forall v, p,\quad [X_v, Z_p] = 0$ since a star & a plaquette overlap on either 0 or 2 edges.

∴ The above generates a valid stabilizer.

② $\prod_v X_v = I$ since edge e is incident to 2 vertices v, w.

∴ X_e appears in X_v, X_w, and no other X_t.

X_e ↑ X on qubit e $\qquad$ X_v, X_w ↑ star ops on vertex v, w

$\prod_p Z_p = I$ similarly.

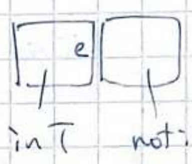
So, the L^2-th X_v and the L^2-th Z_p can be removed from the generator set.

(3) Suppose T = subset of placards s.t. $\prod_{p \in T} Z_p = I$.

Then $T = \phi$ or set of all L^2 placards.

Pf: If $T \neq \phi$, $T \neq$ all placards,

$\exists$ two adjacent placards, one in T and one not.



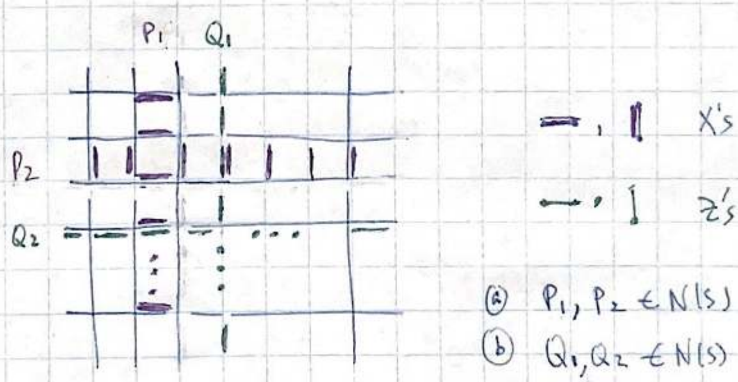
but $\prod_{p \in T} Z_p$ will have Z_e on the edge e incident to both placards, contradicting $\prod_{p \in T} Z_p = I$.

It means L^2-1 Z_p's are multiplicatively independent.

Similarly, any L^2-1 X_v's are multiplicatively indep.

(4) From (3), toric code encodes 2 qubits in $2L^2$ qubits.

(5) Consider the following 4 operators:



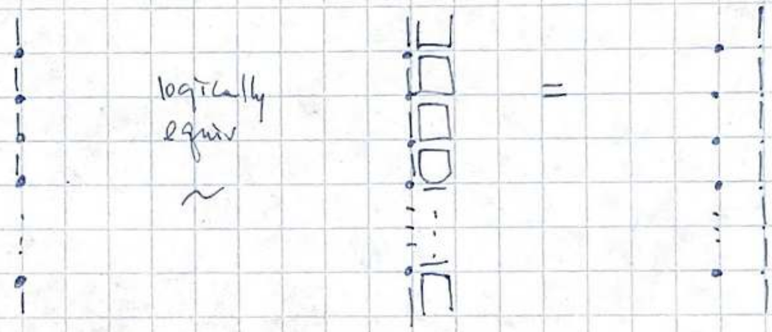
(a) $P_1, P_2 \in N(S)$ since overlap with each Z_p on 2 edges

(b) $Q_1, Q_2 \in N(S)$ - - - - X_v 2 edges

(c) Can shift P_1 left-right using star operators.

Similarly can shift P_2 up-down using star ops.

Can shift Q_1 left-right using placards:



Can shift Q_2 up-down using placard ops.

(d) $[P_1, P_2] = 0$ $\quad\therefore$ Choosing $P_1 = \overline{X}_1$, $P_2 = \overline{X}_2$

$[Q_1, Q_2] = 0$ $\quad Q_1 = \overline{Z}_2$, $Q_2 = \overline{Z}_1$

$[P_1, Q_1] = 0$ $\quad$ satisfies all needed com / anti com relations

$[P_2, Q_2] = 0$

$\{P_1, Q_2\} = 0$ $\quad$ NB we have not yet shown that $P_1, P_2, Q_1, Q_2 \notin S$

$\{P_2, Q_1\} = 0$ $\quad$ but the anticom rel'ns above give a proof.

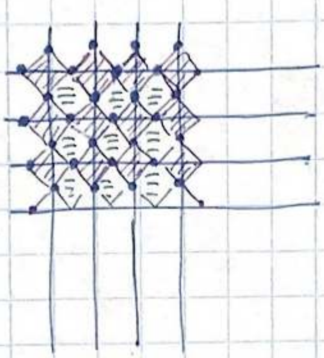
ie $\because Q_2 \in N(S)$ $\therefore P_1 \notin S$

$\because P_1 \in N(S)$ $\therefore Q_2 \notin S$

$\because Q_1 \in N(S)$ $\therefore P_2 \notin S$

$\because P_2 \in N(S)$ $\therefore Q_1 \notin S$.

6 Symmetry between X & Z



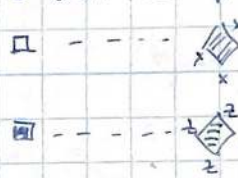
• denotes a qubit

star-op.

plaquette op.

NB: recall the Hypergraph product code in lec5

• corr to $V_1 \times V_2 \cup C_1 \times C_2$



The HGP code has boundary (the surface code) while the toric code does not.

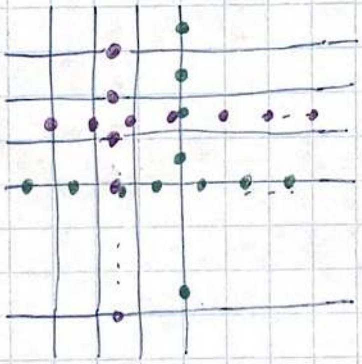
The HGP code only encodes 1 qubit.

(See P7 of lec05 part 2 for example.)

P_1 Q_1

P_2

Q_2



•: X's, • = Z's

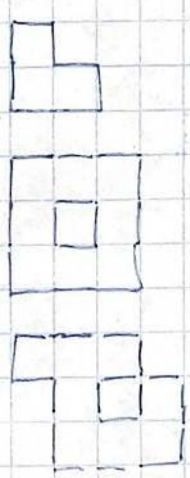
(7) Since the toric is a CSS code, we can consider any Pauli operator by considering the X errors and Z errors separately.

By the symmetry in (6), suffices to consider Z errors.

Using original description that qubits live on edges, a configuration of Z errors corresponds to a subset of edges. The subset can be partitioned into the following :

each edge belongs to only one of the following ←

(a) loops contractable on the torus

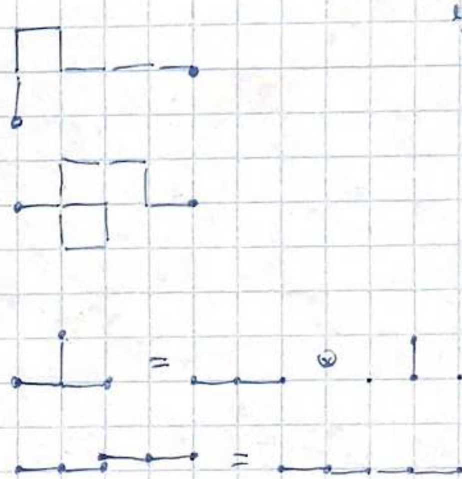


Obs: each is a product of plaquette ops

∴ logical identity ops

- No error !!
- in S

(b) walks with 2 distinct end points



Obs: no repeated edges but vertices may appear in multiple paths

- Detectable errors !!
- not in $N(S)$

if end points are v, w

then error anticom with

X_v and X_w.

(c) loops not contractable on torus

eg P_1, P_2, Q_1, Q_2

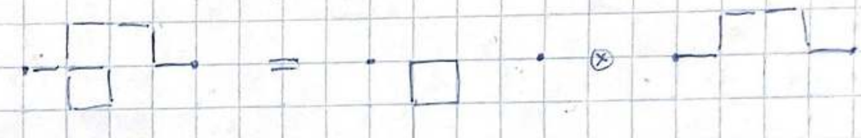
Obs: these are at least weight L.

- Logical error !!

Note we do not replace contractable loops by their logical equivalence; we want to track the weight / prob of the Z error configuration later.

eg. 

But.... we can choose a different partition & move first factor to (a):



[Side note : 

but this is NOT a partition; purple edges are repeated.]

$\therefore$ Can partition the edges into loops and paths, max deg 2,

(8) Recall from lec02- part 2 page (22),

for stabilizer S, distance for $T(S)$ = $\min\{\text{wt } A : A \in N(S) - S\} \geq L$.

since $A = A_X \cdot A_Z$,

at least one of A_X or A_Z has to be in $N(S) - S$, with weight $\geq L$, according to discussion in (7).

$\therefore$ Toric code is a $[[2L^2, 2, L]]$ code.

(correcting arbitrary $\frac{L-1}{2}$ - error channel).

(8) Continuing from (7), for a configuration of Z errors:

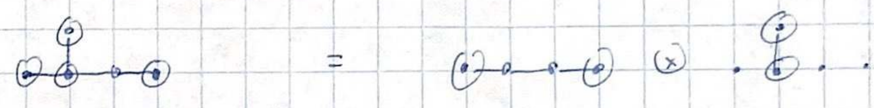
What are the $2L^2-2$ syndrome bits?

(a) meas any Z_p gives +1

(b) meas any X_v gives ± 1 if $\deg(v)$ $\substack{\text{even} \\ \text{odd}}$ for the graph w/ edges rep the Z's.

NB loops in (b) (c) don't contribute.

meas X_v gives -1 if v end pt of a path in (b)



meas X_v gives -1 for all circled vertices (deg 1 or 3)

meas X_v gives -1 for all end pts of paths.

These are called charges.

(Similarly for X errors. p is a vortex if meas Z_p gives -1.)

Charges & Vortices are "excitations".

NB Excitation comes in pairs.

10 Decoding algorithms.

By symmetry of $X \leftrightarrow Z$, consider Z errors.

Let E = set of edges with Z errors, call the error Z_E.

Alg = classical computation returning a set R of edges

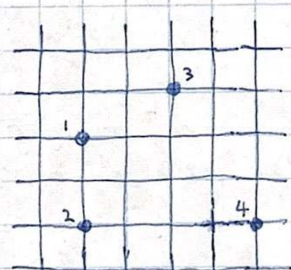
Attempted correction: Z_R (Z on each edge in R)

Correct iff $Z_R Z_E \in S$
$\qquad \qquad = $ product of contractable loops.

ⓐ Minimal weight matching:

Edges in R connect pairs of excitations, min size of R.

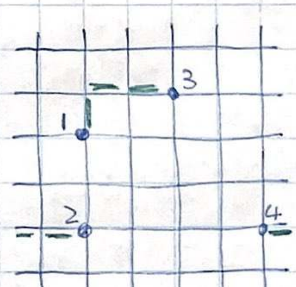
eg.



$L=6$, meas $X_\circ$ gives -1.

dist (1,2) = 2
 " (1,3) = 3
 (2,4) = 2 !!
 (3,4) = 5
 (2,3) = 5
 (1,4) = 4

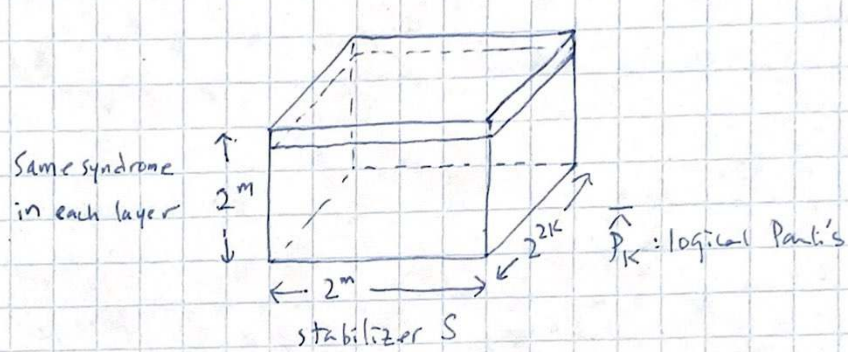
Invalid to connect (1,2), (2,4) (not a matching)
min wt R connects (1,3), (2,4) (matching)



\end{example}

By construction, $Z_R Z_E \in N(S)$ ($\because Z_R, Z_E$ have same syndrome).

Recall from lec 02 - part 2 page 21:



$m = 2L^2 - 2$, $K = 2$ for toric code.

$\therefore$ $Z_R Z_E$ = product of logical Z's (by choice of $\overline{X}$'s, $\overline{Z}$'s)

physical X's Z's

Logical error $\Rightarrow$ $Z_R Z_E$ forms a noncontractable loop

$\Rightarrow$ at least L edges in $R \cup E$

$\Rightarrow$ at least $L/2$ edges in E (else, choosing $R=E$ has strictly lower weight)

$\therefore$ This decoding method, applied to both X & Z errors, corrects arbitrary $\lfloor \frac{L-1}{2} \rfloor$-qubit error channel.

So far we have focused on the distance of a QECC, which is related to the max # of adversarial error that can be corrected.

In lec 01, we showed how independent errors can be corrected using a code with distance $\approx n \cdot \varepsilon \times$ const.

block length   prob error for each qubit.

But the linear distance need not be necessary.

Suppose each qubit has prob ε to have Z error, independently.

$$\text{Prob(logical error)} = \sum_E \text{Prob}(E) \times \text{Prob(logical error} \mid E)$$

$$= \sum_E \text{Prob}(E) \times \underbrace{\text{Prob}\left(Z_R Z_E \text{ contains some non contractable loop} \right)}_{\leq \displaystyle\sum_{\text{non contractable loop } C} \text{Prob}(Z_R Z_E \text{ contains } C)}$$

$$\leq$$

$$= \sum_E \text{Prob}(E) \sum_{l \geq L} (\text{\# of loops of length } l) \, \text{Prob}(Z_R Z_E \text{ contains } \tilde{C})$$

any loop of length l

$$= \sum_{l \geq L} \underbrace{(\text{\# of loops of length } l)}_{\leq L^2 \times 4 \times 3^{l-1}} \times \underbrace{\sum_E \text{Prob}(E) \, \text{Prob}(Z_R Z_E \text{ contains } \tilde{C})}_{\leq 2^l \times \varepsilon^{l/2}}$$

initial pts   # ways to walk.   all possible configuration of Z errors in $\tilde{C}$

$$= \frac{4L^2}{3} \sum_{l \geq L} (36\,\varepsilon)^{l/2} \qquad \left(\varepsilon_0 = \frac{1}{36}\right)$$

$$\leq \frac{4L^2}{3} \cdot \left(\frac{\varepsilon}{\varepsilon_0}\right)^{L/2} \frac{1}{1-\left(\frac{\varepsilon}{\varepsilon_0}\right)^{\frac{1}{2}}} \qquad \text{if } \varepsilon < \varepsilon_0 \approx 2.8\% \text{ !!}$$

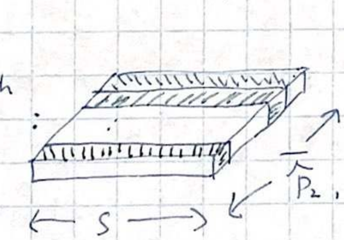
$\therefore$ Can correct $\underbrace{\approx 2.7\% \times 2L^2}_{\gg \text{ distance } L}$ errors with prob $\geq 1 - \exp(-L \times \text{const})$!!

Numerical: 10.6% errors correctable whp !!

(b) Max likelihood decoding.

Suppose E occurs with prob $p(E)$. (nonexisting for adversarial error)

Pauli's with observed syndrome σ



$\overline{P}_2$, 16 elmts $Q_1, \ldots, Q_{16}$

$Q_1 = I$

Each $\overline{Q_i}$ labels a set of 2^{2L^2-2} Paulis.

Let $Pr(\overline{Q_i}) = \sum_E Pr(E)$

contained in the set labeled by Q_i

Find i that max $Pr(\overline{Q_i})$

Revert $\overline{Q_i}$.

NB: for indep errors, max likelihood decoding outperforms min wt matching decoding.

eg. If prob of X error, prob of Z error $< 10.9\%$, prob failure $\sim e^{-\alpha L} \cdot L$.

(11) Surface code ~ toric code

has very good fault-tolerance properties ...

ie if each physical operation (including those used in QECC syndrome meas & correction)

can fail, toric code still works with a reasonable threshold noise rate.

Wait for part 3 for <u>concatenated approach</u>. <u>topological approach</u>

(12) toric code ~ space of min eigenvalue of local Hamiltonian

$$H = -\sum_v X_v - \sum_p Z_p$$

Intuition that gs protected → QECC inspiration

References:

- 9707021

- Keynote : Introduction to surface codes by Kitaev

  youtube channel : Google Quantum AI , Sept 07. 2022

  https:// www.youtube.com / watch? v= M25fBmF9XR0

  number zero.

- Preskill notes (theory. caltech. edu /~preskill / ph 229)

  Toric code recovery , fault-tolerant recovery , fault - tolerate gates
  (2011 handwritten notes) p 1-3.